

$M \in \mathcal{M}_A$ si existen $+$, $\cdot: M \times M \rightarrow M$; $\cdot: A \times M \rightarrow M$ tal que

$(M, +, 0)$ es un grupo abeliano y el producto $\begin{cases} a \cdot (a' \cdot u) = (aa') \cdot u \\ 1 \cdot u = u \end{cases}$

y cumplen con las condiciones de coherencia de $+$ y \cdot siguientes:

$$\begin{cases} (a+b) \cdot u = a \cdot u + b \cdot u \\ 0 \cdot u = 0 \\ a \cdot (m+n) = a \cdot m + a \cdot n \end{cases}$$

$$\begin{cases} m \cdot (a+b) = m \cdot a + m \cdot b \\ (m \cdot a) \cdot a' = m \cdot (aa') \\ \text{Derecha} \end{cases}$$

f es un morfismo de A -módulo si $f(m+n) = f(m) + f(n)$

de $\text{Mod-}A$ $\begin{cases} f(a \cdot m) = a \cdot f(m) \\ f(m \cdot a) = f(m) \cdot a \end{cases}$

Δ^M e tambem $\Delta^M \mathbb{Z}$ $a \cdot u$ $m \cdot z = \underbrace{z + \dots + z}_m$

$$\underbrace{(a \cdot u)}_z \cdot z = a \cdot \underbrace{(m \cdot z)}_z$$
$$\underbrace{a \cdot u + \dots + a \cdot u}_z = a \cdot \underbrace{(m + \dots + m)}_z$$

Es claro entonces que $A^M = A^M \cdot 1$; $1_A = 1_A \cdot 1_A$

Ejemplos menos triviales.

$M_n(\mathbb{R}) \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ $A = M_n(\mathbb{R})$ \mathbb{R}^m vs M
 $X \in M_n(\mathbb{R})$ $m \in \mathbb{R}^m$ $X \cdot m = X m$ - producto de
matriz por vector. Acción izquierda (\cdot) (\cdot)

Si $v \in \mathbb{R}^m$ $X \in M_n(\mathbb{R})$ $v \cdot X = v^t X$ y es claro que
con esas dos acciones \mathbb{R}^m es un $M_n(\mathbb{R})^M$ $M_n(\mathbb{R})$.

El axioma crucial a verificar es $(a \cdot m) \cdot b = a \cdot (m \cdot b)$
y así tenemos:

$$(X \cdot v) \cdot Y = (X v) \cdot Y = X v Y$$

$$X \cdot (v \cdot Y) = X \cdot (v Y) = X v Y$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Producto tensorial de bimódulos

Si (M, N) es un par de bimódulos con A, R y S anillos
 $M \in {}_R \mathcal{M}_A$; $N \in \mathcal{M}_A^S$

$$\Rightarrow M \otimes_A N \in \mathcal{M}_R^S \quad \text{que} \quad \otimes_A : {}_R \mathcal{M}_A \times \mathcal{M}_A^S \rightarrow \mathcal{M}_R^S$$

\downarrow \downarrow \downarrow
 M N $M \otimes_A N$
: A

Propiedad básica. El producto tensorial lleva
bilineales en transformaciones lineales.

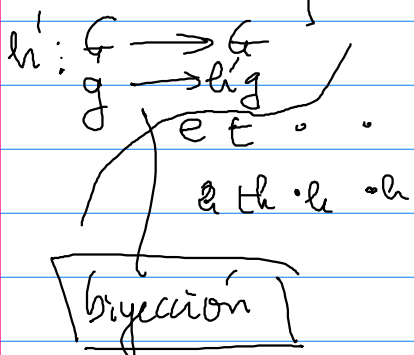
transformaciones

$$H \subset G \quad \frac{1}{|H|} \sum_{h \in H} h = e_H$$

$$(\sum_{g \in G} a_g g) h = \sum_{g \in G} a_g (gh)$$

$$h(\sum_{g \in G} a_g g) = \sum_{g \in G} a_g (hg)$$

$$h' e_H = \frac{1}{|H|} \sum_{h \in H} h' h$$



$$\sum_{h \in H} h = \sum_{h \in H} h' h$$

$$\begin{aligned}
 a_g &= 1 & g \in H \\
 a_g &= 0 & g \notin H \\
 a_g &= \frac{1}{|H|} & g \in H \\
 a_g &= 0 & g \notin H
 \end{aligned}$$

$$h' e_H = e_H$$

$$\begin{aligned}
 e_H &= h' e_H + h' e_H + \dots + h' e_H \\
 &= \sum_{h \in H} h' e_H = \sum_{h \in H} h' e_H = e_H
 \end{aligned}$$

$$(H/e_H = (\sum_{h \in H} h') e_H$$

$$= |H| e_H e_H$$

$$\{ \sum_{g \in G} a_g g \} \in kG$$

$$|H| e_H = |H| e_H^2$$

$$(\sum_{g \in G} a_g g) h = \sum_{g \in G} a_g gh$$

$$1 - \sigma + \frac{1}{7} \sigma^2 \in \mathbb{Q}G$$

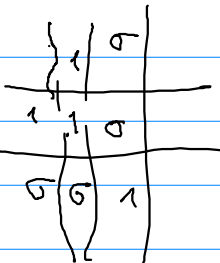
$$e = \frac{1}{3} (1 + \sigma + \sigma^2) = \frac{1}{3} + \frac{1}{3} \sigma + \frac{1}{3} \sigma^2$$

$$G = \{1, \sigma, \sigma^2\} \quad \sigma^3 = 1$$

$$H = \{1\} \quad \{1, \sigma, \sigma^2\} = H$$

$$\begin{aligned}
 & (\sum_{g \in G} a_g g) (\sum_{l \in G} b_l l) \\
 &= \sum_{l \in G} b_l (\sum_{g \in G} a_g gl) \\
 &= \sum_{l \in G} b_l (\sum_{g \in G} a_g) l
 \end{aligned}$$

$$\begin{aligned}
 G &= \{1, \sigma\} \quad \sigma^2 = 1 \\
 \mathbb{Q}G &= \{a_1 \cdot 1 + a_0 \sigma : a_i \in \mathbb{Q}\}
 \end{aligned}$$



$$(a_0 + a_1 X) (b_0 + b_1 X) \quad X^2 \quad X^1 \quad -$$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0) X + a_1 b_1 X^2$$

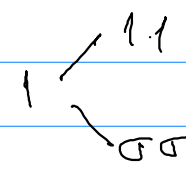
$$(a_1' 1 + a_0' \sigma) (b_1 1 + b_0 \sigma)$$

$$= a_1 b_1 + a_1 b_0 \sigma + a_0 b_1 \sigma + a_0 b_0 \sigma^2 = (a_1 b_1 + a_0 b_0) + (a_1 b_0 + a_0 b_1) \sigma$$

$$\left(\frac{1}{2} + \frac{1}{2} \sigma \right) \left(\frac{1}{2} + \frac{1}{2} \sigma \right)$$

$$= \frac{1}{4} \left(1 + \frac{1}{4} \sigma + \frac{1}{4} \sigma + \frac{1}{4} \right)$$

$$= \frac{1}{2} + \frac{1}{2} \sigma$$



$$\left(\sum_{g \in G} a_g g \right) \left(\sum_{l \in G} b_l l \right) = \sum_{g \in G} \sum_{l \in G} a_g b_l gl$$

$$\left(\sum_{g \in G} a_g g \right) \left(\sum_{l \in G} b_l l \right) = \sum_{x \in G} \left(\sum_{\{g \in G, l \in G : gl = x\}} a_g b_l \right) x$$

$$(a_1 \cdot 1 + a_0 \sigma) \times (b_1 1 + b_0 \sigma) = \sum (a_1 1 + a_0 \sigma) (b_1 1 + b_0 \sigma)$$

$$= \sum a_1 b_1 1 + a_1 b_0 \sigma + a_0 b_1 \sigma + a_0 b_0 \sigma^2$$

$$(a_1 b_1 + a_0 b_0) \cdot 1 + (a_1 b_0 + a_0 b_1) \sigma$$

$$\boxed{1, \sigma, \sigma^2, \sigma^3} \quad \sigma^4 = 1 \quad (a_1 + a_0 \sigma + a_0 \sigma^2 + a_1 \sigma^3) (\dots)$$

$$(a_1 + a_0 \sigma) (b_1 + b_0 \sigma)$$

$$= a_1 b_1 + a_1 b_0 \sigma + \dots \quad G \quad \frac{1}{4} (1 + a + b + c)$$

$$(1, a), (1, b)$$

$$(1, c)$$

$$\begin{matrix} & 1 & a & b & c \\ 1 & 1 & a & b & c \\ a & a & 1 & c & b \\ b & b & c & 1 & a \\ c & c & b & a & 1 \end{matrix}$$

$$\begin{aligned} & \frac{1}{2} (1+a) \\ & \frac{1}{2} (1+b) \\ & \frac{1}{2} (1+c) \end{aligned}$$

$|H| / |G|$

$$(\lambda_1 + \lambda_2 a + \lambda_3 b + \lambda_4 c)$$

$$(\mu_1 + \mu_2 a + \mu_3 b + \mu_4 c)$$

$$= \frac{(\lambda_1 \mu_1 + \lambda_2 \mu_2 a + \lambda_3 \mu_3 b + \lambda_4 \mu_4 c)}{1}$$

$$e \{1, a\} \{1, b\} \{1, c\} \{1, a, b, c\}$$

$$\frac{1}{2} \quad \frac{2}{2} \quad \frac{4}{2} \quad [a][b] = [ab]$$

$$\mathbb{Z}_2 \quad [0] \quad [1] \quad [2] \quad [3] \quad [4] \quad [5]$$

$$\left[\begin{array}{l} [2][4] = 8 = 2 \pmod 4 \\ [2][3] = 6 = 2 \pmod 4 \\ [0]^2 = [0] \quad [1]^2 = 1 \quad [3]^2 = [1] \quad [4]^2 = 1 \end{array} \right]$$