Productos tensoriales de espacios vectoriales. Clase 09/11/2023 / Clase 20

- Relaciones de equivalencia en álgebra y cocientes

$W \subset V$ esp vectoriales (espacio y subespacio)

$\rightsquigarrow \sim_W$, V relación de equivalencia $v \sim_W v' \underset{\text{def}}{\longleftrightarrow} v'-v \in W$

$\rightarrow [v] = \{ v' \in V : v \sim v' \}$ si $v \sim v'$ $v'-v \in W$ o sea

$v' = v + v' - v$ i.e. $v' \in v + W$.

$\Downarrow$

$V \supset [v] = v + W \subset V$. $V/_W = \{ [v] : v \in V \} \subseteq \mathcal{P}(V)$

Además hay operaciones aunque a priori construir un cociente lleva implícito que la nueva estructura tiene operaciones parecidas

$v \sim w$, $v' \sim w'$ $\quad v+v' \sim w+w'$ $\quad \rightarrow [v]+[v'] = [v+v']$, $\lambda[v] = [\lambda v]$

Existe otro ingrediente en todo cociente – la proyección canónica.

$\pi : V \longrightarrow V/_W \quad v \rightsquigarrow [v] = \pi(v) \quad V/_W = \{ \pi(v) : v \in V \}$

y prop univ $\forall\, T : V \longrightarrow U : T(W) = 0 \ \exists!\, \hat{T} : V/_W \longrightarrow U$: el diagrama

$\boxed{W \subset \operatorname{Ker} T}$

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\pi"'] & U \\ V/_W \arrow[ur, "\hat{T}"'] & \end{tikzcd}$$

una conmuta. ($\hat{T} \circ \pi = T$

$\hat{T}([v]) = T(v)$

Entonces Puede ser complejo conocer $V/_W$ pero podemos construir mapas en $V/_W$

Lo central de la construcción de un cociente es:

a) Construir $V/_W$

b) Construir $\pi : V \longrightarrow V/_W$

c) Construir mapas en el cociente mediante la prop universal

$(+,\cdot,1,0)$

$B \subset A \supset (+,\cdot,1,0)$

subanillo anillo

$A \times A \xrightarrow{+} A \qquad (a,a') \longrightarrow a+a'$

$B \times B \xrightarrow{+} B \qquad (b,b') \longrightarrow b+b' \in B$

$\quad\qquad\qquad\qquad\qquad\qquad \in B \ \ \in B$

$A \times A \xrightarrow{\cdot} A$

$B \times B \xrightarrow{\cdot} B$

$a \underset{B}{\sim} a' \overset{def}{\longleftrightarrow} a'-a \in B$

$a, a' \in A$

$a \underset{B}{\sim} a, \quad a \underset{B}{\sim} a' \to a' \underset{B}{\sim} a, \quad a \underset{B}{\sim} a' \underset{B}{\sim} a'' \Longrightarrow a \underset{B}{\sim} a''$

$[a] = \{a' : a \underset{B}{\sim} a'\}$

$a \underset{B}{\sim} a' \leftrightarrow a'-a \in B$

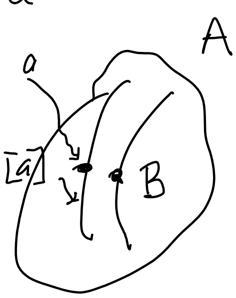
$a' = a + (a'-a)$

$a' = a + b, \quad \boxed{a' \in a+B}$

$a \underset{B}{\sim} a' \leftrightarrow a'-a \in B$

$a' = a + B$

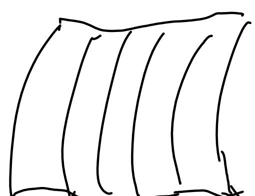
$\boxed{[a] = a+B} \qquad \boxed{a=0} \qquad [0] = \boxed{0 = B}$



$a+B = [a]$

$[a] \subset A$

$\left\{ \begin{array}{l} B = [0] \subset A \\ a+B = [a] \subset A \end{array} \right\}$



subanillo anillo

$a'+B \ \cdots \ a+B \quad B = 0+B$

$B \subset A, \ \underset{B}{\sim}, \ [a] = \{a' : a' \sim a\} = a+B \qquad \boxed{A/B = \{[a] : a \in A\}}$

$\pi : A \to A/B$ función.

$a \longrightarrow [a]$

A

$\left(A/B = \{[a] : a \in A\} \right.$

$[a] \subset A$

$A/B \subset \mathcal{P}(A)$

$[a] \subset A$

$A = \bigcup_{a \in A} [a] \longleftarrow$

$\left([a] = [a+b] \quad \forall b \in B \right.$

2 elementos de un cociente de tipo algebraico son $\boxed{A/B;\ \pi : A \longrightarrow A/B}$

$I \subset A \qquad A$ anillo I ideal $\ \ldots \ A/I$

$$T(W) = 0$$

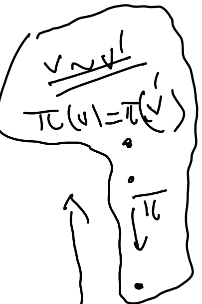
Propiedad universal

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\pi"'] & U \\ V/W \arrow[ur, "\hat{T}"'] & \end{tikzcd}$$

dado U y T con ciertas condiciones

$\exists ! \hat{T} : V/W \to U$

: el diagrama conmuta



Producto tensorial

Es la misma idea pero con algunas diferencias que son

La "relación de equivalencia" se da en $V \times W$ los pares de vectores de V y W

$$\begin{tikzcd} V \times W \arrow[d, "\otimes"'] \\ V \otimes W \end{tikzcd}$$

¡No se da explícitamente se da indirectamente diciendo que $\otimes$ es bilineal - ya se lo dá la propiedad. que es:

$\otimes(v,w) = v \otimes w$ y la bilinealidad dice que

$$\begin{cases} \lambda v \otimes w = v \otimes \lambda w = \lambda(v \otimes w) \text{ en } V \otimes W \\ (v_1+v_2)\otimes w = v_1 \otimes w + v_2 \otimes w \\ v \otimes (w_1+w_2) = v \otimes w_1 + v \otimes w_2 \end{cases}$$

$\otimes(v_1+v_2, w) = \otimes(v_1, w) + \otimes(v_2, w)$

$\otimes(\lambda v, w) = \lambda \otimes(v, w)$

La prop universal

$\exists ! \ \hat{c}$

$$\begin{tikzcd} V \times W \arrow[r, "c"] \arrow[d, "\otimes"'] & U \\ V \otimes W \arrow[ur, "\hat{c}"'] & \end{tikzcd}$$

c bilineal

lineal :

el diagrama conmuta.

o sea $\hat{c}(v \otimes w) = c(v,w)$ estamos diciendo que si

c verifica $c(\lambda v, w) = \lambda c(v,w) = c(v, \lambda w)$

$c(v_1+v_2, w) = c(v_1,w) + c(v_2,w)$

$c(v, w_1+w_2) = c(v,w_1) + c(v,w_2)$

$\Rightarrow \exists \ \hat{c} : \hat{c}(v \otimes w) = c(v,w)$

$$\begin{array}{ccc} V\times W & \xrightarrow{c} & U \\ \downarrow{\otimes} & \nearrow{\hat{c}} & \\ V\otimes W & & \end{array}$$

c biliniară în U esp. vect. $\exists!\ \hat{c}$

$\hat{c}$ liniară: el diagrama comută

biliniară $\quad \hat{c}(v\otimes w) = c(v,w)$

$$\begin{array}{ccc} V & \xrightarrow{T} & U \\ \downarrow{\pi} & \nearrow{\hat{T}} & \\ V/W & & \end{array}$$

$[v]$

Si

$$\begin{array}{ccc} V\times W & \xrightarrow{c} & U \\ \downarrow{\otimes} & \nearrow{\hat{c}} & \\ V\otimes W & & \end{array}$$

c biliniară

$\exists!$ liniară

liniară

liniară

liniară

sobre

liniară

$$\begin{array}{ccc} V & \xrightarrow{T} & U \\ \downarrow{\pi} & \nearrow{\hat{T}} & \\ V/W & & \end{array}$$

$\forall T$ liniară

$\exists!\ \hat{T}$

$\hat{T}: V/W \to U$

și liniară

Ejemplo El producto escalar y vectorial

$(av+bv')\cdot \omega = a(v,\omega)+b(v',\omega)$

$$\begin{array}{ccccc} \mathbb{R}^2 \times \mathbb{R}^2 & \xrightarrow{\cdot} & \mathbb{R} \\ \Psi \quad \Psi & & \\ v \quad \omega & \rightsquigarrow & v\cdot\omega \\ (v_1,v_2) \; (\omega_1,\omega_2) & \rightsquigarrow & v_1\omega_1+v_2\omega_2 \end{array}$$

$$\mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$(v,\omega) = v\wedge\omega$

$$\begin{pmatrix} v_1\\ v_2\\ v_3\end{pmatrix} \wedge \begin{pmatrix} \omega_1\\ \omega_2\\ \omega_3\end{pmatrix} = \begin{pmatrix} v_2\omega_3 - v_3\omega_2 \\ v_3\omega_1 - v_1\omega_3 \\ v_1\omega_2 - v_2\omega_1\end{pmatrix}$$

$(v+v')\wedge\omega = v\wedge\omega + v'\wedge\omega$
$v\wedge(\omega+\omega') = v\wedge\omega + v'\wedge\omega$
etc.

$v\otimes\omega \to v\wedge\omega$

$\mathbb{R}^2\times\mathbb{R}^2 \xrightarrow{\cdot} \mathbb{R}$, $\otimes$, $\mathbb{R}^2\otimes\mathbb{R}^2$, $\tilde{\cdot}$

$v,\omega \to v\cdot\omega$, $\otimes$, $v\otimes\omega$, $\tilde{\cdot}$

$$\begin{array}{ccc} \mathbb{R}^2\otimes\mathbb{R}^2 & \xrightarrow{\hat{\cdot}} & \mathbb{R} \\ \Psi & & \\ v\otimes\omega & \rightsquigarrow & v\cdot\omega \end{array}$$

es lineal

En base $\mathbb{R}^2\otimes\mathbb{R}^2$ tiene dim 4 y la base es $\{e_1\otimes e_1, e_1\otimes e_2, e_2\otimes e_1, e_2\otimes e_2\}$

$e_1\otimes e_1 \to 1$, $e_1\otimes e_2 = 0$, $e_2\otimes e_1 \to 0$, $e_2\otimes e_2 \to 1$

Con el producto vect tenemos dim 9 y sea t.l dim 9 $\longrightarrow$ dim 3

La base es $\{e_1\otimes e_1,\ e_1\otimes e_2,\ e_1\otimes e_3,\ e_2\otimes e_1,\ e_2\otimes e_2,\ e_2\otimes e_3,\ e_3\otimes e_1, e_3\otimes e_2, e_3\otimes e_3\}$

$e_1\otimes e_1 \longrightarrow 0$ $\qquad e_2\otimes e_1 \longrightarrow -e_3$

$e_1\otimes e_2 \longrightarrow e_3$ $\qquad e_2\otimes e_2 \longrightarrow 0$

$e_1\otimes e_3 \longrightarrow -e_2$ $\qquad e_2\otimes e_3 \longrightarrow e_1$

$e_3\otimes e_1 \longrightarrow e_2$

$e_3\otimes e_2 \longrightarrow -e_1$

$e_3\otimes e_3 \longrightarrow 0$

Ej Matriz asociada.

Ejercicio. Sean

$e_1 e_2 e_3 \qquad e_1 e_2 e_3 \qquad e_1 e_2 e_3$

$\mathbb{R}^3 \otimes \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \qquad v \otimes w \to v \wedge w$ lineal

$\mathbb{R}^9 \longrightarrow \mathbb{R}^3$ lineal cuya matriz asociada en la base natural y la estándar

$(e_1 \otimes e_1, e_1 \otimes e_2, e_1 \otimes e_3, e_2 \otimes e_1, e_2 \otimes e_2, e_2 \otimes e_3, e_3 \otimes e_1, e_3 \otimes e_2, e_3 \otimes e_3)$ $\mathcal{B} \subset \mathbb{R}^3 \otimes \mathbb{R}^3$

$\mathcal{G} \subset \mathbb{R}^3$

(e_1, e_2, e_3)

$(e_1 \wedge e_1 = 0 \quad e_1 \wedge e_2 = e_3 \quad e_1 \wedge e_3 = -e_2, -e_3, 0, e_1, e_2, -e_1, 0)$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\mathbb{R}^3 \otimes \mathbb{R}^3 \xrightarrow{\bigcirc} \mathbb{R}^3$

$v \otimes w \longrightarrow v \wedge w$

bilineal

Ejercicio $\quad V_1 \xrightarrow{T} V_2 \qquad W_1 \xrightarrow{S} W_2$

bilineal Conclusión

$$\begin{tikzcd} V_1 \times W_1 \arrow[r, "T\times S"] \arrow[d, "\otimes"] \arrow[dr, "\otimes \circ (T\times S)"] & V_2 \times W_2 \arrow[d, "\otimes"] \\ V_1 \otimes W_1 \arrow[r, dotted, "\otimes \circ (T\times S)"'] & V_2 \otimes W_2 \end{tikzcd}$$

bilineal

$(v_1, w_1) \to (Tv_1, Sw_1) \to Tv_1 \otimes Tw_1$

$$\begin{tikzcd} U \times V \arrow[r, "b"] & W \\ U' \times V' \arrow[u, "A\times B"] \arrow[ur, "b\circ(A\times B)"'] & \end{tikzcd}$$

lineal

$A: U' \to U$

$B: V' \to V$

$$\begin{tikzcd} U \times V \arrow[d, "\otimes"] \arrow[dr, "b"] & \\ U \otimes V \arrow[r, dotted, "\hat{b}"] & W \end{tikzcd}$$

$$\begin{tikzcd} V_1 \times W_1 \arrow[r, "T\times S"] \arrow[d, "\otimes"] \arrow[dr, dashed, "\otimes\circ(T\times S)"] & V_2 \times W_2 \arrow[d, "\otimes"] \\ V_1 \otimes W_1 \arrow[r, "T\otimes S"] & V_2 \otimes W_2 \end{tikzcd}$$

$\hat{b} =$

$$\begin{tikzcd} V_1 \times W_1 \arrow[r, "T\times S"] \arrow[d, "\otimes"] \arrow[dr, "\exists"] & V_2 \times W_2 \arrow[d, "\otimes"] \\ V_1 \otimes W_1 \arrow[r, "?"] & V_2 \otimes W_2 \end{tikzcd}$$

$= \sum$

$U \otimes V$

$\sum u_i \otimes v_i \qquad \mathcal{U}_0 = \langle u_1 \cdots u_n \rangle$

$$U \otimes V \ni \sum_{i=1}^{m} u_i \otimes v_i \qquad \langle u_1 \cdots u_m \rangle = \langle \bar{u}_1, \ldots, \bar{u}_t \rangle$$

$$\sum_{l=1} \bar{u}_l$$

$$\| \quad \sum_{j=1}^{t} \bar{u}_j \otimes \bar{v}_j \quad \text{li}$$

$$\boxed{u_1 \otimes v_1 + u_2 \otimes v_2} \qquad \langle u_1, u_2 \rangle = \langle \alpha_1, \alpha_2 \rangle$$

$$u_1 = A_1 \alpha_1 + A_2 \alpha_2$$

$$u_2 = B_1 \alpha_1 + B_2 \alpha_2$$

$$(A_1\alpha_1 + A_2\alpha_2) \otimes v_1 + (B_1\alpha_1 + B_2\alpha_2) v_2$$

$$= A_1(\alpha_1 \otimes v_1) + A_2(\alpha_2 \otimes v_1) + B_1(\alpha_1 \otimes v_2) + B_2(\alpha_2 \otimes v_2)$$

$$= \alpha_1 \otimes A_1 v_1 + \alpha_2 \otimes A_2 v_1 + \alpha_1 \otimes B_1 v_2 + \alpha_2 \otimes B_2 v_2$$

$$= \alpha_1 \otimes (A_1 v_1 + B_1 v_2) + \alpha_2 \otimes (A_2 v_1 + B_2 v_2)$$

$$\sum_{i=1}^{m} u_i \otimes v_i \overset{?}{=} \sum_{j=1}^{t} \alpha_j \otimes \beta_j \qquad \langle u_1 \cdots u_m \rangle = \langle \alpha_1, \ldots \alpha_t \rangle$$

$\alpha_1 \ldots \alpha_t$ li

$$\| \quad \underbrace{\sum_{i=1}^{m-1} (u_i \otimes v_i)}_{\sum_{j=1}^{t} (\alpha_j \otimes \beta_j)} + u_m \otimes v_m$$

$$\langle \alpha_1 \ldots \alpha_t, u_m \rangle \qquad u_m = a_1\alpha_1 + \cdots + a_t\alpha_t$$

$$\sum_{j=1}^{t} (\alpha_j \otimes \beta_j) + (u_m \otimes v_m)$$

li

$$= \sum_{j=1}^{t} \alpha_j \otimes \beta_j + a_1\alpha_1 \otimes v_m + \cdots + a_t\alpha_t \otimes v_m$$

$$= \sum_{j=1}^{t} \alpha_j \otimes \beta_j + \alpha_1 \otimes a_1 v_m + \cdots + \alpha_t \otimes a_t v_m$$

$$= \alpha_1 \otimes (\beta_1 + a_1 v_m) + \cdots + \alpha_t \otimes (\beta_t + a_t v_m)$$

$$0 = \sum_{i=1}^{m} u_i \otimes v_i \qquad (u_1 \ldots u_m) \text{ li} \implies v_1 = \cdots = v_m = 0$$

$$(u_j^* \otimes \mathrm{id})\left(\sum u_i \otimes v_i\right) = \cdots$$

$$\langle u_1, \ldots u_m \rangle \quad u_1^* \ldots u_m^*$$

$$\begin{array}{c} U \\ \downarrow f \\ \mathbb{R} \end{array}$$

$$U \otimes V \ni \xi = \sum u_i \otimes v_i$$

$$\begin{array}{c} U \otimes V \\ \downarrow f \otimes id \\ \mathbb{R} \otimes V \\ \shortparallel \\ V \end{array}$$

$$\sum f(u_i) \otimes v_i = \sum f(u_i)\, v_i \in V$$

$$\begin{array}{c} U \otimes V \\ \downarrow f \otimes id \\ V \end{array}$$

$$\boxed{U \otimes V} \xrightarrow[f \otimes id]{} \mathbb{R} \otimes V \xrightarrow{\approx} \boxed{V} \ni$$

$$\sum u_i \otimes v_i \longrightarrow \boxed{\sum_{i=1} f(u_i)\, v_i}$$

$$\mathbb{R} \otimes V \cong V$$

$$\lambda \otimes v \longrightarrow \lambda v$$

$$1 \otimes v \longleftarrow v$$