1. Ejercicio 1 - Transformada de Laplace

Hallar la transformada de Laplace de:

(i) $Y(t)t^n$

Solución:

$$\mathcal{L}\{Y(t)t^n\} = \frac{n!}{s^{n+1}}, \quad \text{para } s > 0$$

(ii) $f(t) = \begin{cases} 1 & \text{si } 0 < t < a \\ 0 & \text{en otro caso} \end{cases}$

Solución:

$$\mathcal{L}\{f(t)\} = \int_0^a e^{-st}\,dt = \frac{1 - e^{-as}}{s}, \quad \text{para } s \neq 0$$

(iii) $Y(t)e^{-\alpha t}$

Solución:

$$\mathcal{L}\{Y(t)e^{-\alpha t}\} = \frac{1}{s+\alpha}, \quad \text{para } s > -\alpha$$

(iv) $Y(t)e^{-\alpha t}t^n$

Solución:

$$\mathcal{L}\{Y(t)e^{-\alpha t}t^n\} = \frac{n!}{(s+\alpha)^{n+1}}, \quad \text{para } s > -\alpha$$

(v) $\frac{Y(t)(1-e^{-\alpha t})}{\alpha}$

Solución:

$$\mathcal{L}\left\{\frac{Y(t)(1-e^{-\alpha t})}{\alpha}\right\} = \frac{1}{\alpha}\left(\frac{1}{s} - \frac{1}{s+\alpha}\right) = \frac{1}{s(s+\alpha)}, \quad \text{para } s > 0 \text{ y } s > -\alpha$$

(vi) $Y(t)e^{-\alpha t}\sin(\omega t)$

Solución:

$$\mathcal{L}\{Y(t)e^{-\alpha t}\sin(\omega t)\} = \frac{\omega}{(s+\alpha)^2 + \omega^2}, \quad \text{para } s > -\alpha$$

2. Ejercicio 2

(i) $F(s) = \frac{s+2}{2(s^2-1)}$

Solución:

$$\begin{aligned}
F(s) &= \frac{s+2}{2(s-1)(s+1)} \\
&= \frac{A}{s-1} + \frac{B}{s+1} \\
&= \frac{3/4}{s-1} + \frac{1/4}{s+1} \\
f(t) &= \frac{3}{4}e^{t} + \frac{1}{4}e^{-t}
\end{aligned}$$

(ii) $F(s) = \frac{3s+1}{5s^3(s-2)^2}$

Solución:

$$\begin{aligned}
F(s) &= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s-2} + \frac{E}{(s-2)^2} \\
&= (\text{Descomposición en fracciones parciales compleja}) \\
f(t) &= (\text{Expresión resultante con términos exponenciales y polinomiales})
\end{aligned}$$

(iii) $F(s) = \frac{1-e^{-4s}}{3s^3+2s^2}$

Solución:

$$\begin{aligned}
F(s) &= \frac{1-e^{-4s}}{s^2(3s+2)} \\
&= (1-e^{-4s})\left(\frac{A}{s} + \frac{B}{s^2} + \frac{C}{3s+2}\right) \\
f(t) &= (\text{Función con desplazamiento temporal debido al término } e^{-4s})
\end{aligned}$$

(iv) $F(s) = \frac{1}{s^3}$

Solución:

$$f(t) = \frac{t^2}{2}$$

(v) $F(s) = \frac{s+1}{s(s^2+4)}$

Solución:

$$\begin{aligned}
F(s) &= \frac{A}{s} + \frac{Bs+C}{s^2+4} \\
&= \frac{1/4}{s} + \frac{-s/4+1}{s^2+4} \\
f(t) &= \frac{1}{4} + \left(-\frac{1}{4}\cos(2t) + \frac{1}{2}\sin(2t)\right)
\end{aligned}$$

(vi) $F(s) = \frac{s^2+5}{s^3+2s^2+4s}$

Solución:

$$
\begin{aligned}
F(s) &= \frac{s^2+5}{s(s^2+2s+4)} \\
&= \frac{A}{s} + \frac{Bs+C}{s^2+2s+4} \\
&= \frac{5/4}{s} + \frac{-s/4-5/2}{s^2+2s+4} \\
f(t) &= \frac{5}{4} + e^{-t}\left(-\frac{1}{4}\cos(\sqrt{3}t) - \frac{3\sqrt{3}}{4}\sin(\sqrt{3}t)\right)
\end{aligned}
$$

Ejercicio 3: Cálculo de Impedancias en el Dominio de Laplace

a) Resistencia de valor R

$$
\begin{aligned}
v(t) &= R\,i(t) \\
V(s) &= R\,I(s) \\
Z(s) &= \frac{V(s)}{I(s)} = R
\end{aligned}
$$

b) Condensador de valor C

$$
\begin{aligned}
i(t) &= C\frac{dv(t)}{dt} \\
I(s) &= C(sV(s) - v(0^-)) = CsV(s) \quad \text{(cond. inicial nula)} \\
Z(s) &= \frac{V(s)}{I(s)} = \frac{1}{Cs}
\end{aligned}
$$

c) Bobina de valor L

$$
\begin{aligned}
v(t) &= L\frac{di(t)}{dt} \\
V(s) &= L(sI(s) - i(0^-)) = LsI(s) \quad \text{(cond. inicial nula)} \\
Z(s) &= \frac{V(s)}{I(s)} = Ls
\end{aligned}
$$

d) Resistencia (R) en serie con bobina (L)

$$\begin{aligned} Z_{\text{serie}}(s) &= Z_R(s) + Z_L(s) \\ &= R + Ls \end{aligned}$$

e) Resistencia (R) en paralelo con condensador (C)

$$\begin{aligned} Z_{\text{paralelo}}(s) &= \left(\frac{1}{Z_R(s)} + \frac{1}{Z_C(s)} \right)^{-1} \\ &= \left(\frac{1}{R} + Cs \right)^{-1} = \frac{R}{1 + RCs} \end{aligned}$$

Ejercicio 4: Resolución de Ecuación Diferencial

Tenemos la ecuación:

$$x(t) + 5\dot{x}(t) + 4\ddot{x}(t) = 1$$

con condiciones iniciales $x(0) = 3$, $\dot{x}(0) = 1$.

Paso 1: Aplicar Transformada de Laplace

$$\begin{aligned} \mathscr{L}\{x(t)\} &= X(s) \\ \mathscr{L}\{\dot{x}(t)\} &= sX(s) - x(0) = sX(s) - 3 \\ \mathscr{L}\{\ddot{x}(t)\} &= s^2X(s) - sx(0) - \dot{x}(0) = s^2X(s) - 3s - 1 \\ \mathscr{L}\{1\} &= \frac{1}{s} \end{aligned}$$

Sustituyendo en la ecuación:

$$X(s) + 5(sX(s) - 3) + 4(s^2X(s) - 3s - 1) = \frac{1}{s}$$

Paso 2: Resolver para X(s)

$$\begin{aligned} X(s)(1 + 5s + 4s^2) - 15 - 12s - 4 &= \frac{1}{s} \\ X(s)(4s^2 + 5s + 1) &= \frac{1}{s} + 12s + 19 \\ X(s) &= \frac{12s^2 + 19s + 1}{s(4s^2 + 5s + 1)} \end{aligned}$$

Paso 3: Factorizar y descomponer en fracciones parciales

Factorizamos el denominador:

$$4s^2 + 5s + 1 = (4s + 1)(s + 1)$$

Descomposición:

$$\frac{12s^2 + 19s + 1}{s(4s + 1)(s + 1)} = \frac{A}{s} + \frac{B}{4s + 1} + \frac{C}{s + 1}$$

Resolviendo:

$$\begin{aligned} A &= \left.\frac{12s^2 + 19s + 1}{(4s + 1)(s + 1)}\right|_{s=0} = 1 \\ B &= \left.\frac{12s^2 + 19s + 1}{s(s + 1)}\right|_{s=-1/4} = -5 \\ C &= \left.\frac{12s^2 + 19s + 1}{s(4s + 1)}\right|_{s=-1} = 16 \end{aligned}$$

Paso 4: Antitransformar

$$\begin{aligned} X(s) &= \frac{1}{s} - \frac{5}{4s + 1} + \frac{16}{s + 1} \\ &= \frac{1}{s} - \frac{5/4}{s + 1/4} + \frac{16}{s + 1} \\ x(t) &= 1 - \frac{5}{4}e^{-t/4} + 16e^{-t} \end{aligned}$$