PARTIAL FRACTION EXPANSIONS: DISTINCT COMPLEX POLES

Distinct complex roots present challenges different from those for the repeated root case. Since the roots are distinct but not real, the methods of equations 12.25 and 12.29 apply. Unfortunately, the resulting partial fraction expansion has complex residues, and the resulting inverse transform has complex exponentials multiplied by complex constants. Such imaginary time functions lack meaning in the real world unless their imaginary parts cancel to yield real-time functions. When they do, our goal is to find a direct route for computing the associated real-time signals. To do this, consider a rational function having a pair of distinct complex poles as in the following equation:

$$F(s) = \frac{n(s)}{[(s+a)^2 + \omega^2]d(s)} = \frac{n(s)}{(s+a+j\omega)(s+a-j\omega)\,d(s)} \tag{12.31}$$

Since the poles $-a - j\omega$ and $-a + j\omega$ are distinct, the partial fraction expansion of equation 12.24 is valid. Since the poles are complex conjugates of each other, the residues of each pole are complex conjugates. Therefore, it is possible to write the partial fraction expansion of $F(s)$ as

$$F(s) = \frac{A+jB}{s+a+j\omega} + \frac{A-jB}{s+a-j\omega} + \frac{n_1(s)}{d(s)} \tag{12.32}$$

for appropriate polynomials $n_1(s)$ and $d(s)$. As per equation 12.25b, the first residue in equation 12.32 is

$$A + jB = (s + a + j\omega)F(s)\,]_{s=-a-j\omega} \tag{12.33}$$

With A and B known, executing a little algebra on equation 12.32 to eliminate complex numbers results in an expression more amenable to inversion, i.e.,

$$F(s) = \frac{C_1 s + C_2}{(s+a)^2 + \omega^2} + \frac{n_1(s)}{d(s)} = F_0(s) + \frac{n_1(s)}{d(s)} \tag{12.34}$$

where

$$C_1 = 2A \tag{12.35a}$$

and

$$C_2 = 2aA + 2\omega B \tag{12.35b}$$

with A and B specified in equation 12.33. With C_1 and C_2 given by equations 12.35, it is straightforward to show that

$$F_0(s) = \frac{C_1 s + C_2}{(s+a)^2 + \omega^2} = C_1 \frac{s+a}{(s+a)^2 + \omega^2} + \left(\frac{C_2 - C_1 a}{\omega}\right)\frac{\omega}{(s+a)^2 + \omega^2} \tag{12.36}$$

From Table 12.1, item 19, or a combination of items 10 and 11,

$$f_0(t) = e^{-at}\left[C_1 \cos(\omega t) + \left(\frac{C_2 - C_1 a}{\omega}\right)\sin(\omega t)\right]u(t)$$

Exercise. Suppose $F(s) = \dfrac{C_1 s + C_2}{s^2 + 4}$. Compute $f(t)$.

ANSWER: $C_1 \cos(2t)u(t) + 0.5C_2 \sin(2t)u(t)$

The following example illustrates the algebra for computing C_1 and C_2 without using complex arithmetic.

EXAMPLE 12.16. Find $f(t)$ when

$$F(s) = \frac{3s^2 + s + 3}{(s+1)(s^2+4)} = \frac{D}{s+1} + \frac{A + jB}{s + j2} + \frac{A - jB}{s - j2} = \frac{D}{s+1} + \frac{C_1 s + C_2}{s^2 + 4} \tag{12.37}$$

Step 1. *Compute the coefficients D, C_1, and C_2 in the partial fraction expansion of equation 12.37.* First we find D by the usual techniques:

$$D = \left.\frac{3s^2 + s + 3}{s^2 + 4}\right]_{s=-1} = 1$$

Given that $D = 1$, to find C_2 we evaluate $F(s)$ at $s = 0$, in which case $0.75 = 1 + 0.25C_2$, or $C_2 = -1$. With $D = 1$ and $C_2 = -1$, we evaluate $F(s)$ at $s = 1$ to obtain $0.7 = 0.5 + 0.2(C_1 - 1)$ or, equivalently, $C_1 = 2$. Thus,

$$F(s) = \frac{1}{s+1} + 2\frac{s}{s^2+4} - 0.5\frac{2}{s^2+4} \tag{12.38}$$

Step 2. *Compute $f(t)$.* Using Table 12.1, items 8 and 9, to compute the inverse transform yields

$$f(t) = [\, e^{-t} + 2\cos(2t) - 0.5\sin(2t)\,]u(t)$$

Alternative Step 1. *Compute A and B in equation 12.37 by hand or with MATLAB.* In MATLAB,

```
»num = [3,1 3];
»den = conv([1 1],[1 0 4])
den = [1 1 4 4]
»[r, p, k] = residue(num,den)
r =
 1.0000 + 0.2500i
 1.0000 - 0.2500i
 1.0000 + 0.0000i
p =
 -0.0000 + 2.0000i
 -0.0000 - 2.0000i
 -1.0000
k = 0
```

This implies that

$$F(s)=\frac{D}{s+1}+\frac{A+jB}{s+j2}+\frac{A-jB}{s-j2}=\frac{1}{s+1}+\frac{1-j0.25}{s+j2}+\frac{1+j0.25}{s-j2} \quad (12.39)$$

Alternative Step 2. One must exercise caution here and note the difference between the MATLAB output and the form of the partial fraction expansion. From equation 12.39, $\omega = +2$, $A = 1$, and $B = -0.25$. Again using MATLAB to obtain the form needed in item 20 of Table 12.1,

```
»K = 2*sqrt(A^2 + B^2)
K = 2.0616
»theta = atan2(B,A)*180/pi
theta = –14.0362
```

Thus

$$f(t) = [\, e^{-t} + 2.0616\cos(2t) + 14.04^{o}\,)]u(t)$$

Example 12.16 illustrates not only the computation of an inverse transform having complex poles, but also the computation of C_1 and C_2 without resorting to complex arithmetic, as was needed in equation 12.32. The trick again was to evaluate $F(s)$ at two distinct s-values different from the poles of $F(s)$. This yields two equations that can be solved for the unknowns C_1 and C_2.

Exercises. 1. Find $f(t)$ when $F(s)=\dfrac{5s^2-8s+4}{s(s^2+4)}$.

ANSWER: $f(t) = [1 + 4\cos(2t) - 4\sin(2t)]u(t)$

2. Find $f(t)$ when $F(s)=\dfrac{5s^2-2s+5}{s(s^2+2s+5)}$.

ANSWER: $f(t) = u(t) + 4e^{-t}[\cos(2t) - \sin(2t)]u(t)$

7. MORE TRANSFORM PROPERTIES AND EXAMPLES

Another handy property of the Laplace transform is the frequency shift property, which permits one to readily compute the transform of functions multiplied by an exponential. With knowledge of the transforms of $u(t)$, $\sin(\omega t)$, and other functions, computation of $e^{-at}u(t)$ and $e^{-at}\sin(\omega t)u(t)$ becomes quite easy.

Frequency shift property: Let $F(s) = \mathcal{L}[f(t)]$. Then

$$\mathcal{L}[\, e^{-at} f(t)] = F(s+a) \quad (12.40)$$