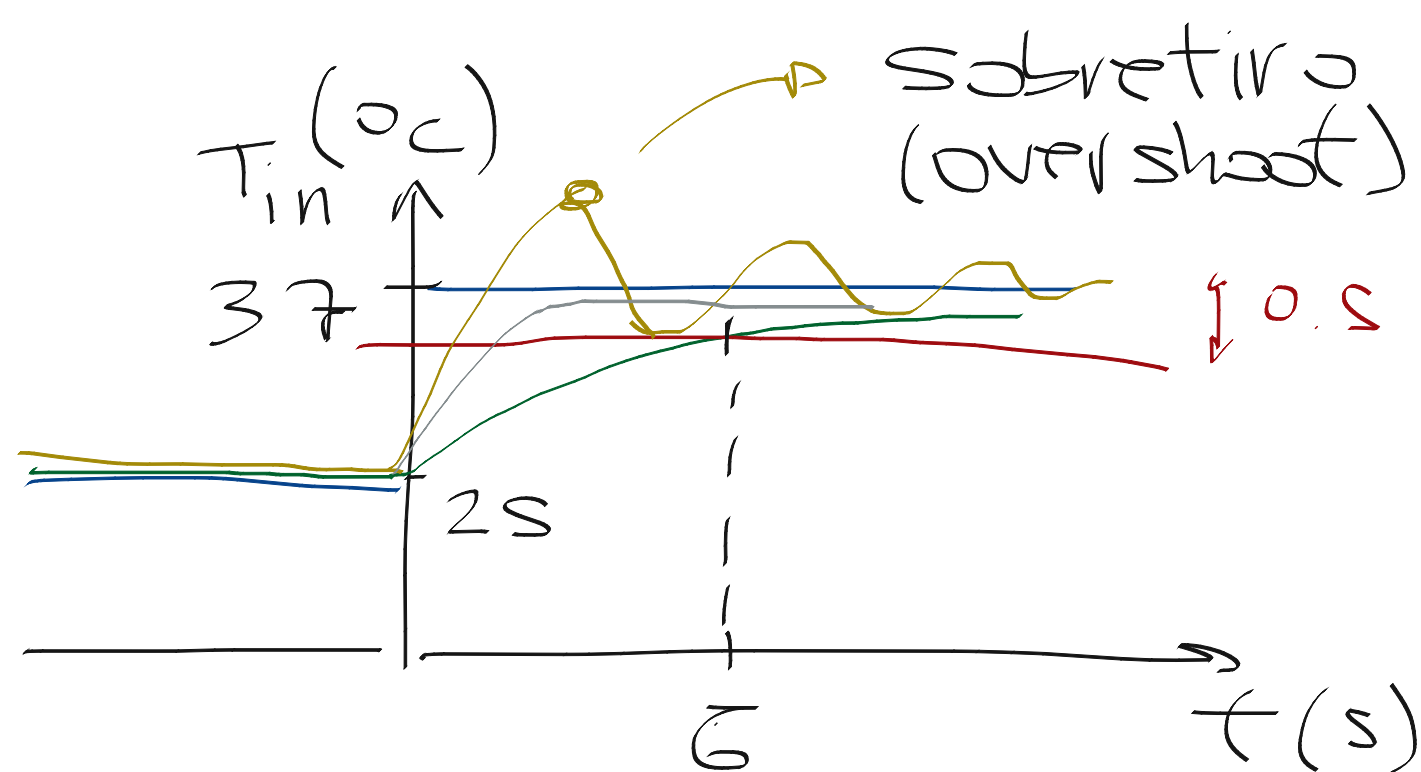


CLASE 0

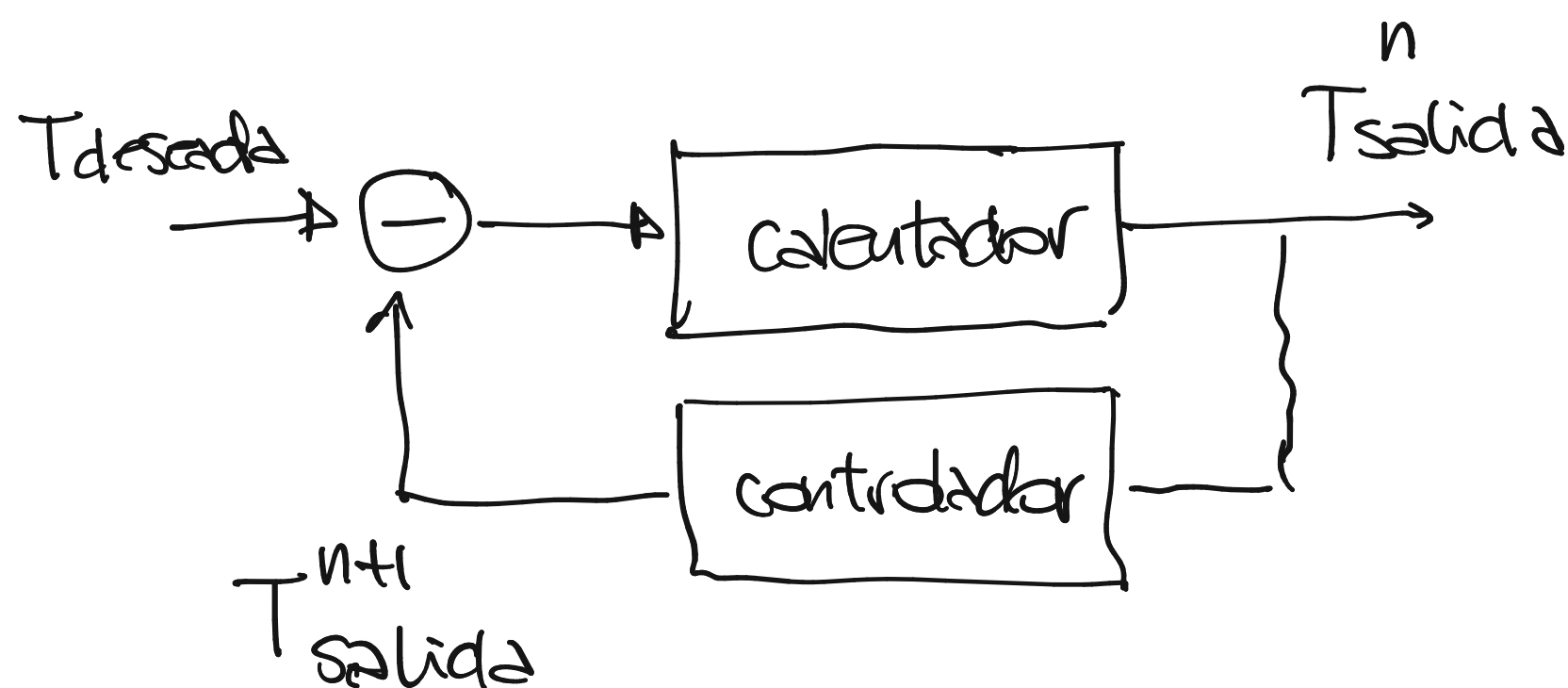
Introducción

Ejemplo: incubadora de huevos

objetivo: "mantener la temperatura cte"

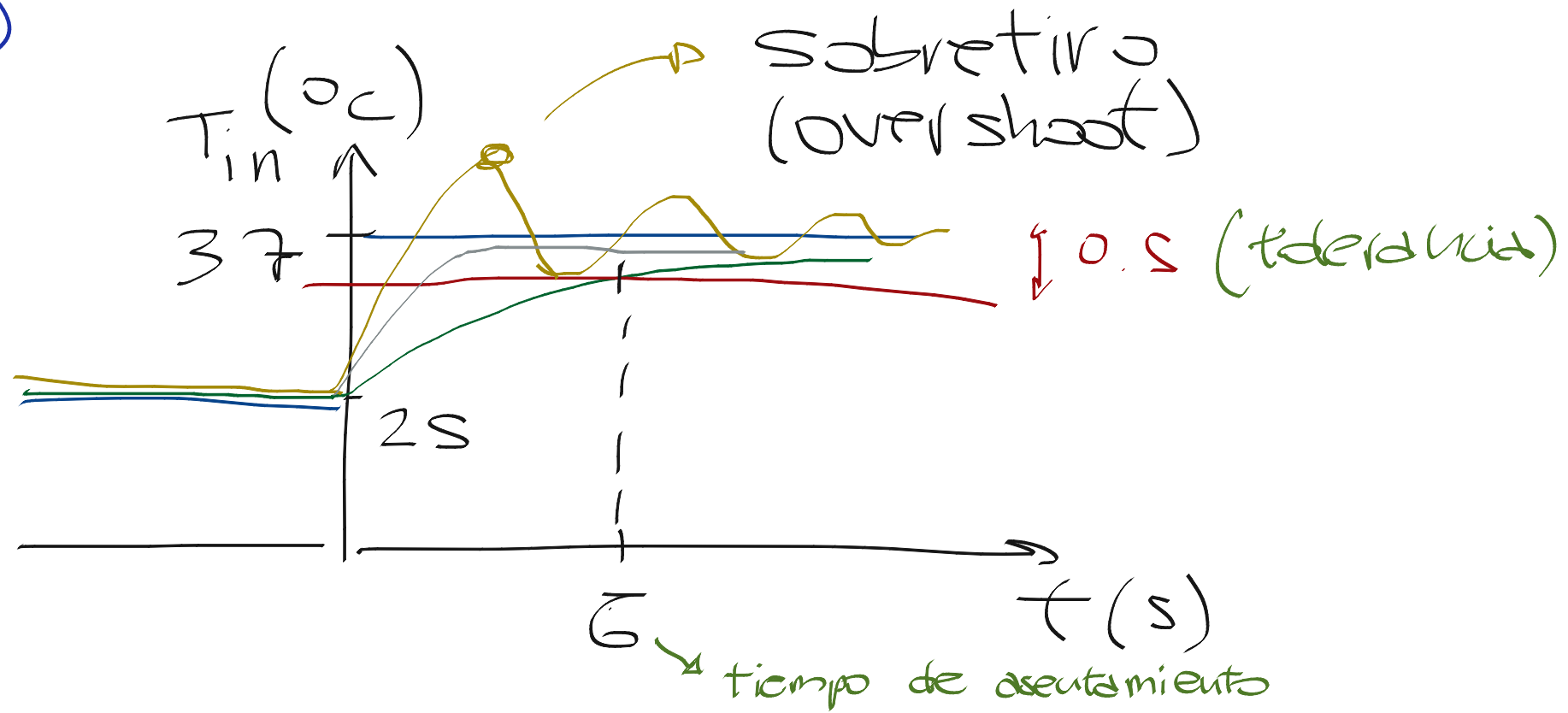


Diagrama

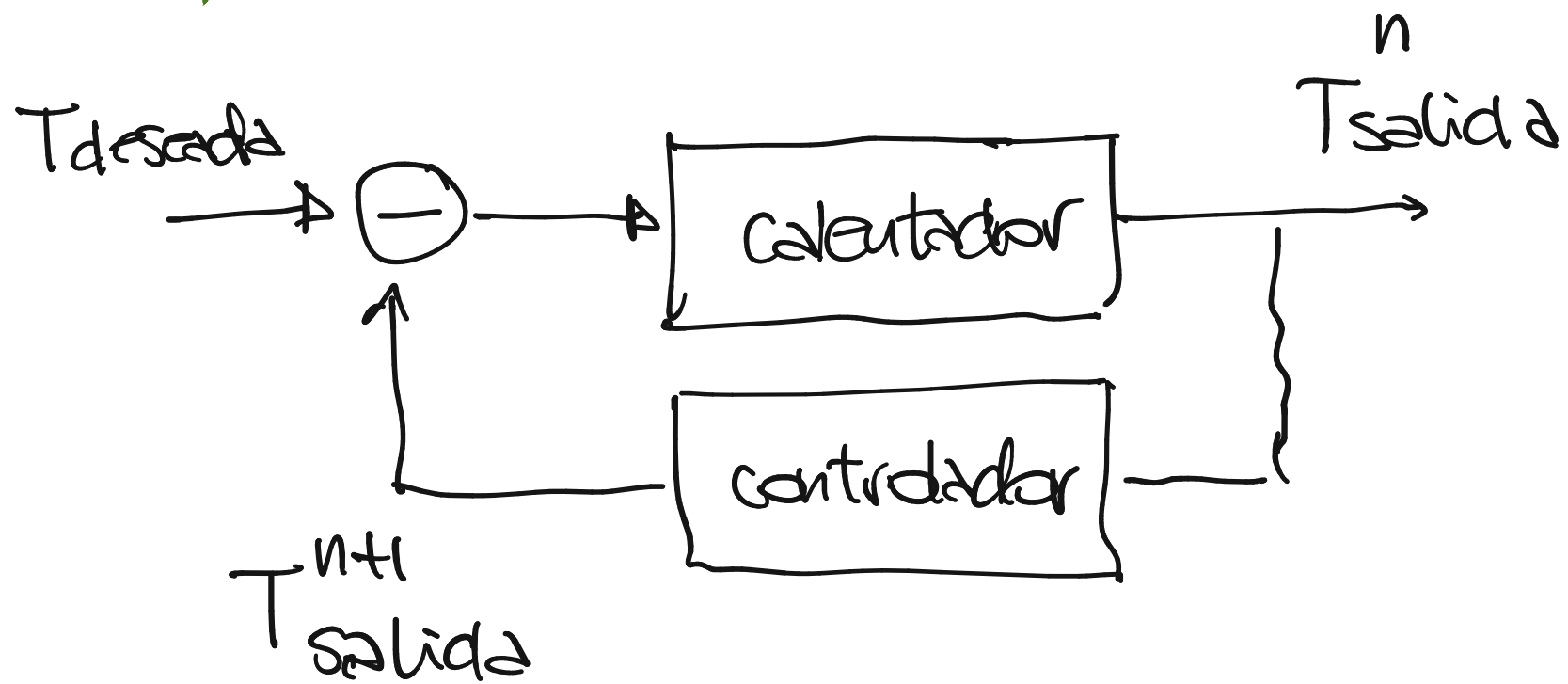


Ejemplo: "invernadero automático"

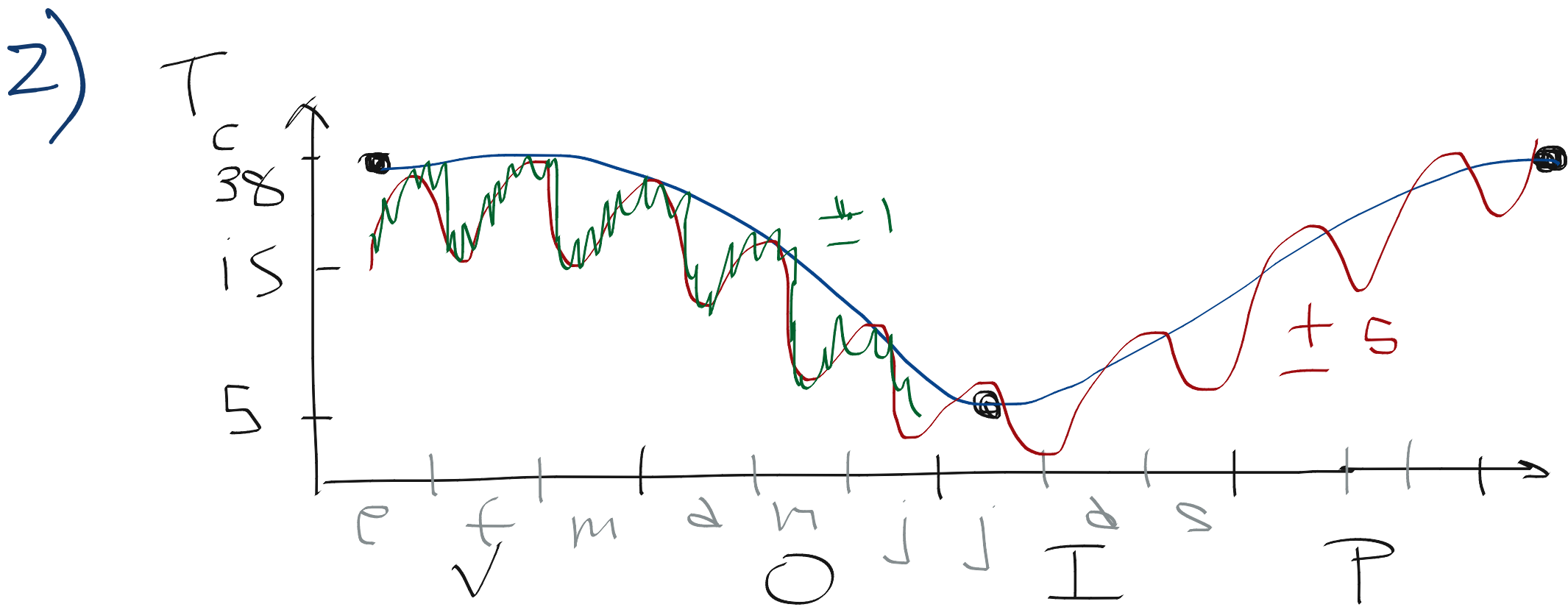
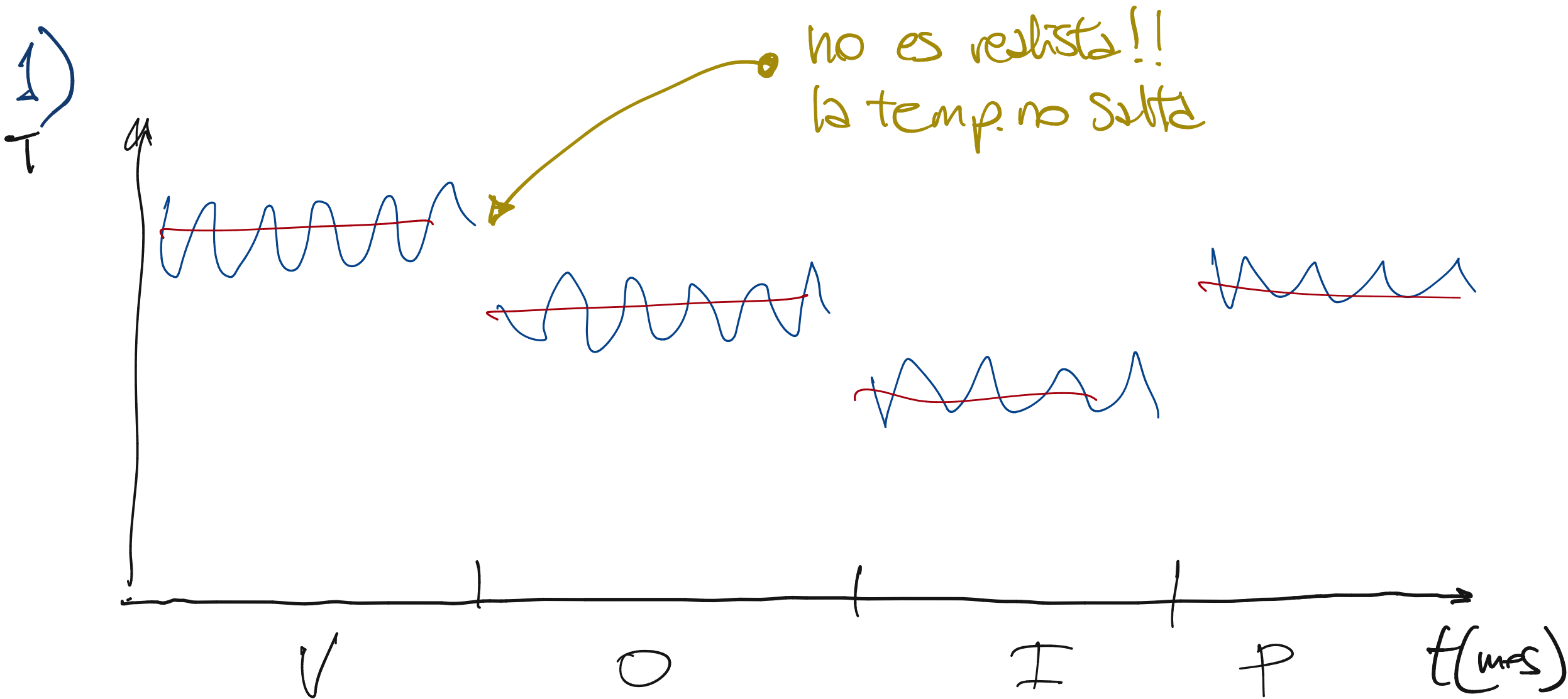
objetivo: "mantener la temperatura cte"
(v1)



Diagrama



Ejemplo: "señal de temperatura"



Ejemplo (v2)

Teoría de números

Naturales: Axiomas (de Peano)

- 1) $0 \in \mathbb{N}$
- 2) $\forall n \in \mathbb{N}, S(n) \in \mathbb{N}$
- 3) $\forall m, n \in \mathbb{N}, m = n \iff S(m) = S(n)$
- 4) $\nexists n \in \mathbb{N} / S(n) = 0$

1 2

Operación (suma)

$$+ : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} (?)$$

* estructura: $(\mathbb{N}, +)$

* grupos, anillo, cuerpo.

* grupo: $\rightarrow$ clausura: $a, b \in \mathbb{N} \Rightarrow a+b \in \mathbb{N}$ ✓
 $\rightarrow$ asociatividad: $(a+b)+c = a+(b+c)$ ✓
 $\rightarrow$ elemento identidad: $\exists e \in \mathbb{N} / \forall n \in \mathbb{N}$
 $n+e = n$ ✓

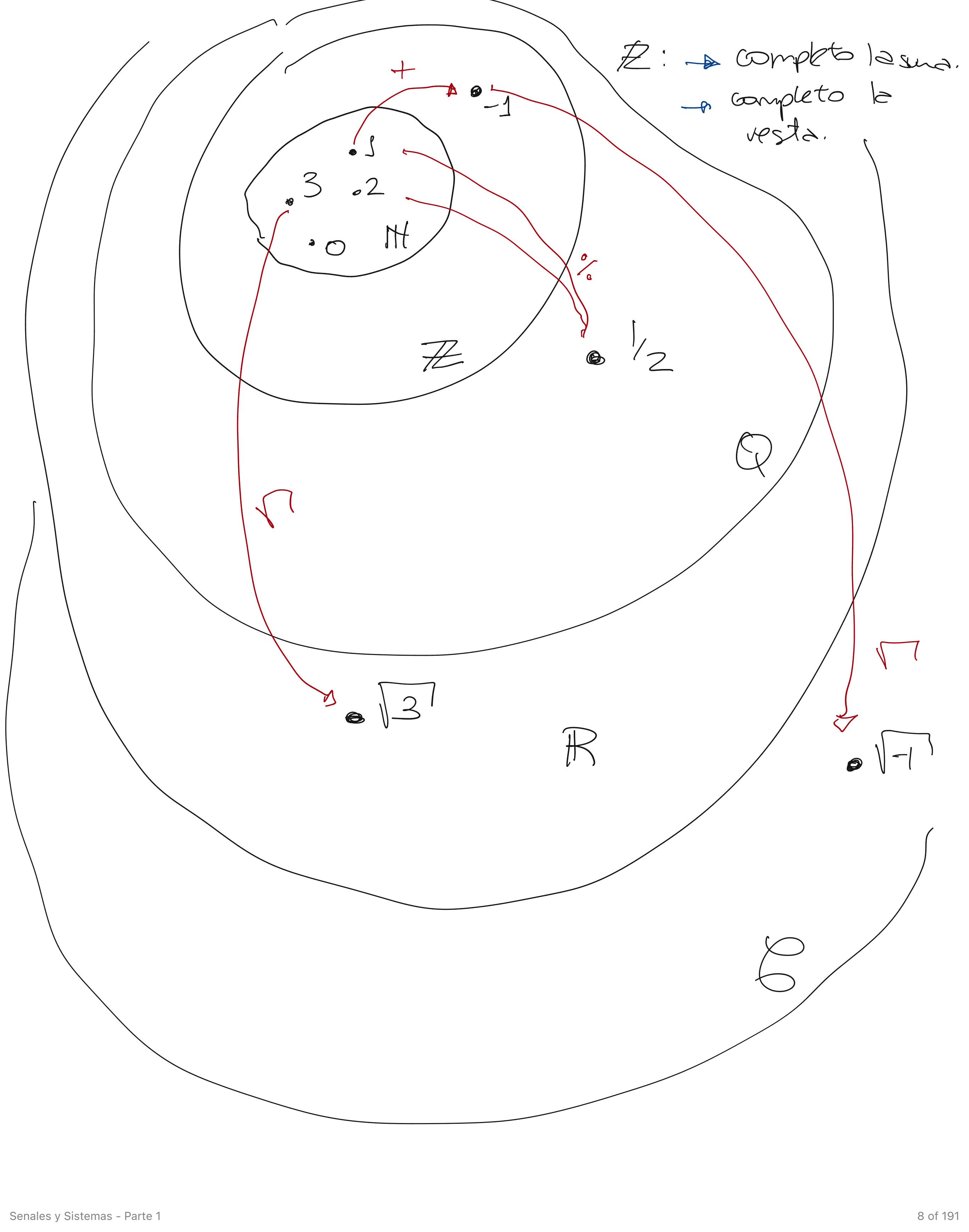
$\rightarrow$ elemento inverso

$$\forall n \in \mathbb{N}, \exists n^{-1} \in \mathbb{N} / n + n^{-1} = e$$

ej: $x \in \mathbb{N} / x + 1 = 0$ ✗

✓
 $2 \oplus 2$
✓
 $2 \oplus 3$

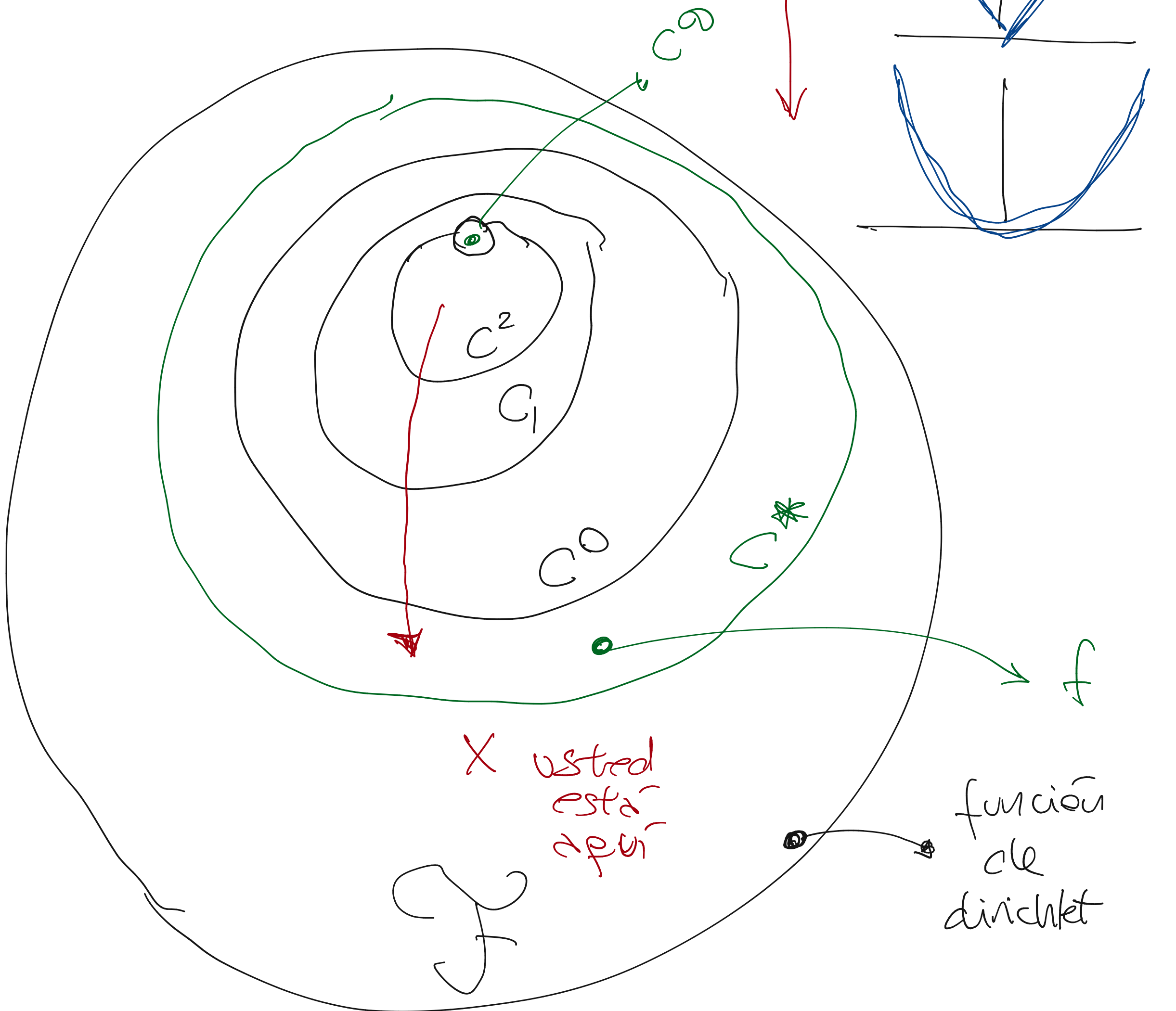
$\mathbb{Z}$: $\rightarrow$ completo la suma.
 $\rightarrow$ completo la resta.



TIPOS DE FUNCIONES

regularidad: "five times over"

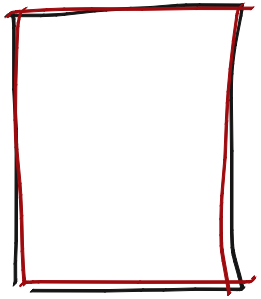
Conjunto: * agrupación en base a una propiedad



Balance : simplicity vs universality

EJEMPLOS

Marcos para Cuadros: "relación de aspecto 2:1"



1) Ancho igual a la mitad que el largo

* variables: a : ancho

l : largo

* modelo: $l = 2a$

2) Cálculo directo: ¿cuánta madera necesito para un marco de 5 cm de ancho?

$$P = (l + a) \cdot 2$$

$$a = 5 \Rightarrow l = 2 \cdot 5 = 10$$

$$\Rightarrow P = 2 \cdot (10 + 5) = 30$$

3) Tengo una madera de 72 cm de largo, como me queda el ancho

$$\begin{aligned} l &= 2a \\ 2(l + a) &= 72 \end{aligned} \Rightarrow \begin{aligned} 2(2a + a) &= 72 \\ 6a &= 72 \end{aligned}$$

$$l = 24$$

$$a = 12$$

Hamburguesas:

* variables: - h : cantidad de hamburguesas
- p : cantidad de panes

* modelado: - $2h = p$
- v : precio producto
 $v = 20$

$$- V = 20 \cdot (h + p)$$

* Pregunta: - Si vendi 720 \$
cuantas unidades
vendi?

CLASE 1

Introducción

[ver transparencias]

CLASE 2

Señales

Definición y clasificación

- ① Clasificación básica
- ② Clasificación avanzada
- ③ Transformaciones dominio
- ④ Transformaciones codominio
- ⑤ Señales pares e impares

CLASIFICACIÓN DE SEÑALES

Def: "señal" $x: \mathbb{R} \rightarrow \mathbb{R}$ ($x: \mathbb{R} \rightarrow \mathbb{E}$)
función $x(t)$ t : tiempo.

clasificación:

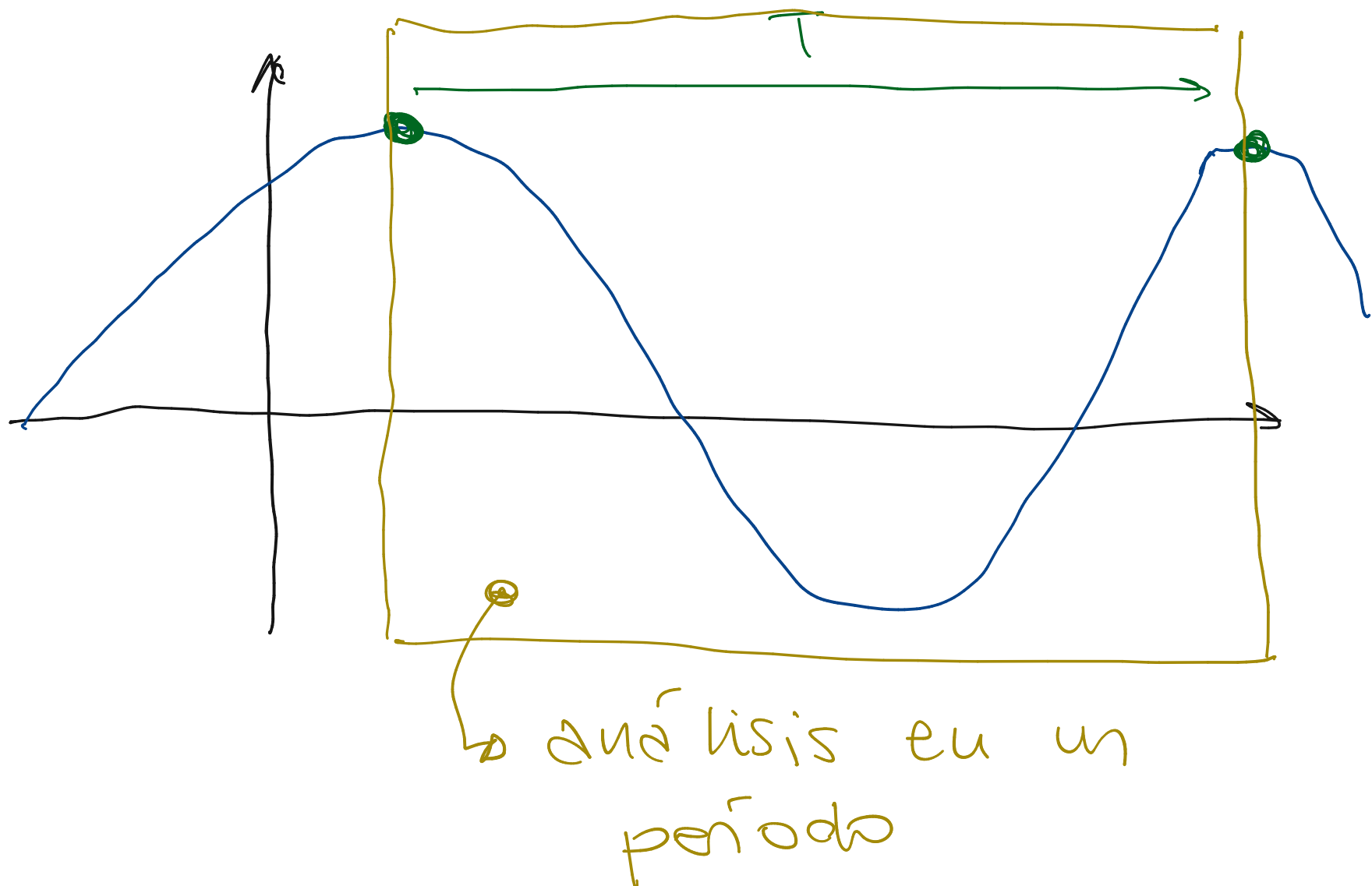
- * dividir un conjunto en clases o categorías
- * siempre hay un criterio: en base a atributos o propiedades

Determinísticas:

- * determinística: tengo una forma fija de saber la correspondencia.
- * aleatoria: existe un componente de azar en la correspondencia.

Periódicas

- * repite exactamente su valor cada cierto tiempo.



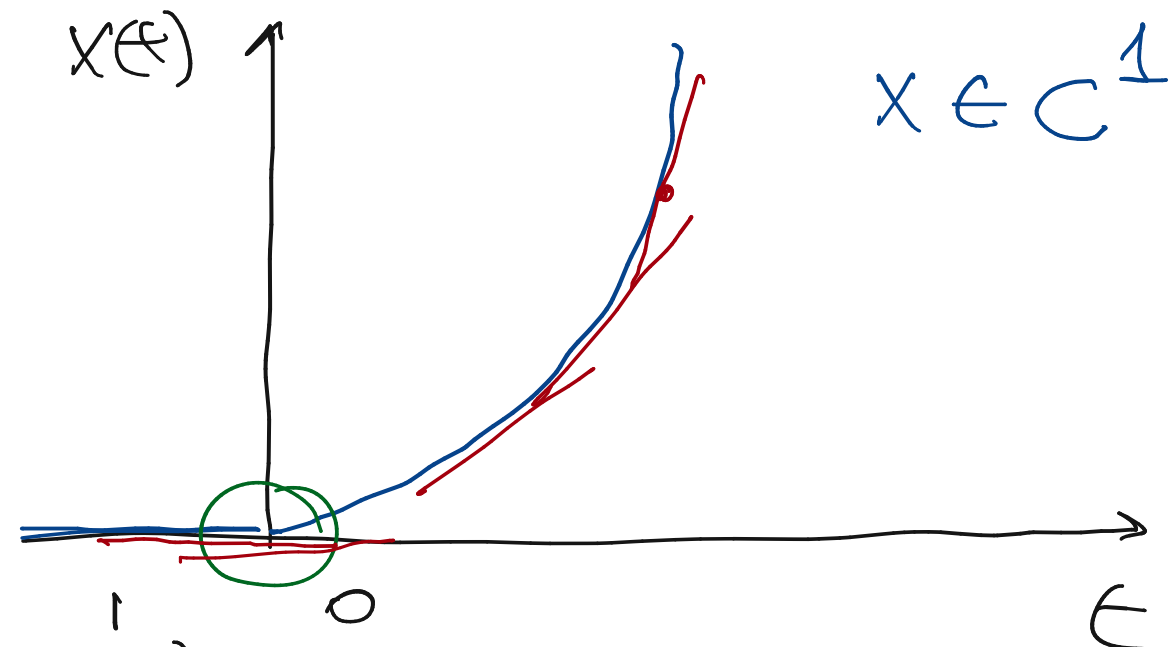
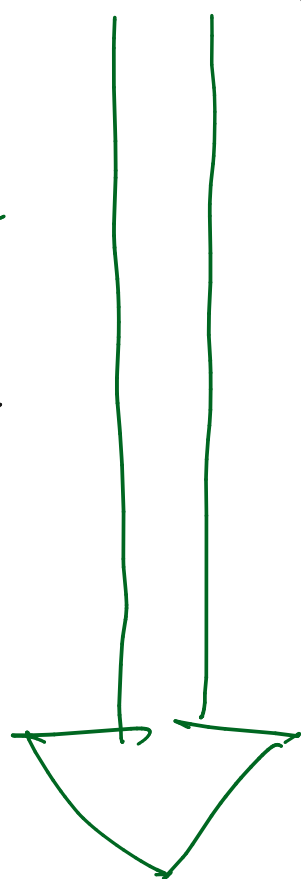
Regularidad

* regulares (suaves): continuas con derivadas continuas

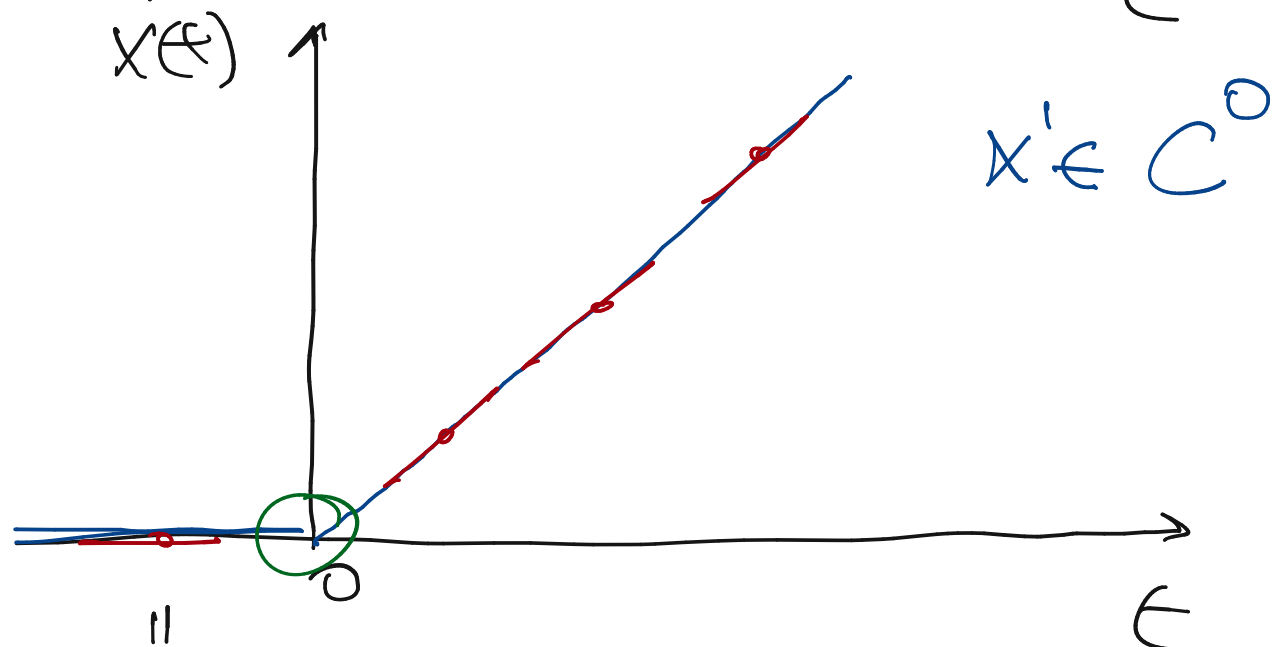
* irregulares: discontinuas o no derivables

$$x(t) = \begin{cases} t^2, & t > 0 \\ 0, & t \leq 0 \end{cases} \quad (\text{e.o.c.})$$

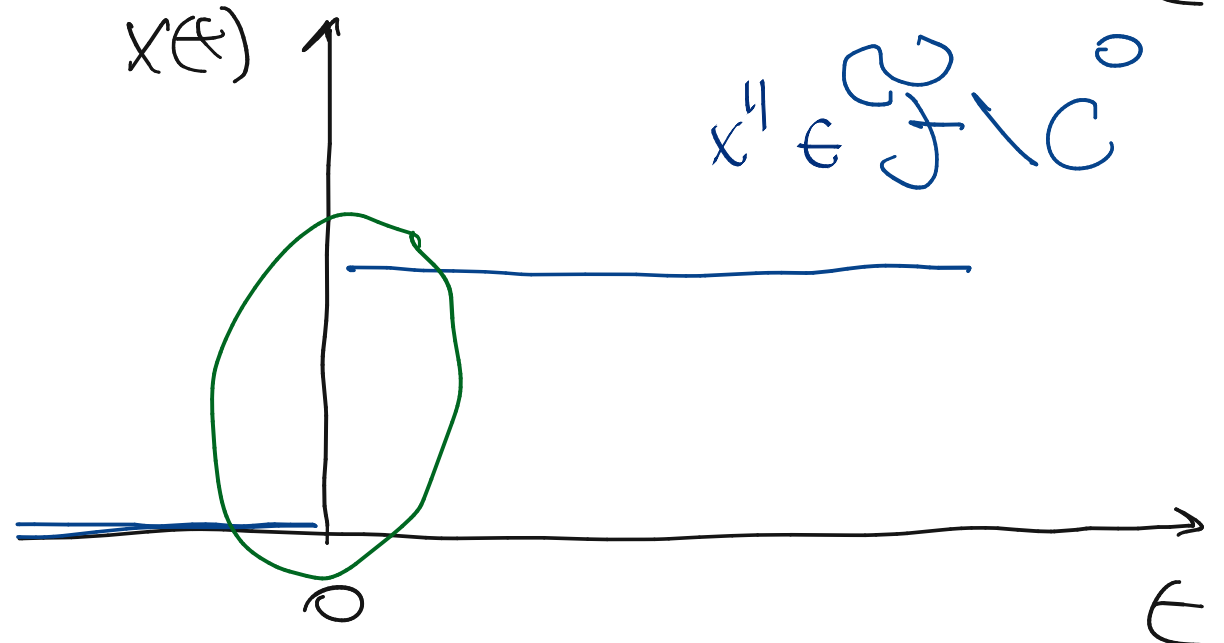
derivada
"reduce suavidad"



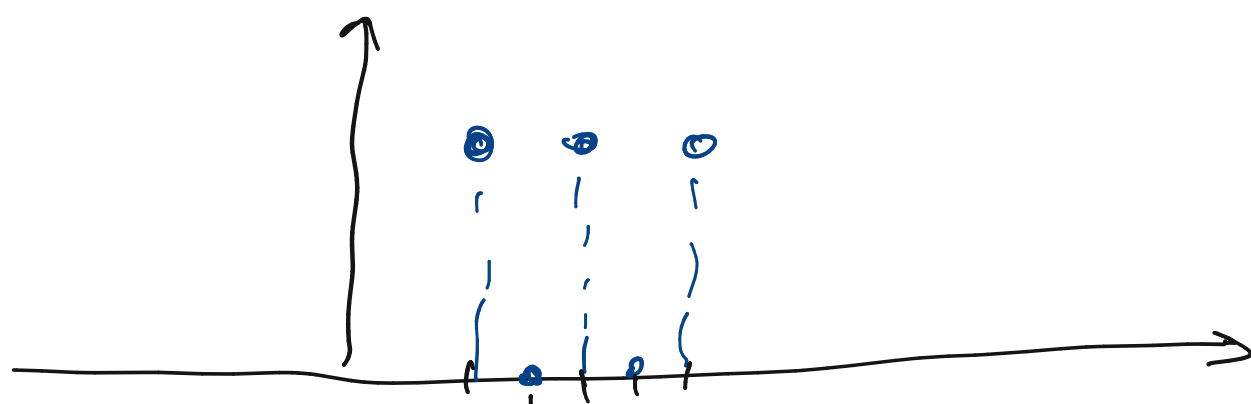
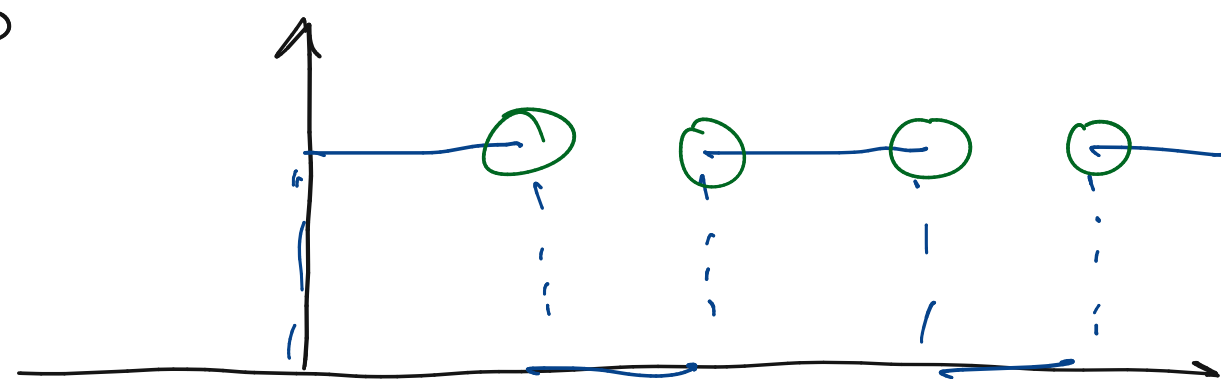
$x \in C^1$



$x' \in C^0$



$x'' \in C^0 \setminus C^1$



obs:

- * no es binaria, es gradual
- * que tan discontinua es.

* propiedad que "se cumple en casi todo punto"

obs: función de Dirichlet

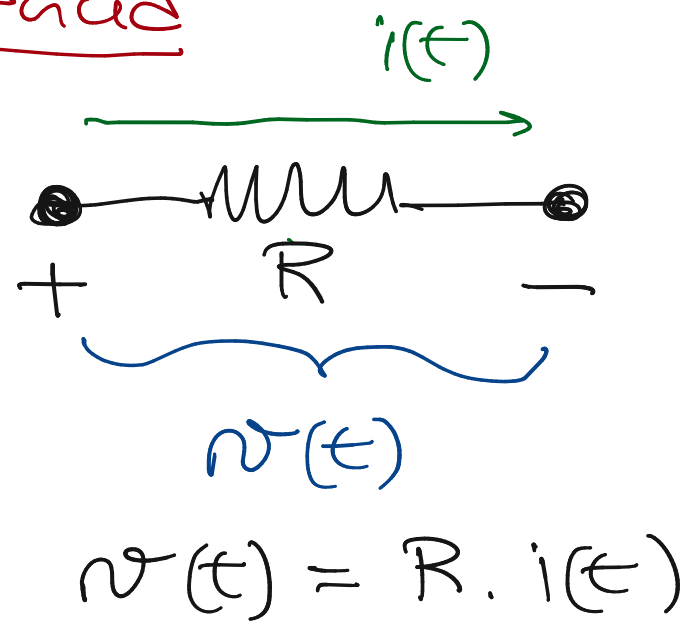
$$y(t) = \begin{cases} 1 & t \in \mathbb{Q} \\ 0 & t \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Señales de energía y de potencia

Ejemplo:

* potencia instantánea: $p(t) = v(t) \cdot i(t)$

$$p(t) = \frac{1}{R} \cdot v^2(t)$$



$$v(t) = R \cdot i(t)$$

* Energía disipada en $[t_1, t_2]$

$$E(t_1, t_2) = \int_{t_1}^{t_2} p(t) \cdot dt = \int_{t_1}^{t_2} \frac{1}{R} \cdot v^2(t) \cdot dt$$

* Potencia promedio:

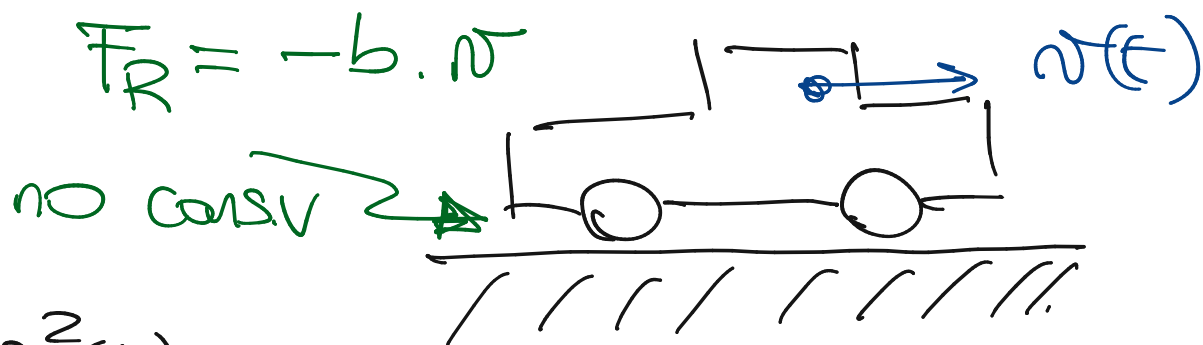
$$P(t_1, t_2) = \frac{E(t_1, t_2)}{t_2 - t_1}$$

$$P(t_1, t_2) = \frac{1}{t_2 - t_1} \cdot \int_{t_1}^{t_2} \left(\frac{1}{R} \right) \cdot v^2(t) \cdot dt$$

→ promedio

Ejemplo:

$$\Delta E = T_r \Rightarrow T_r = F_r \cdot d$$



P.I. $p_r(t) = \underline{f_r(t)} \cdot v(t) = b \cdot v^2(t)$

P.M. $P(t_1, t_2) = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} b \cdot v^2(t) \cdot dt$

Definiciones

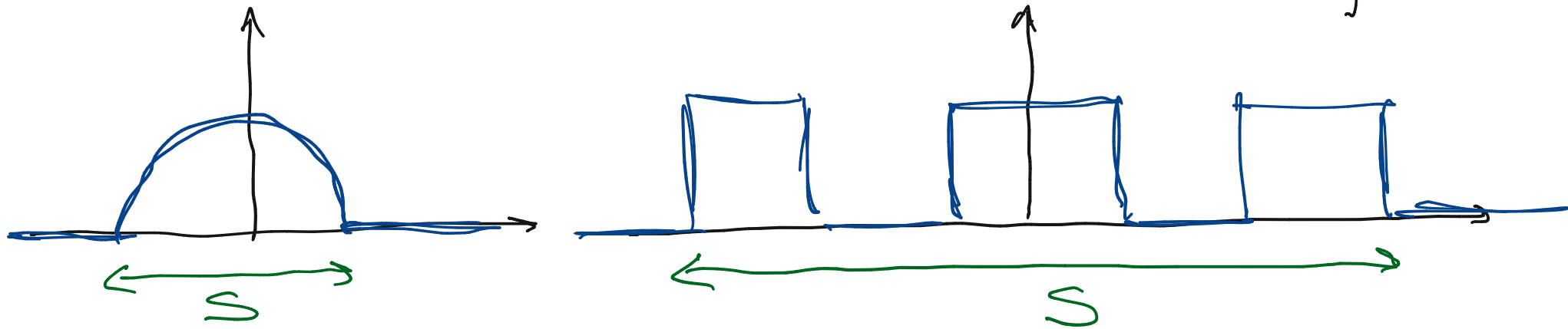
* Energía: $\rightarrow E(t_1, t_2) = \int_{t_1}^{t_2} |x(t)|^2 dt$ ↗ módulo. si $x \in \mathbb{R}$

$\rightarrow E[n_1, n_2] = \sum_{n=n_1}^{n_2} |x[n]|^2$

* Potencia media: $\rightarrow P(t_1, t_2) = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 dt$

$\rightarrow P[n_1, n_2] = \frac{1}{n_2 - n_1} \cdot \sum_{n=n_1}^{n_2} |x[n]|^2$

* Soporte de una función: $S: S = \{t \in \mathbb{R} / f(t) \neq 0\}$



"el conjunto cerrado más chico, que contiene todos los valores no nulos"

* Energía total: $E_\infty = \int_{-\infty}^{\infty} |x(t)|^2 dt = \lim_{T \rightarrow \infty} \int_{-T}^T |x(t)|^2 dt$ (the integral is circled in green and labeled E_T)

* Potencia media total: $P_T = \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt$

$P_\infty = \lim_{T \rightarrow \infty} P_T$

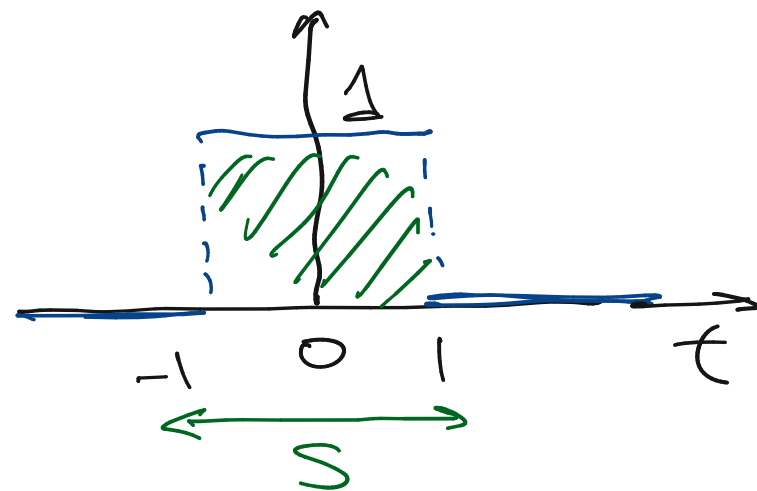
Definiciones:

* señal de energía (finita): $E_{\infty} < \infty$

* señal de potencia (finita): $P_{\infty} < \infty$

Ejemplos

1) $x(t) = \begin{cases} 1 & t \in [-1, 1] \\ 0 & \text{e.o.c.} \end{cases}$

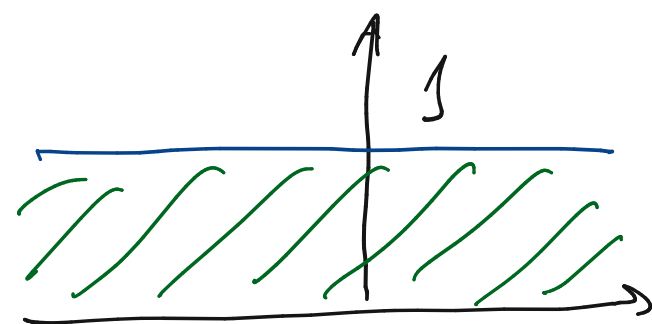


→ $E_{\infty} = \int_{-1}^1 |x(t)|^2 dt = 2$ ✓

→ de soporte acotado " $S < \infty$ "

→ $P_{\infty} = 0$ ✓

2) $x(t) = 1, \forall t \in \mathbb{R}$



→ $E_{\infty} = \infty$

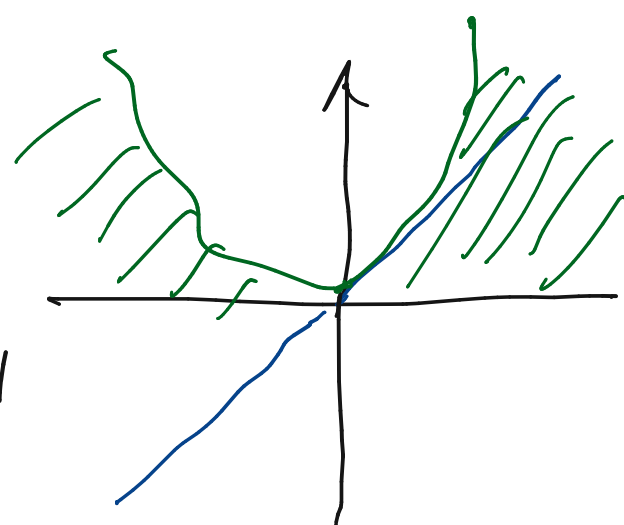
→ $P_{\infty} = 1$

3) $x(t) = t, \forall t \in \mathbb{R}$

→ $E_{\infty} = \infty$

→ $P_{\infty} = \infty$

" $\int t^2 = \frac{t^3}{3}$ "

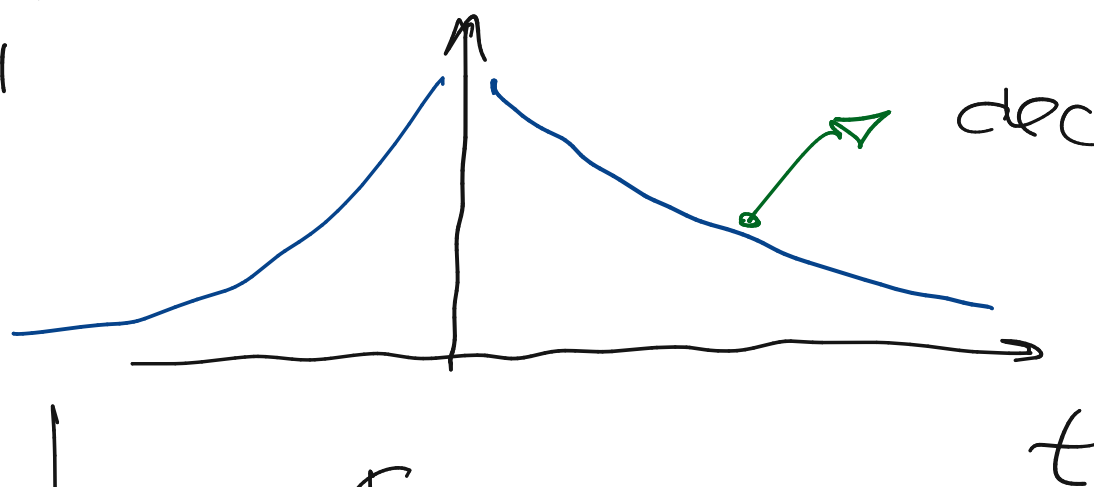


Obs: "cómo tiene que ser una señal para ser de energía"

$\lim_{t \rightarrow \pm\infty} |x(t)| = 0$

$\int_{-\infty}^{\infty} \left| \frac{1}{t^{\alpha}} \right|$

→ ∞ si $\alpha > 1$



TRANSFORMACIONES

Idea:

catalogar las transformaciones que puede sufrir una señal para simplificar el análisis de los sistemas

T. de la variable independiente (t)

Corrimiento temporal

$$y[n] = x[n - n_0]$$

$$y[n] = x[n - 2]$$

$$n_0 = 2$$

$$y[1] = x[1 - 2] = x[-1]$$

$$y[2] = x[2 - 2] = x[0]$$

* si $n_0 > 0$ retardo

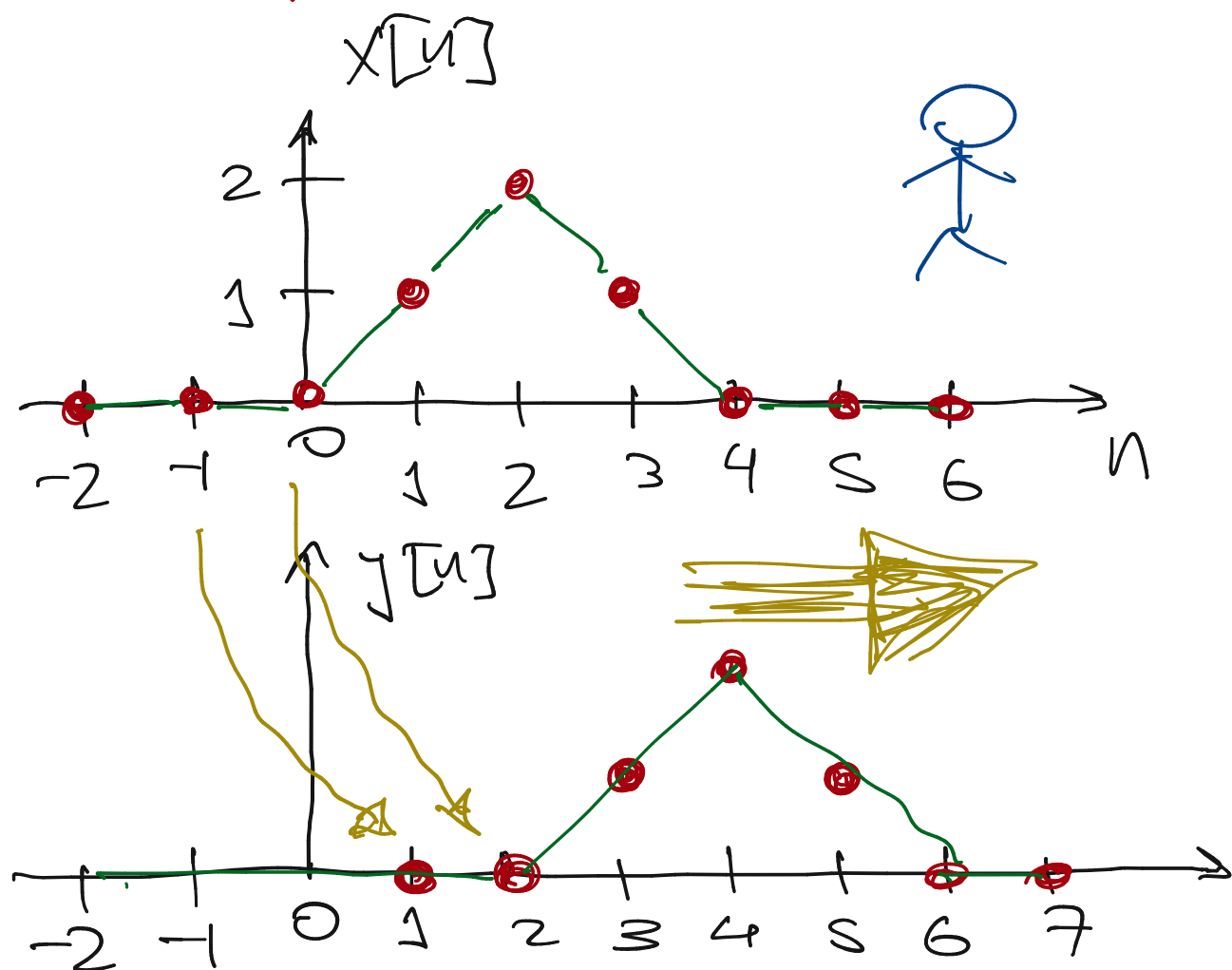
* si $n_0 < 0$ adelanto

idea: "descripción de una señal"

* ancho (soporte) } forma.

* alto (amplitud)

* posición (ubicación en el tiempo)



Compresión (expansión) temporal

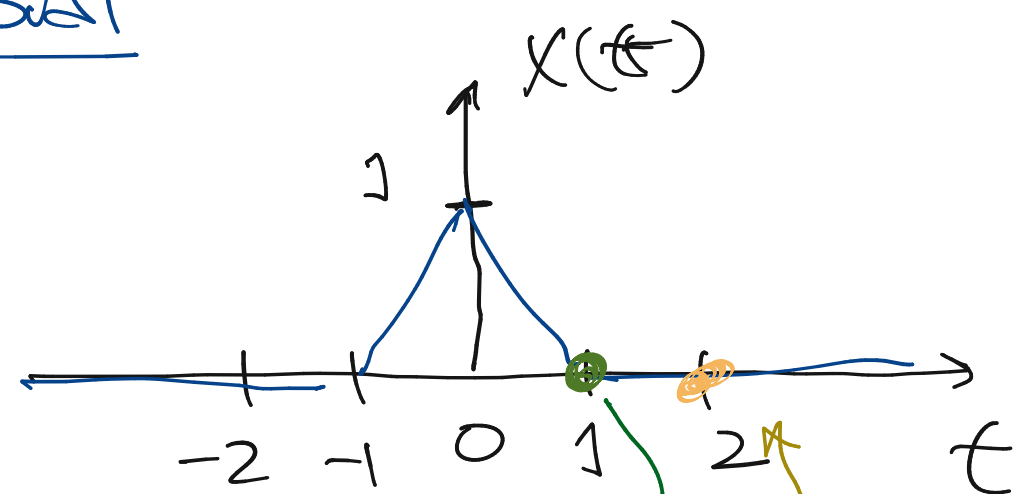
$$y(t) = x(\alpha \cdot t), \quad \alpha \in \mathbb{R}$$

$$\boxed{\alpha = 2}$$

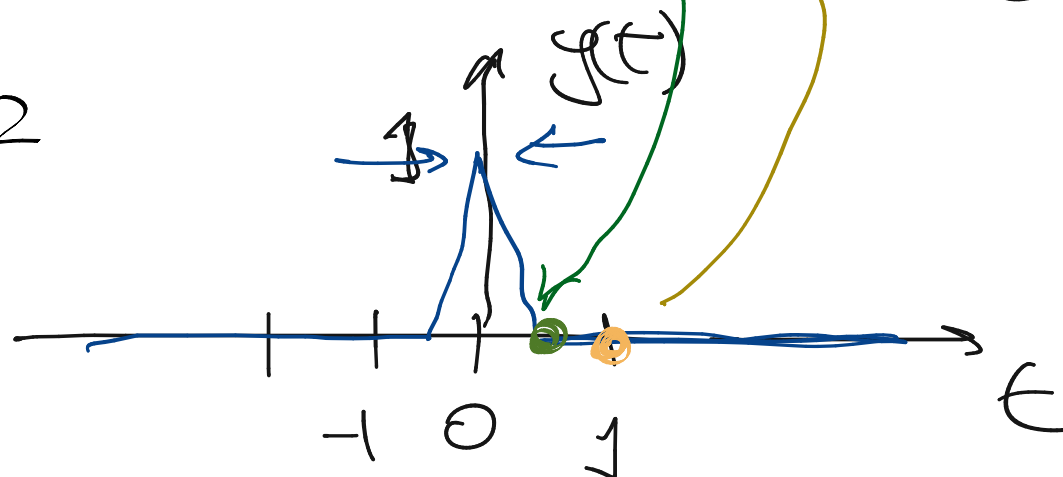
$$1) y(t) = x(2t)$$

$$y(1) = x(2)$$

$$y(0) = x(0)$$

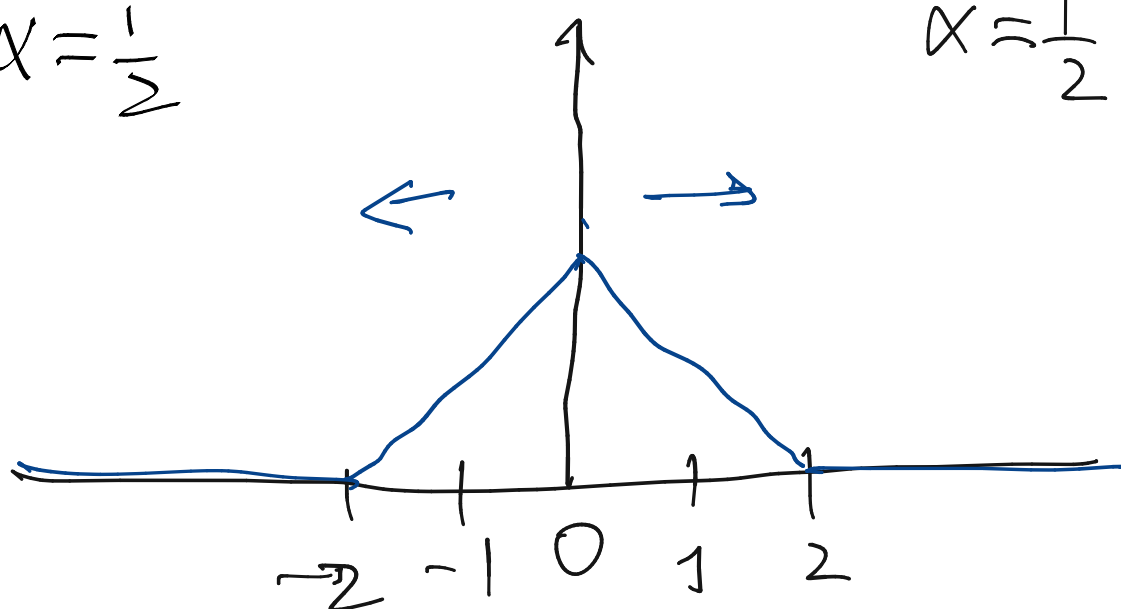


$$1) \alpha = 2$$



$$2) \alpha = \frac{1}{2}$$

$$\alpha = \frac{1}{2}$$



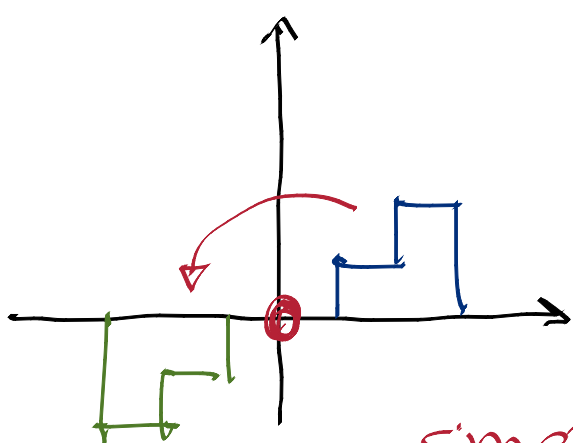
Obs: * $\alpha > 1$: compresión

* $\alpha < 1$: expansión

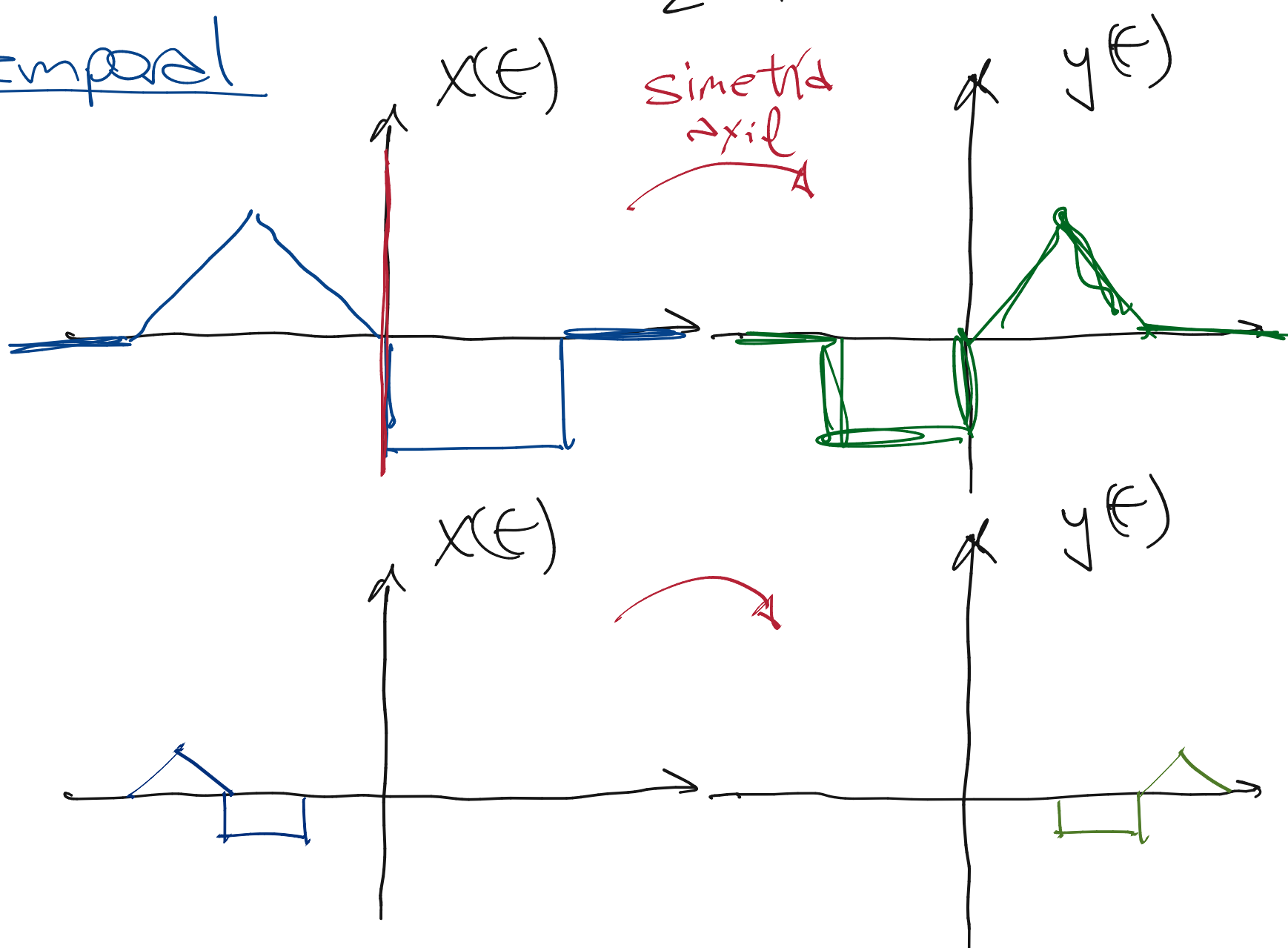
Inversión temporal

$$y(t) = x(-t)$$

$$y(t) = -x(-t)$$



simetría
central



Señales Periódicas

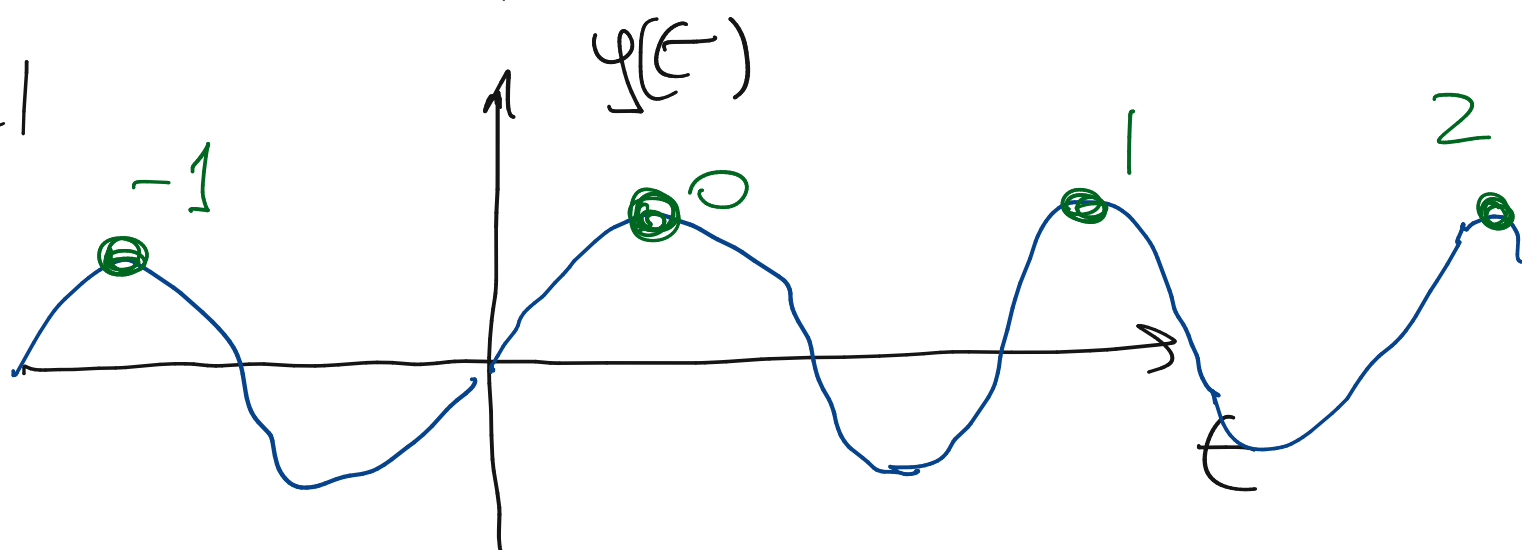
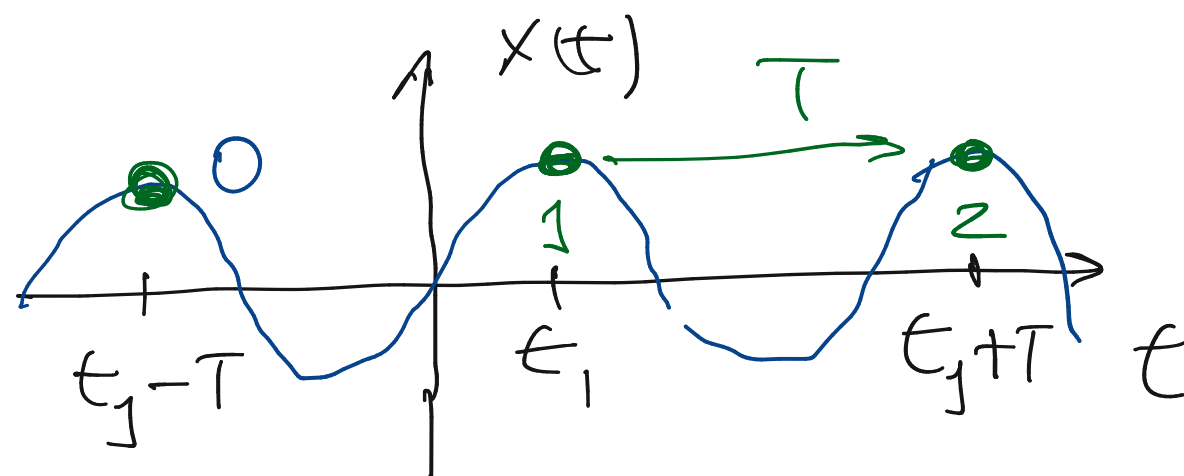
$$\exists T \in \mathbb{R} / x(t) = x(t+T)$$

$$\text{ó } x(t) = x(t-T)$$

* corrimiento temporal

$$y(t) = x(t-T)$$

invariante!!



* obs! $T_k = k \cdot T, k \in \mathbb{Z}$

→ T_k también es período.

→ $x(t)$ tiene infinitos períodos

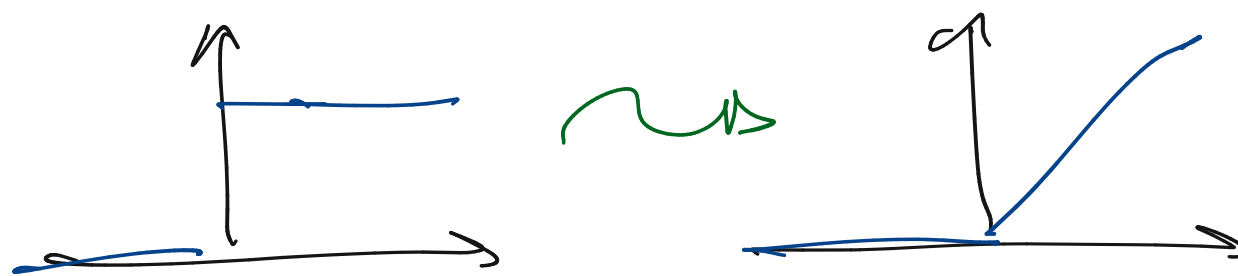
* período fundamental: $T_0 = \min \{ |T_k| \}$

* importancia para sistemas:

→ entrada periódica

→ salida periódica

→ con el mismo período.



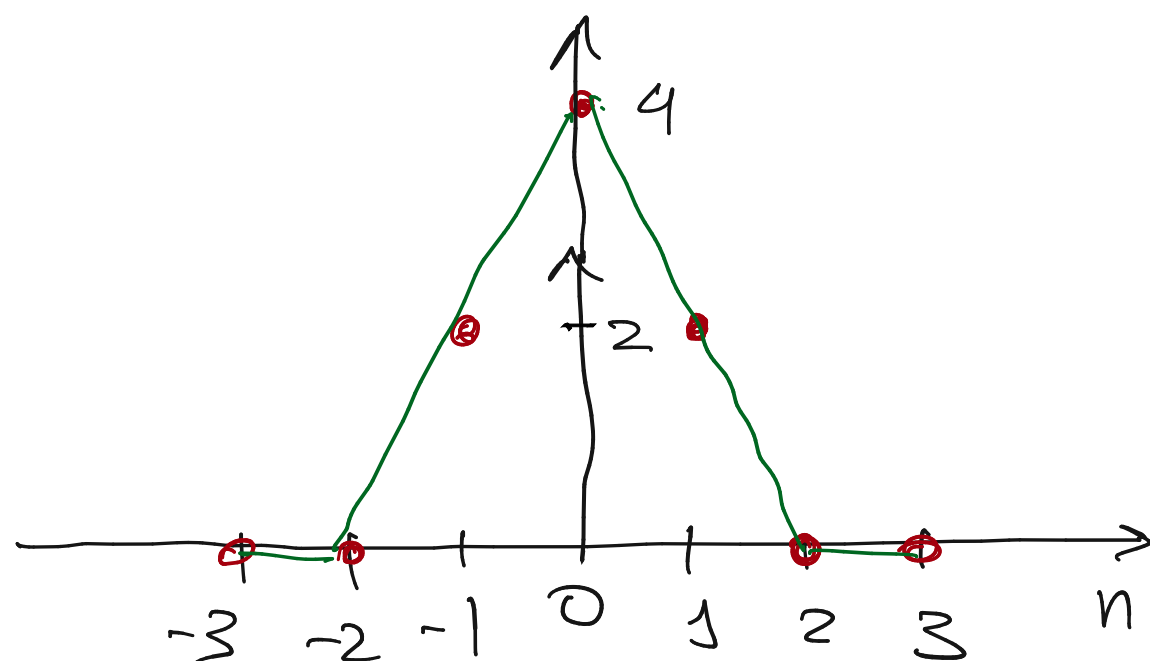
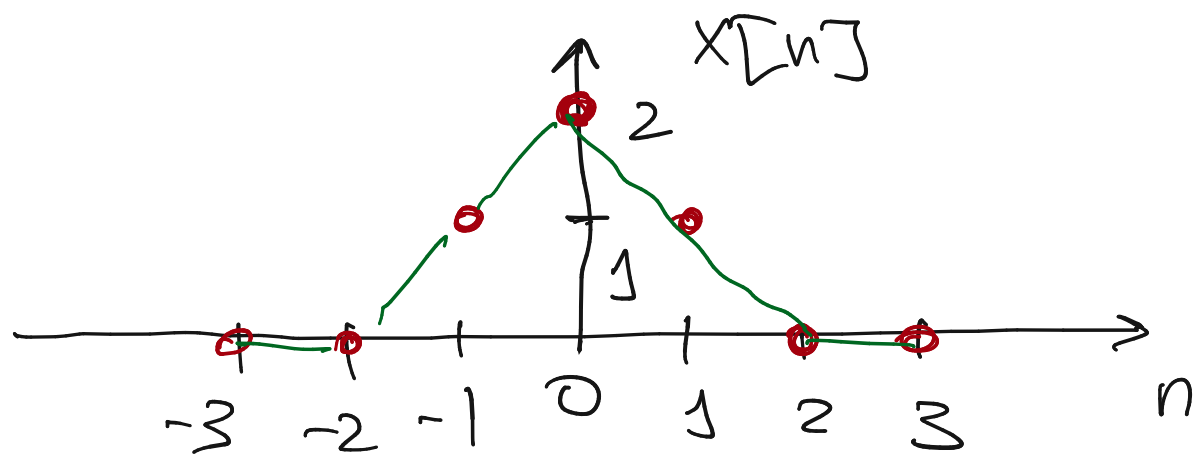
TRANSFORMACIONES DE LA VARIABLE DEPENDIENTE

Escala (Amplificación)

$$y[n] = \alpha \cdot x[n]$$

$$\alpha = 2$$

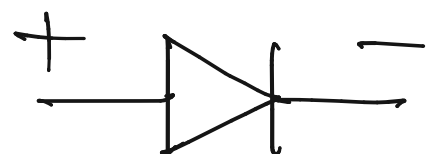
$$y[n] = 2 \cdot x[n]$$



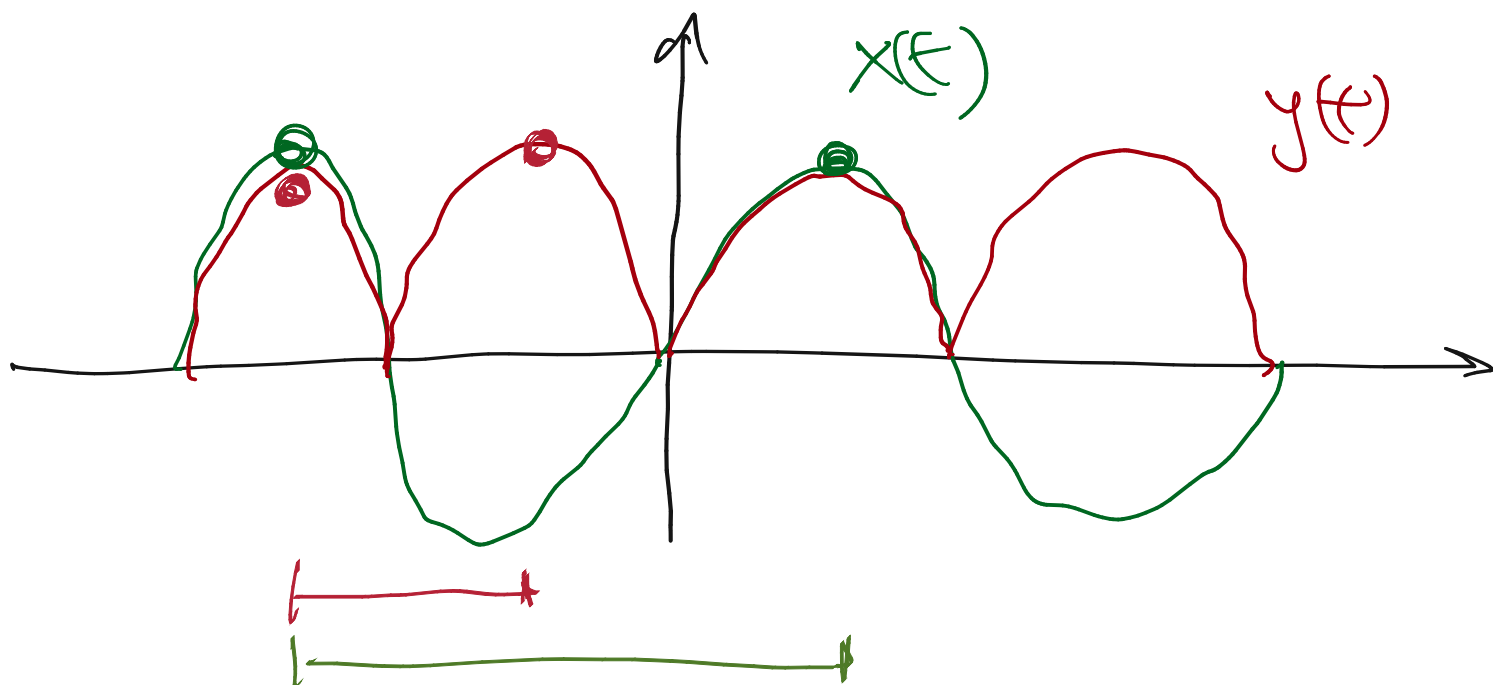
Rectificación

"borro lo negativo"

$$y(t) = |x(t)|$$



"diodo"



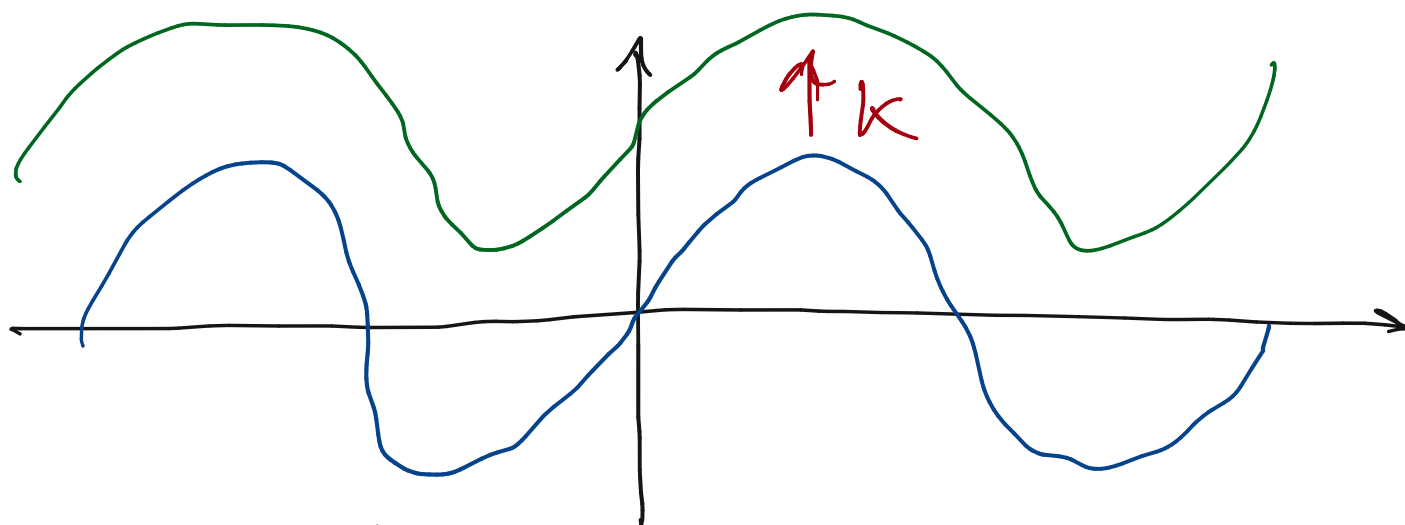
Q&S:

* cambia la velocidad !!

Offset

$$y(t) = x(t) + k$$

* cambia posición vertical.



señales pares e impares

* par: $x(t) = \underbrace{x(-t)}_{y(t)}$

* impar: $x(t) = -x(-t)$

"no cambian con la transformación"

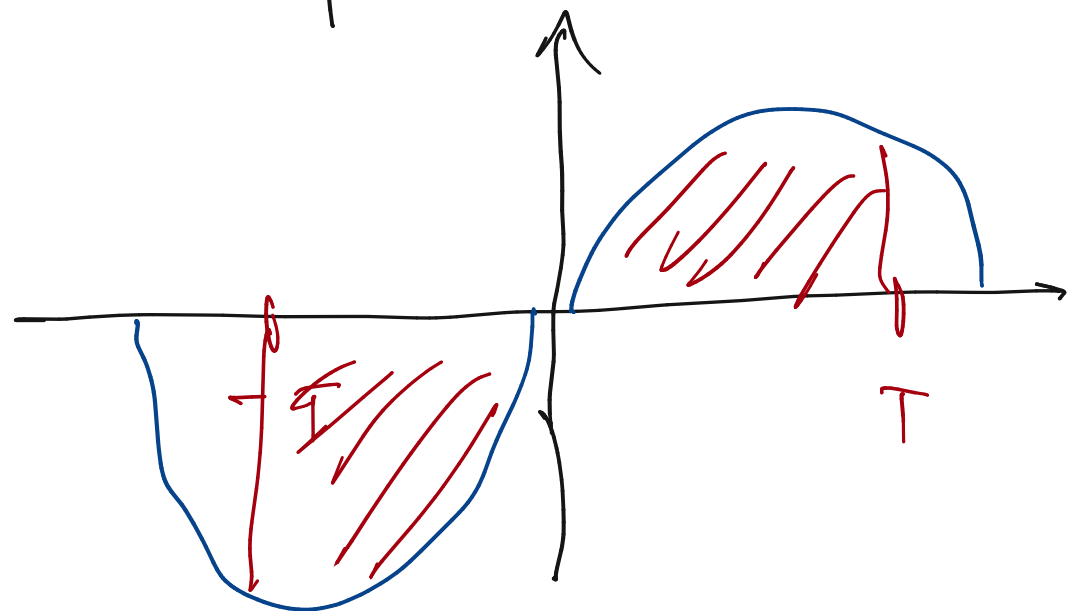
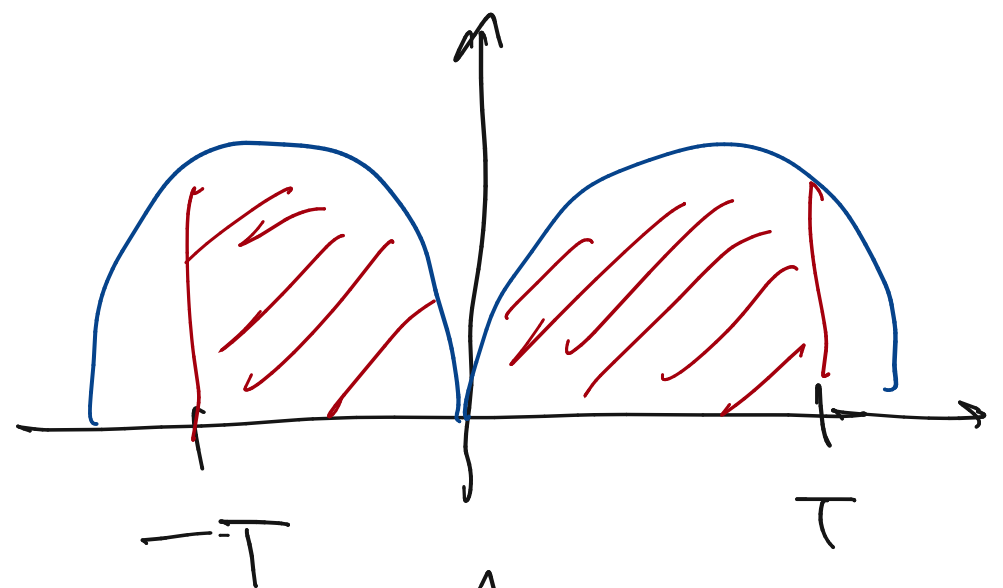
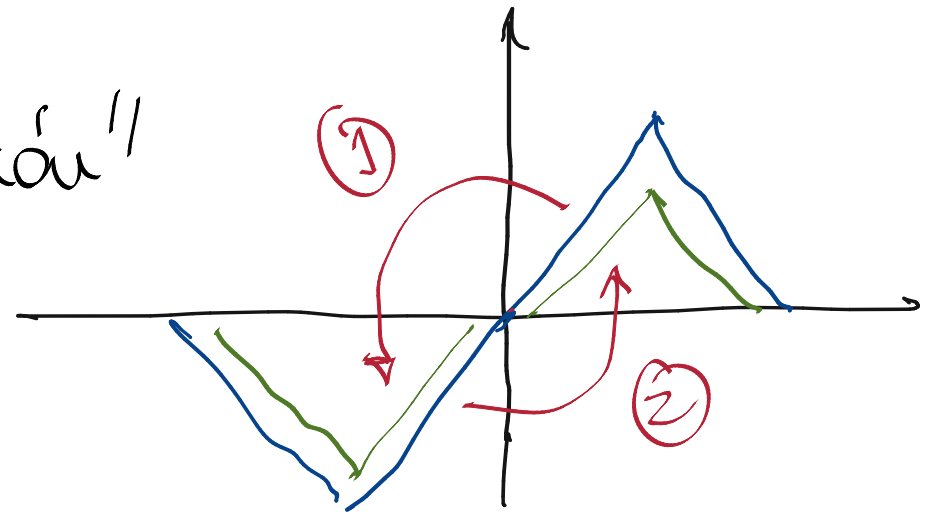
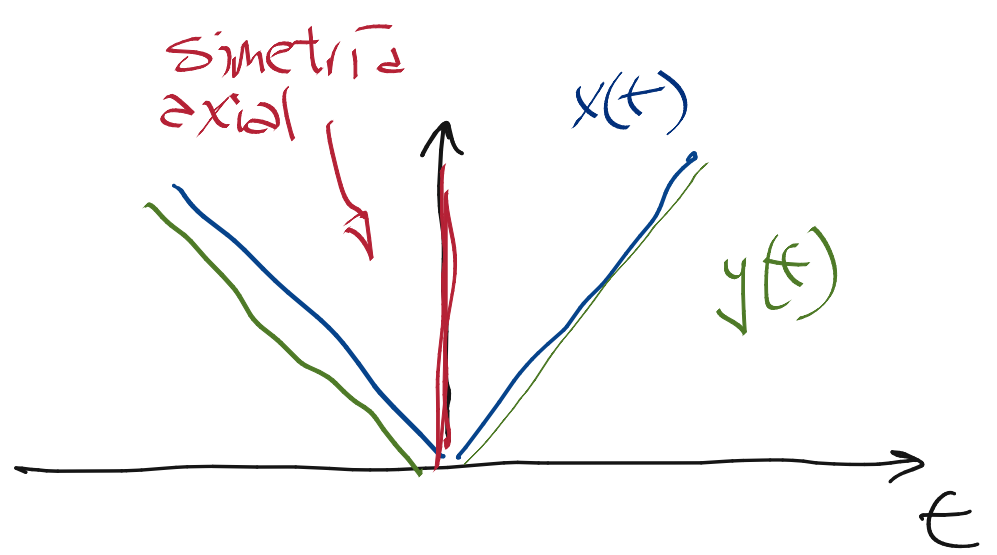
* integral par:

$$\int_{-T}^T x(t) \cdot dt = 2 \cdot \int_0^T x(t) \cdot dt$$

* integral impar:

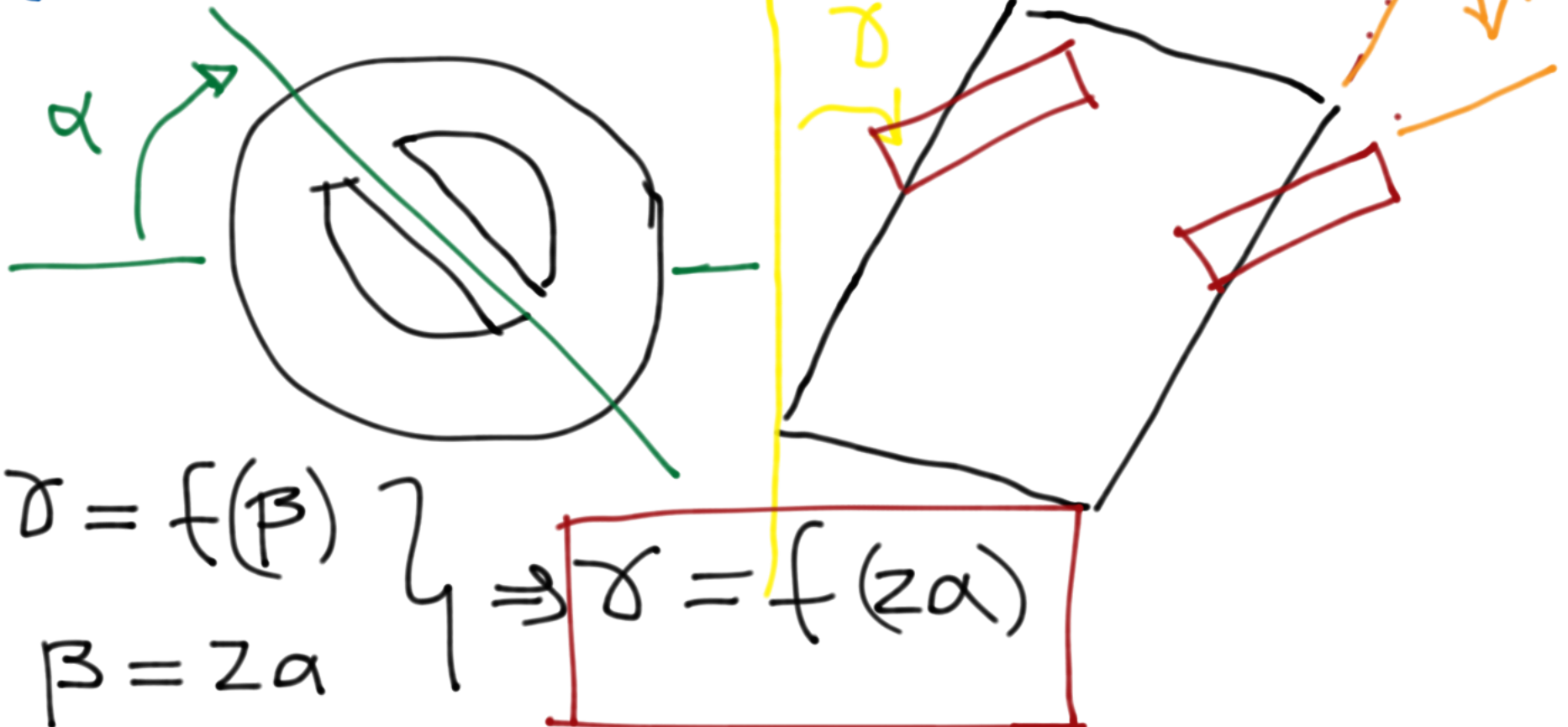
$$\int_{-T}^T x(t) \cdot dt = 0$$

"facilitan las cuentas"



TRANSFORMACIONES DE LA VARIABLE INDEPENDIENTE

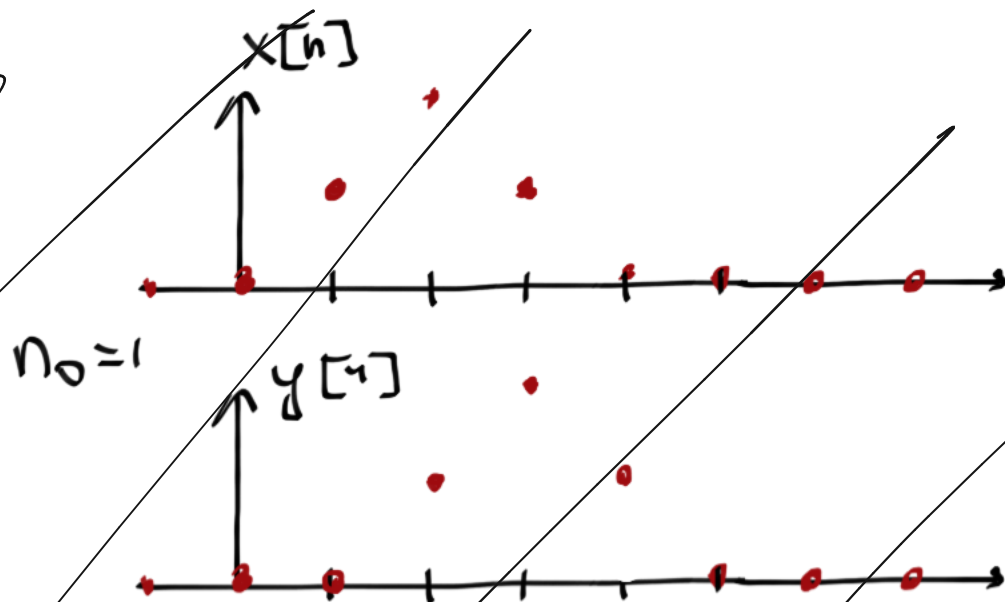
Ejemplo: "lento"



Corrimiento temporal

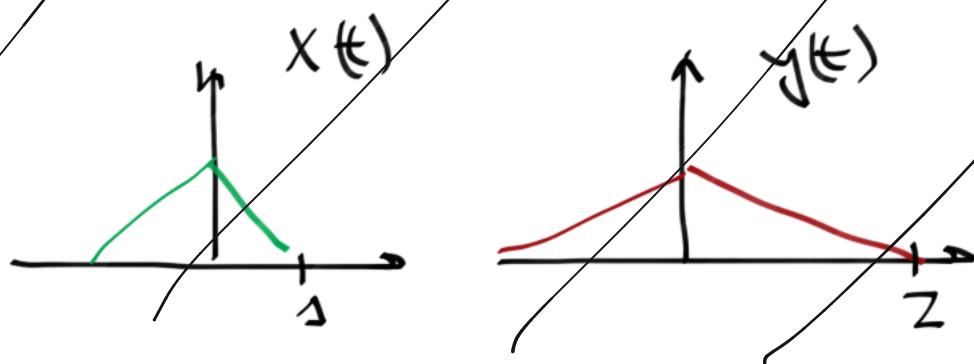
$$y[n] = x[n - n_0]$$

$n_0 > 0$ retardo
 $n_0 < 0$ adelanto



Compresión

$$y(t) = x(2t)$$



CLASE 3

Señales especiales

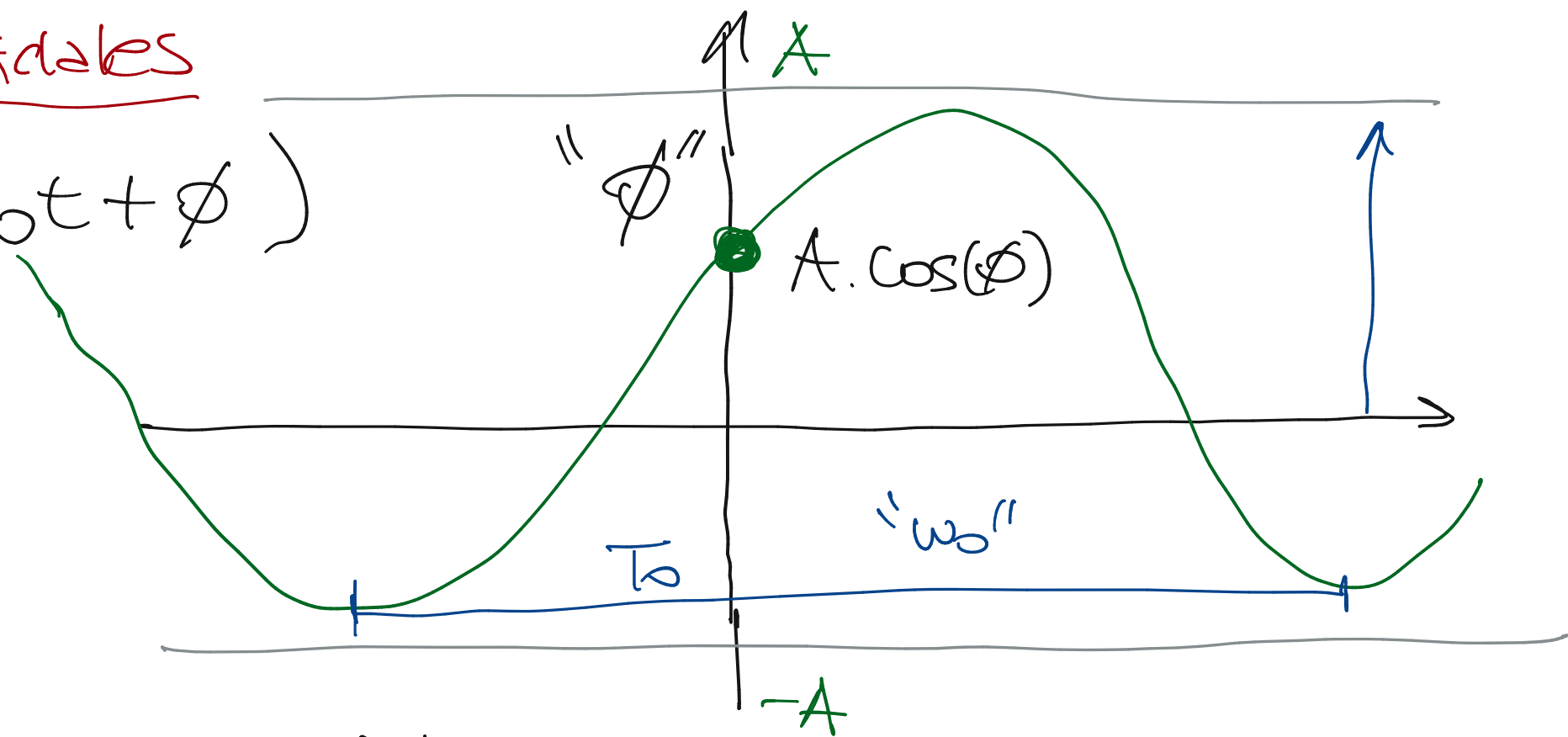
- ① señales especiales (sinusoidales)
- ② señales especiales (singulares)
- ③ Espacios de funciones

SEÑALES ESPECIALES

Señales sinusoidales

$$x(t) = A \cdot \cos(\omega_0 t + \phi)$$

$$A, \omega_0, \phi \in \mathbb{R}$$



- * coseno:
 - determinística
 - periódica
 - par ($\phi = 0$)
 - regularidad: C^∞

- * parámetros:
 - período: T_0 (s)
 - frecuencia: $f_0 = \frac{1}{T_0}$ (Hz)
 - frecuencia angular $\omega_0 = 2\pi \cdot f_0 = \frac{2\pi}{T_0}$
 - Fase inicial: ϕ

- * seno:
 - determinística
 - periódica
 - impar ($\phi = 0$)
 - regularidad: C^∞

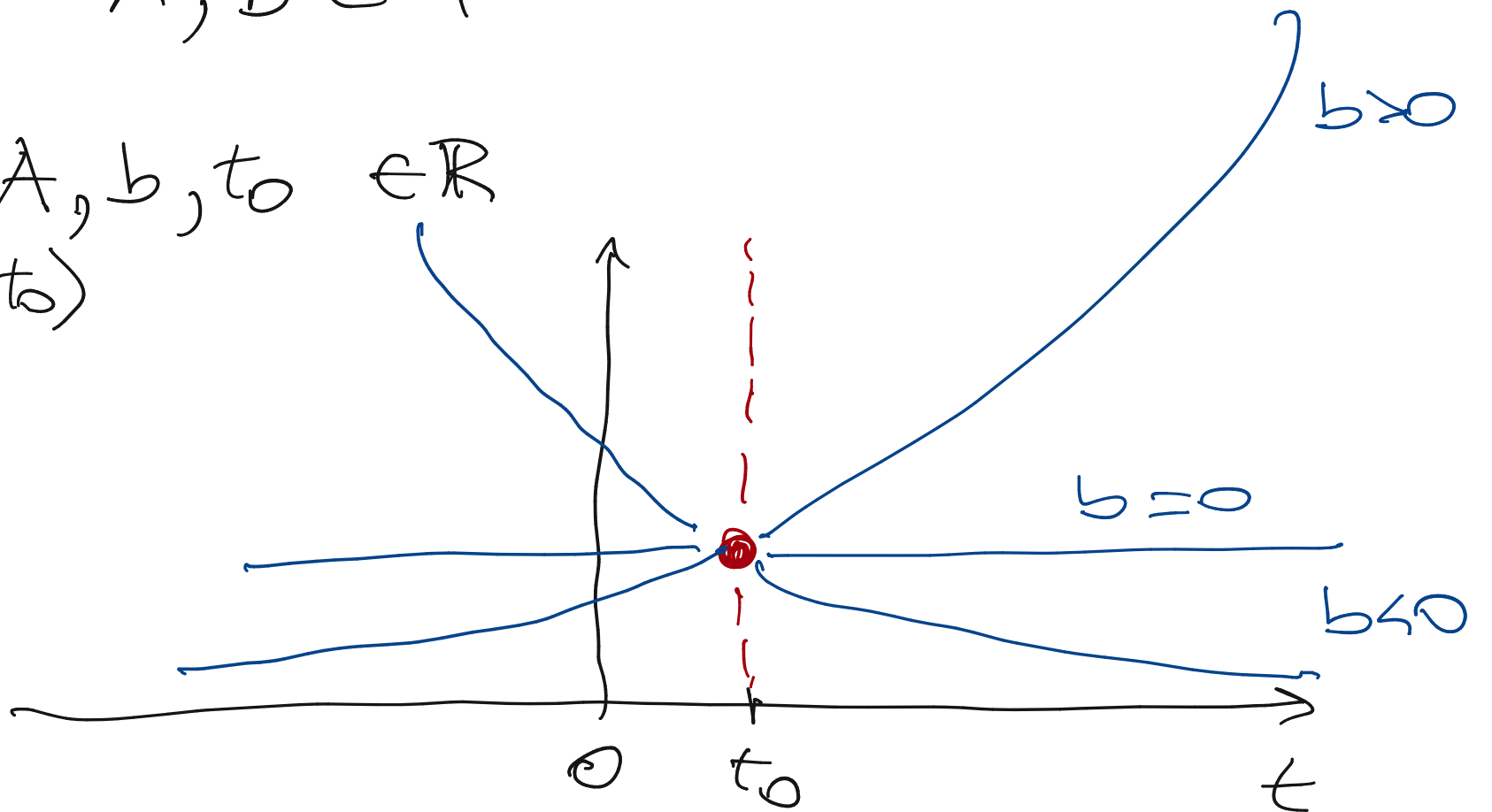
Exponenciales

$$y(t) = A \cdot e^{bt} \quad A, b \in \mathbb{C}$$

Caso real

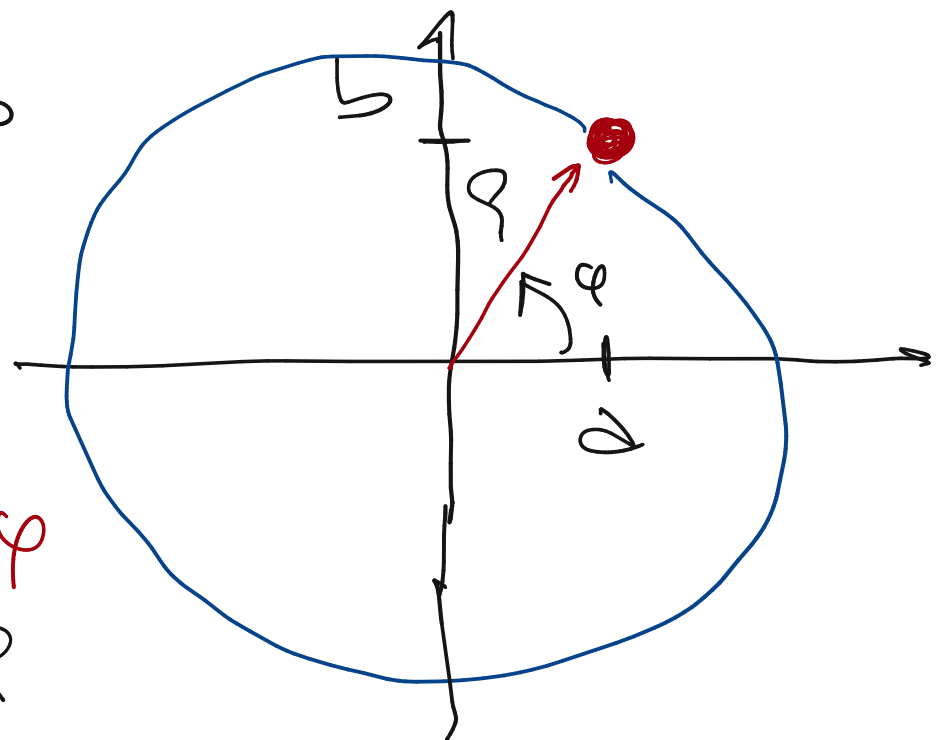
$A, b, t_0 \in \mathbb{R}$

$$y(t) = A \cdot e^{b(t-t_0)}$$



Repaso de complejos

- * rep. cartesiana: $C = a + j \cdot b$
- * rep. vectorial: $C = (a, b)$
- * rep. polar: $C = \rho \cdot \underline{e^{j\varphi}}$
- * Euler: $\rightarrow e^{j\varphi} = \cos \varphi + j \cdot \sin \varphi$
 $\rightarrow \cos \varphi = \frac{e^{j\varphi} + e^{-j\varphi}}{2}$
 $\rightarrow \sin \varphi = \frac{e^{j\varphi} - e^{-j\varphi}}{2j}$



- * conversión: $\rightarrow a = \rho \cdot \cos \varphi$
 $\rightarrow b = \rho \cdot \sin \varphi$

Caso imaginario puro

$$y(t) = \underbrace{A}_{\text{módulo}} e^{j \underbrace{\omega_0 t}_{\text{fase}}}$$

$$A \in \mathbb{R}$$

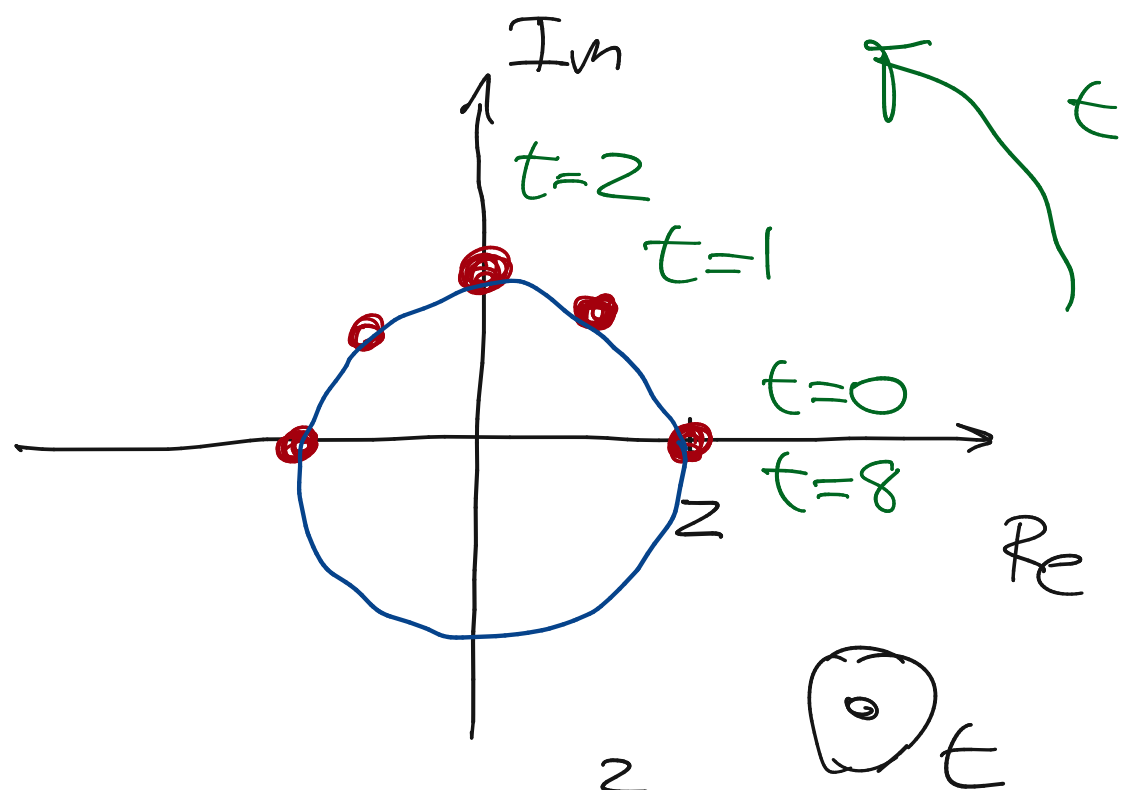
$$b \in \mathbb{C} \setminus \mathbb{R}$$

$$b = j\omega_0$$

$$y(t) = \underbrace{A}_{\text{módulo}} \cdot \underbrace{[\cos(\omega_0 t) + j \cdot \sin(\omega_0 t)]}_{\text{oscila}}$$

ejemplo: $\omega_0 = \frac{\pi}{4}$, $A = 2 \Rightarrow y(t) = 2 \cdot e^{j \frac{\pi}{4} t}$

t	$ y(t) $	$\angle y(t)$
0	2	0
1	2	$\frac{\pi}{4}$
2	2	$\frac{\pi}{2}$
3	2	$\frac{3\pi}{4}$
4	2	π
5	2	$\frac{5\pi}{4}$
6	2	$\frac{3\pi}{2}$
7	2	$\frac{7\pi}{4}$
8	2	2π



$$y: \mathbb{R} \rightarrow \mathbb{R}^2$$

Nota: ver espiral en presentación

obs: * siempre periódicas

* mayor $\omega_0 \Rightarrow$ mayor velocidad de osc

Caso complejo:

$$\tilde{A}, \tilde{b} \in \mathbb{C}$$

$$\tilde{A} = A \cdot e^{j\phi}$$

$$\tilde{b} = b + j\omega_0$$

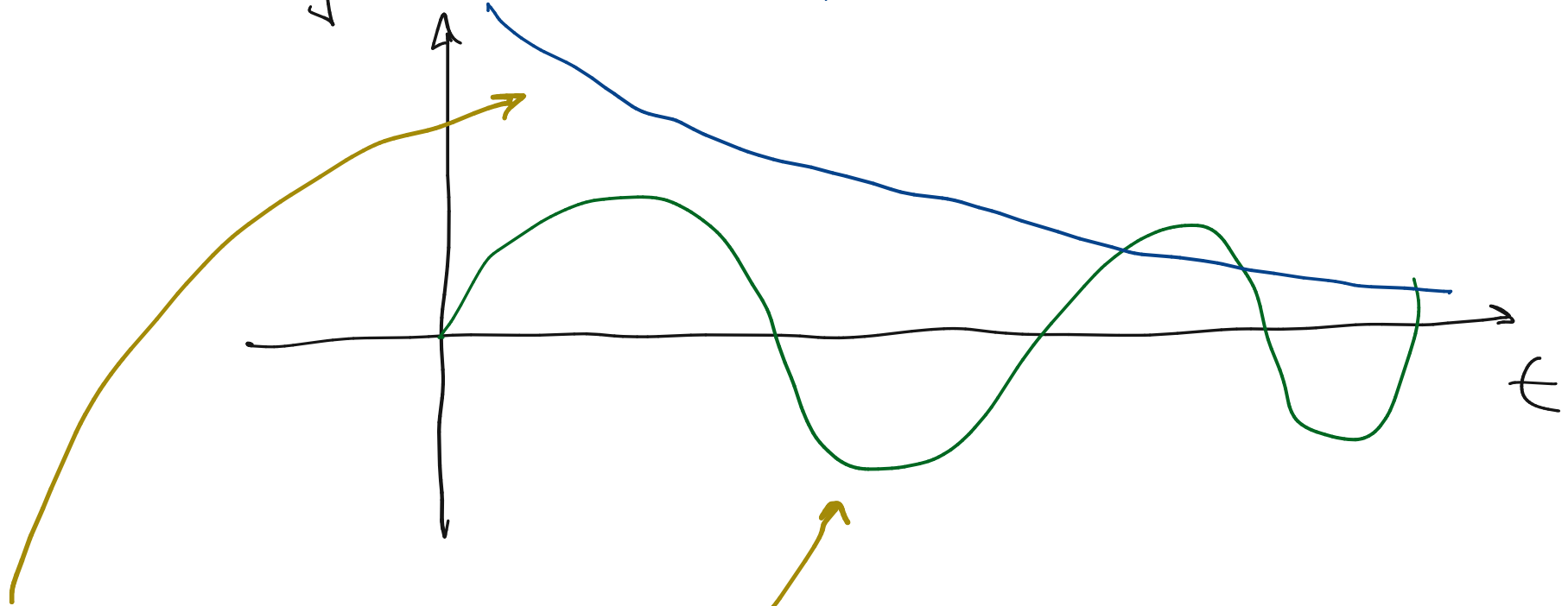
$$y(t) = \tilde{A} \cdot e^{\tilde{b}t}$$

$$y(t) = A \cdot e^{j\phi} \cdot e^{(b+j\omega_0)t}$$

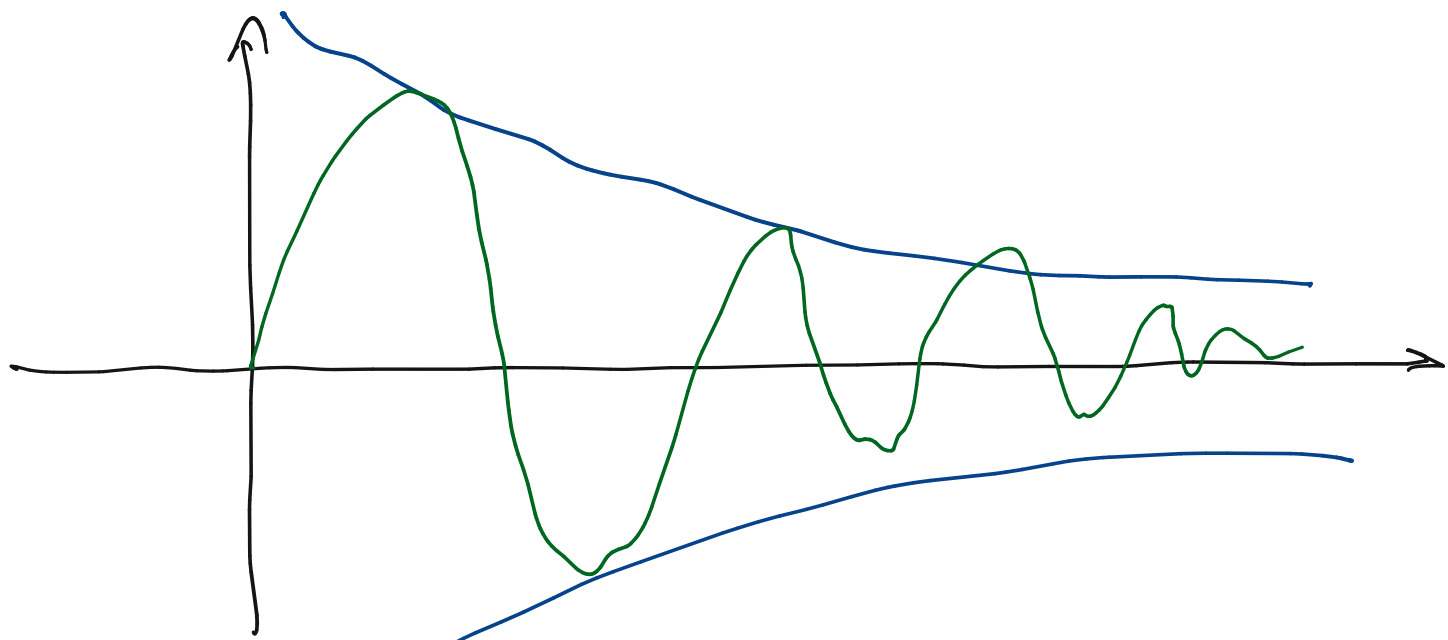
$$y(t) = A \cdot e^{bt} \cdot e^{j\phi} \cdot e^{j\omega_0 t}$$

$$= \underbrace{A \cdot e^{bt}}_{\text{módulo}} \cdot \underbrace{e^{j(\omega_0 t + \phi)}}_{\text{fase}}$$

$\text{Re}\{y(t)\}$



$$\text{Re}\{y(t)\} = A \cdot e^{bt} \cdot \cos(\omega_0 t + \phi)$$



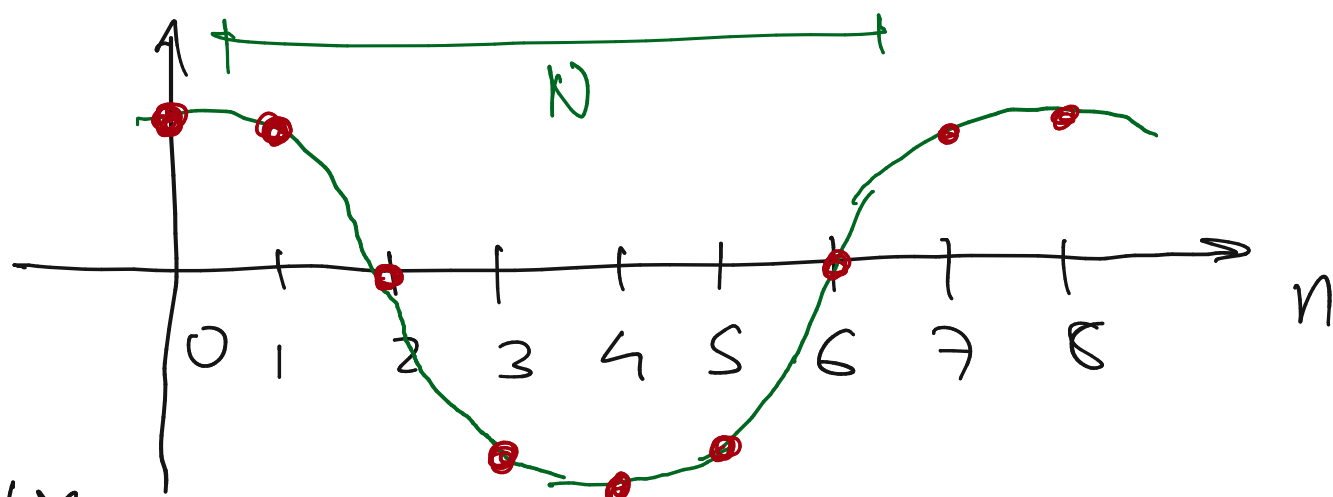
obs:

- * no es periódico (pero casi)
- * modulación no cambia la velocidad
- * módulo y fase separados
- * regularidad: C^∞

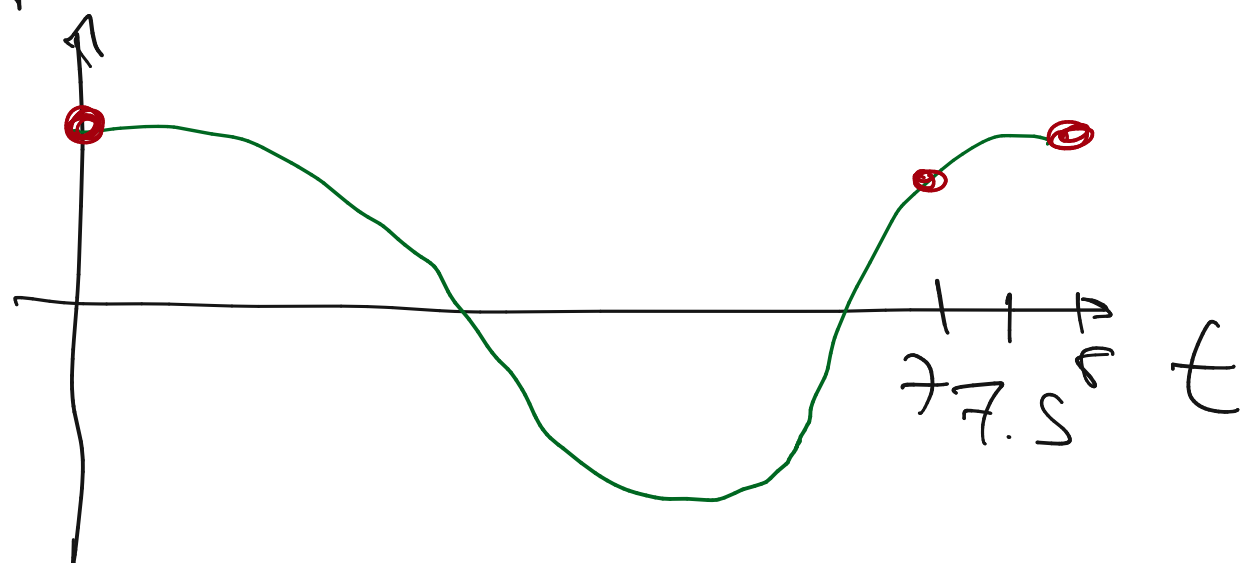
Señales sinusoidales (discretas)

$$x[n] = A \cdot \cos[\omega_0 n]$$

$$A=2, \omega_0 = \frac{\pi}{4}$$



Si $x[n]$ fuera periódica
 $\exists N \in \mathbb{Z} / x[n] = x[n+N]$



$$\cancel{A} \cdot \cos[\omega_0 n] = \cancel{A} \cdot \cos[\omega_0 (n+N)]$$

$$\omega_0 \cdot n + 2k\pi = \omega_0 (n+N), \quad k \in \mathbb{Z}$$

$$2k\pi = \omega_0 \cdot N$$

$$N = \frac{2k\pi}{\omega_0} \quad \xrightarrow{k=1}$$

$$N = \frac{2\pi}{\omega_0} \quad \omega_0 = 1$$

produce una

Obs! * no $\omega \neq 1$ fuer
 señal periódica

$$\omega_0 = k \cdot \frac{2\pi}{N}$$

Exponenciales discretas

Caso imaginario

$$x[n] = A \cdot e^{j\omega_0 n}$$

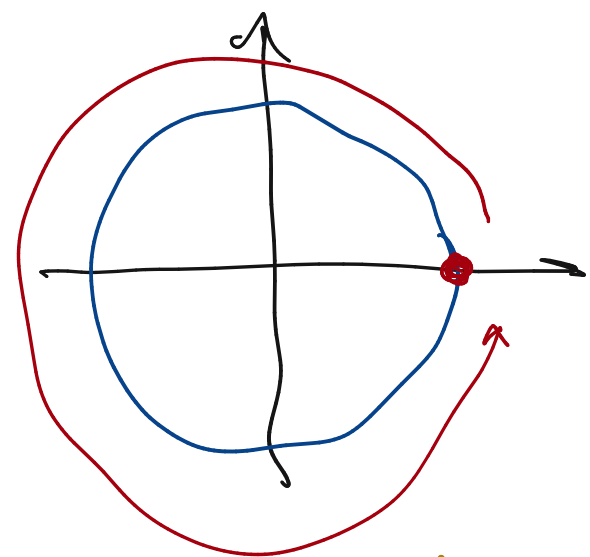
* obs: ¿cuál es la velocidad máxima?

→ en el caso continuo $\omega_0 \in [0, \infty)$

→ $e^{j\omega_0 n}$

$$e^{j(\omega_0 + 2\pi)n} = e^{j\omega_0 n} + e^{j2\pi n}$$

* la señal con frecuencia ω_0 es la misma que aquella con $\omega_0 + 2k\pi$



* la frec. discreta tiene máximo.

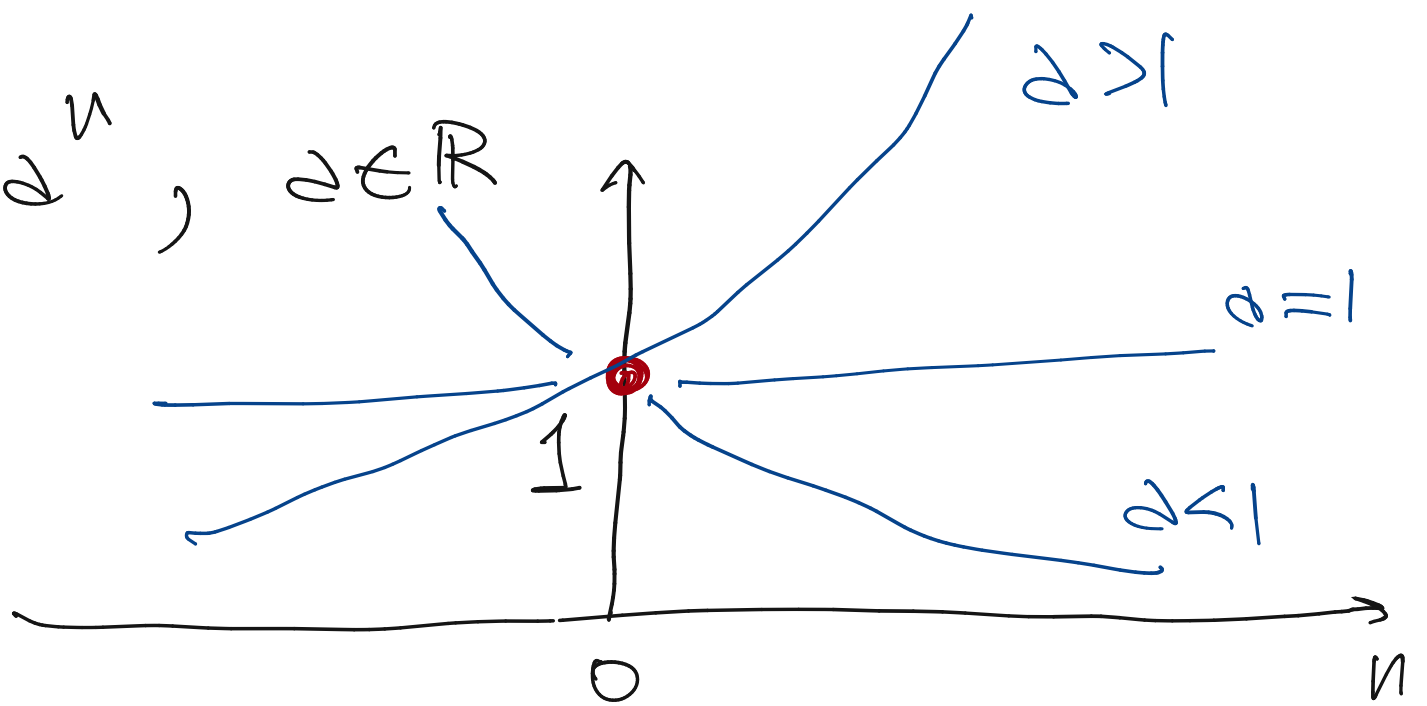
→ cont: $\omega_0 \in [0, \infty) \rightarrow (-\infty, \infty)$

→ disc: $\omega_0 \in [0, \pi) \rightarrow (-\pi, \pi)$

después una vuelta.

Caso real

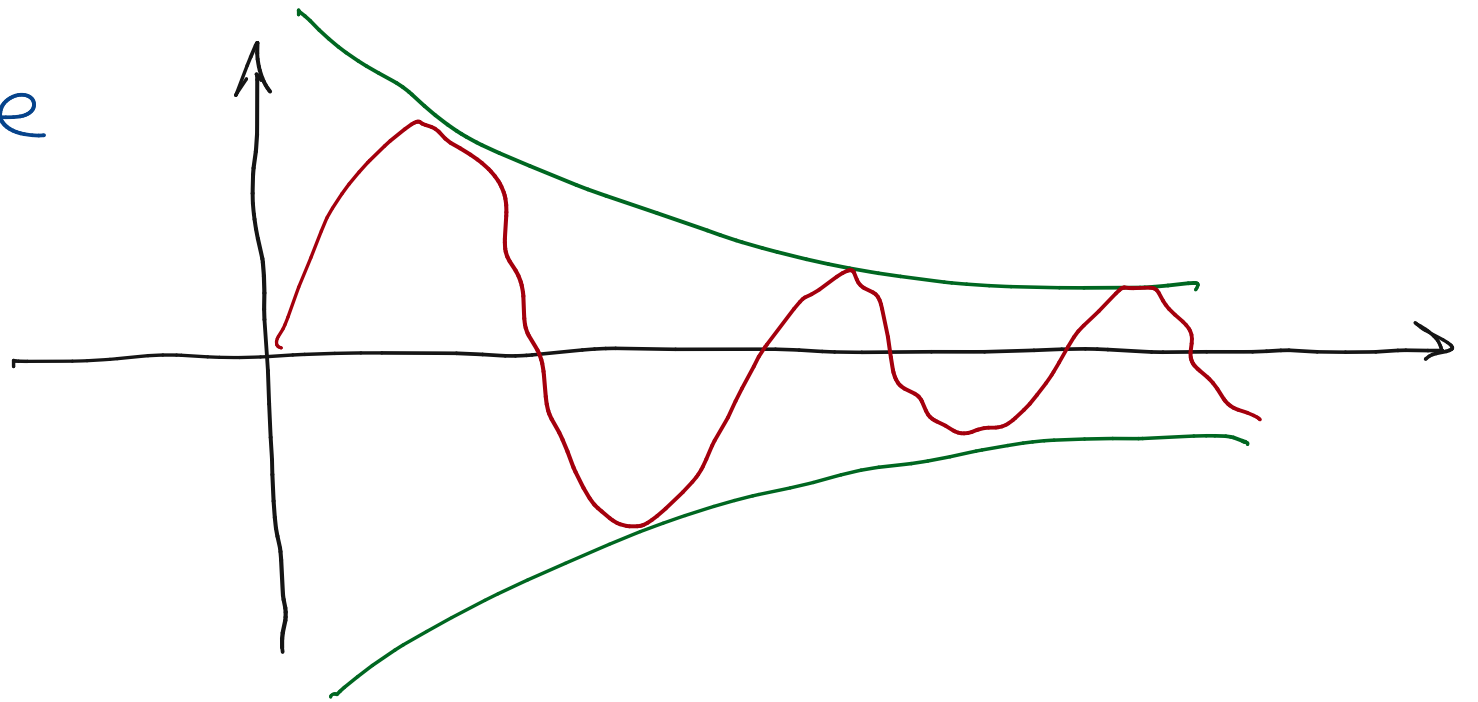
$$x[n] = a^n, \quad a \in \mathbb{R}$$



Caso general

$$x[n] = c^n, \quad c \in \mathbb{C}, \quad c = \rho \cdot e^{j\phi}$$

$$x[n] = \underbrace{\rho^n}_{\text{módulo}} \cdot \underbrace{e^{jn\phi}}_{\text{fase}}$$

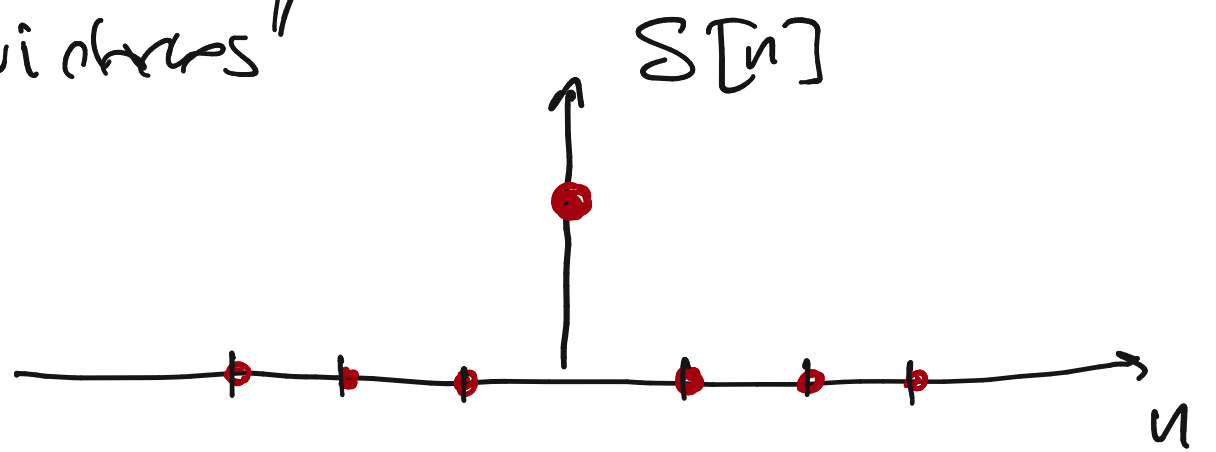


FUNCIONES SINGULARES

idea: "tienen discontinuidades"

Impulso unitario

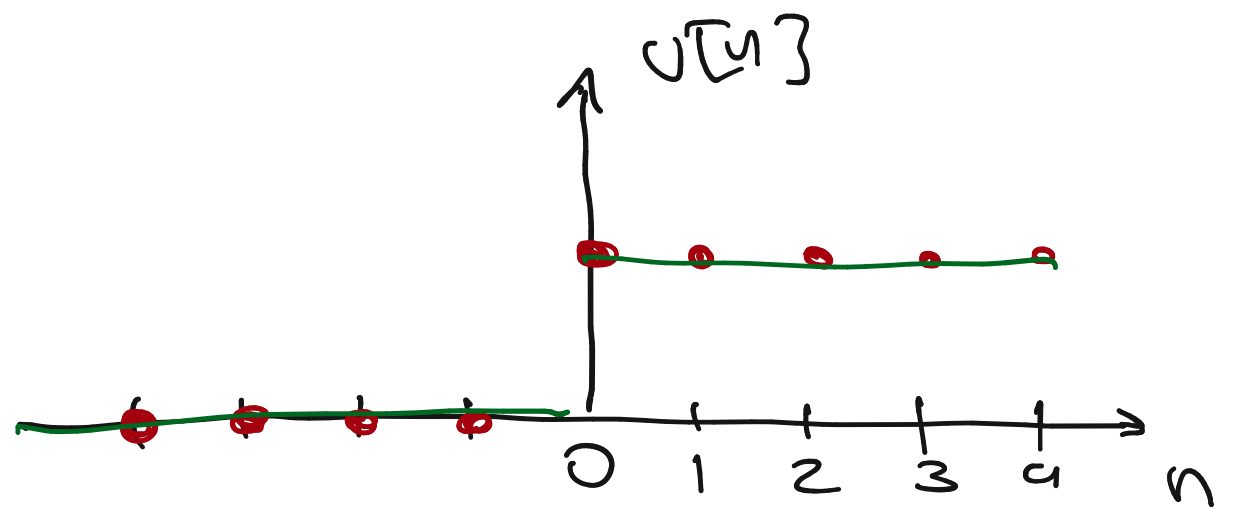
$$\delta[n] = \begin{cases} 1 & n=0 \\ 0 & n \neq 0 \end{cases}$$



* obs: → delta de Kronecker
→ "señal más sencilla"

Escala unitario

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



* obs: modela el
"encendido de una llave"

Relación

1)

"directa"

$$v[n] = u[n-1]$$

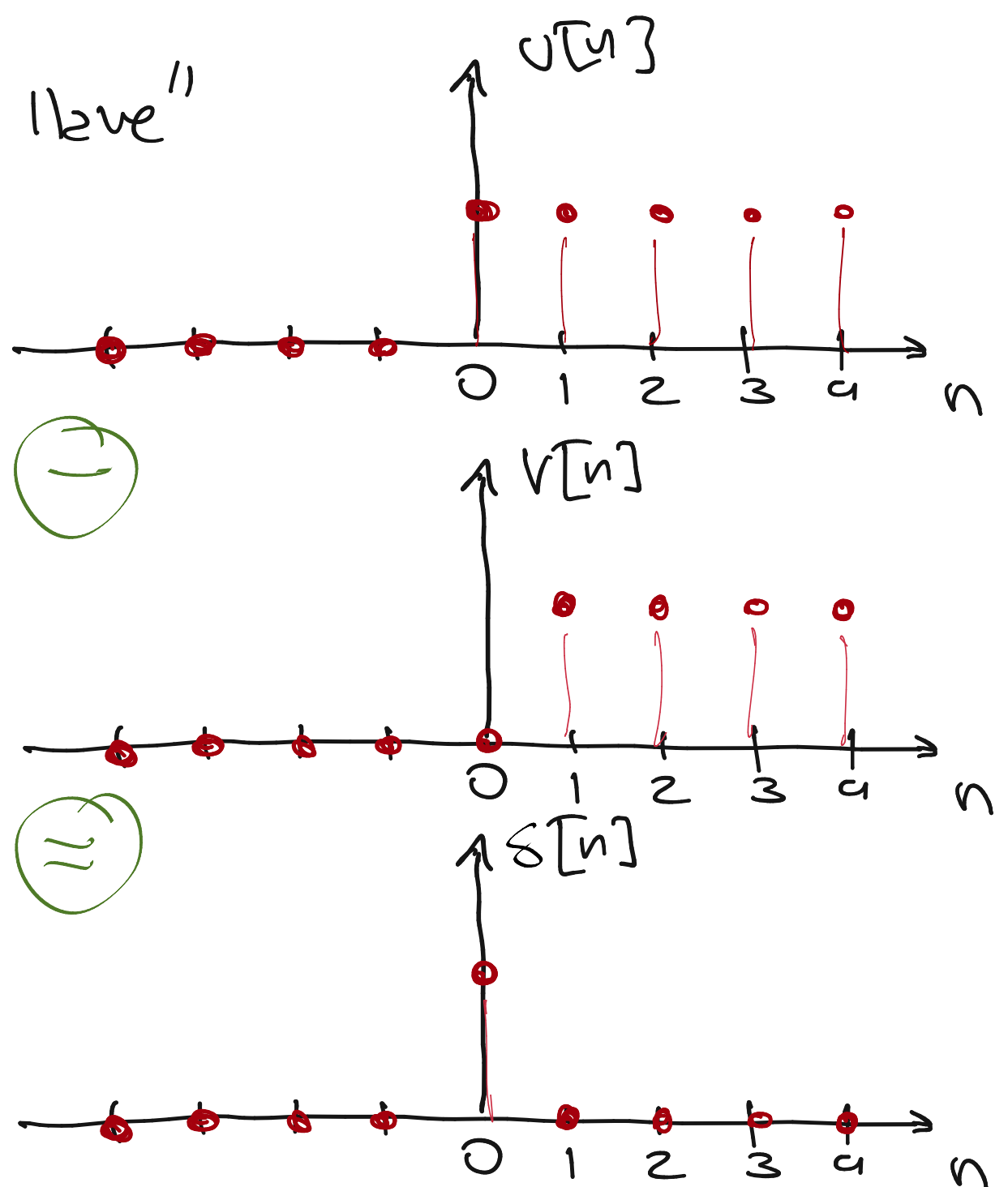
$$v[n] = u[n] - \delta[n-1]$$

$$v[n] = \begin{cases} 1 & n \geq 1 \\ 0 & n < 1 \end{cases}$$

$$u[n-1] = u[n] - \delta[n-1]$$

$$\delta[n] = u[n] - u[n-1]$$

primera diferencia.



$$\frac{v[n] - v[n-1]}{1} \rightarrow$$

" $v'[n]$ "

$$\frac{f(t) - f(t-h)}{h} \rightarrow$$

$f'(t)$

* obs: $\rightarrow$ primera diferencia es el análogo discreto a la derivada primera.

$\rightarrow$ $\delta[n]$ es la derivada del escalón $u[n]$

2) "inversa"
 $u[n]$ es la "integral" de la $\delta[n]$

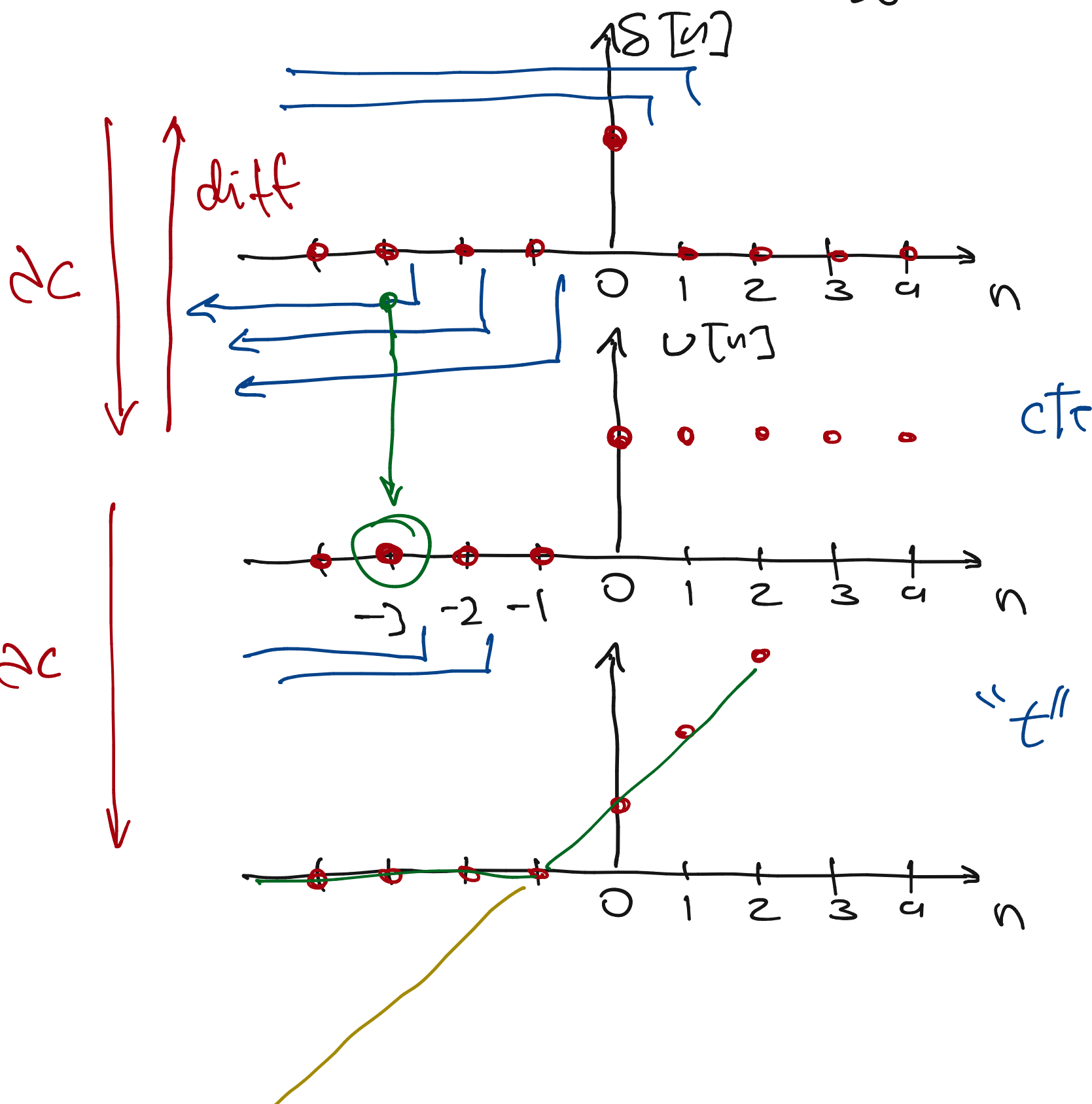
* Acumulación:
 $u[n] = \sum_{k=-\infty}^n \delta[k] \leftrightarrow f(t) = \int_{-\infty}^t f(\tau) d\tau$

Ejemplo:

$$w[n] = \sum_{k=-\infty}^n u[k]$$

$$w[n] = \begin{cases} n+1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

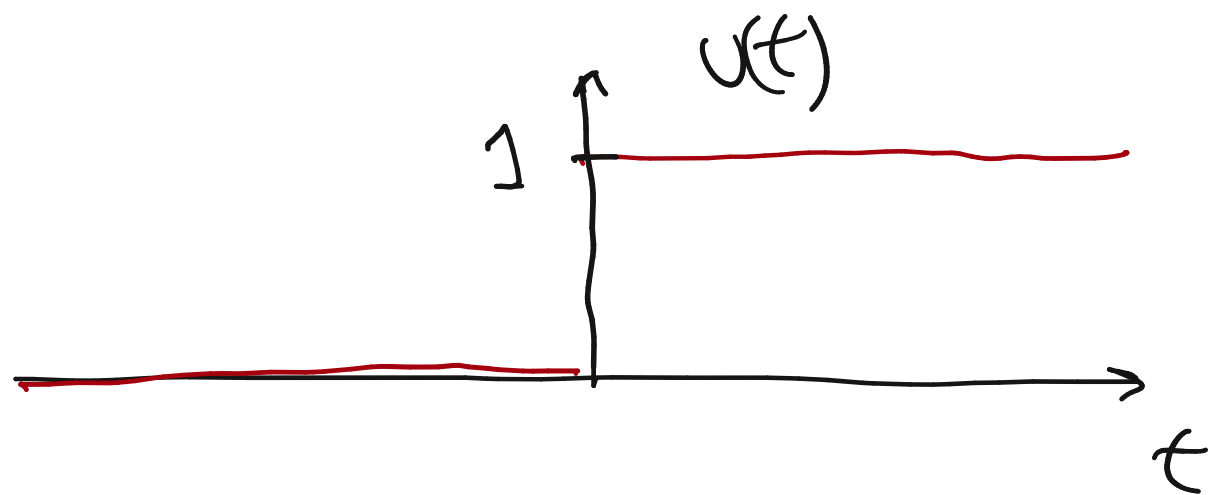
$$w[n] = u[n] \cdot (n+1)$$



Funciones singulares (continuas)

Escalón unitario

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



* obs: función de Heaviside

Impulso unitario

* $\delta(t) = ?$ delta de Dirac

* $\delta(t) = u'(t)$

* Pregunta: . cómo derivar en la discontinuidad?

- * obs:
- donde no es discontinua, es como
 - enfoque opuesto: "lo interesante es la discontinuidad"
 - cuando no puedo calcular $\Rightarrow$ aproximo



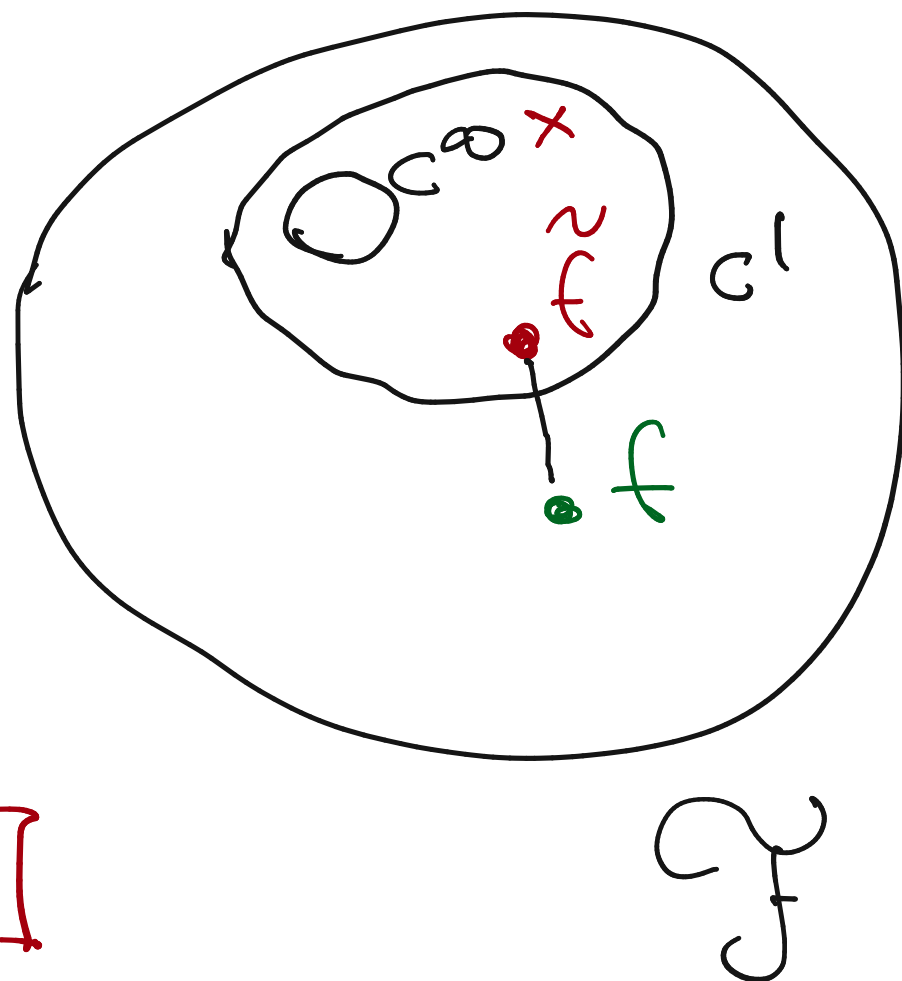
Apox.
continua.

ESPACIOS DE FUNCIONES

Aproximación de funciones

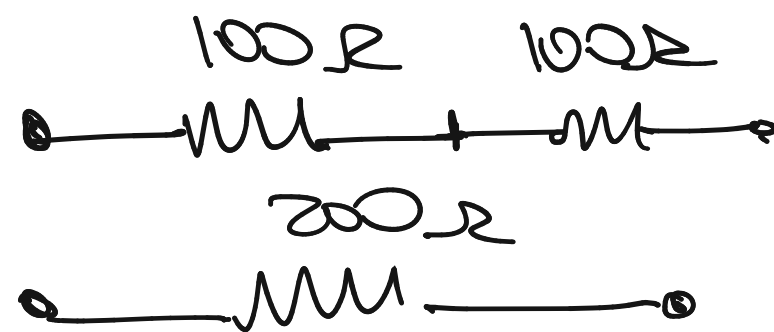
Idea:

$$\begin{array}{l} \overbrace{\frac{1}{n+1} \sim \frac{1}{n}}^{\text{pescados}} \not\sim \frac{1}{n^2} \\ \frac{n+1}{n^2+n} \sim \frac{n}{n^2} = \frac{1}{n} \quad \text{dif.} \end{array}$$



* Obs: $\rightarrow$ son iguales? **NO**

$\rightarrow$ para la tarea den lo mismo, son equivalentes



Aproximación continua

Dado $f \in \mathcal{F} \setminus C^1$ quiero encontrar una $\tilde{f} \in C^1 / d(f, \tilde{f}) \sim 0$

* Obs. $\rightarrow$ necesito medir "distancias" entre funciones
 $\rightarrow$ las distancias se miden en espacios vectoriales

Espacios vectoriales

- * conjuntos: $\rightarrow V$: vectores
 $\rightarrow E$: escalares (números)

- * operaciones: $\rightarrow$ suma ($\geq$ vectores): $+: V \times V \rightarrow V$
 $\rightarrow$ producto (vector-escalar): $\bullet: V \times E \rightarrow V$

Ejemplo: $V = \mathbb{R}^2$, $E = \mathbb{R}$ "vectores euclidianos"

Base: $B = \{e_1, e_2\}$

- * L.I.

- * generador

- * ortogonal: $\perp$ y $\| \| = 1$ (largo)

- * operaciones: $\rightarrow$ descomposición
 $\rightarrow$ generación

Producto interno

$$\langle v, w \rangle : V \times V \rightarrow E$$

$$\langle v, w \rangle = v_1 \cdot w_1 + v_2 \cdot w_2$$

- * propiedades:

- $\rightarrow$ simetría conjugada: $\langle x, y \rangle = \overline{\langle y, x \rangle}$

$$x, y \in \mathbb{R} \Rightarrow \langle x, y \rangle = \langle y, x \rangle$$

- $\rightarrow$ linealidad: $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

- $\rightarrow$ definido positivo: $\rightarrow \langle x, x \rangle \geq 0$

$$\rightarrow \langle x, x \rangle = 0 \Leftrightarrow x = 0$$

- * Obs: $\rightarrow$ "sirve para medir el ángulo entre dos vectores"

- $\rightarrow$ perpendicularidad: si $x \perp y \Rightarrow \langle x, y \rangle = 0$

Norma: "largo de un vector" $\| \cdot \| : V \rightarrow \mathbb{R}$

- * propiedades:
- homogeneidad (abs): $\|a \cdot \vec{v}\| = |a| \cdot \|\vec{v}\|$
 - sub-aditividad: $\| \vec{u} + \vec{v} \| \leq \| \vec{u} \| + \| \vec{v} \|$
 - no negatividad: $\| \vec{v} \| \geq 0, \forall \vec{v} \in V$
 - definida: $\| \vec{v} \| = 0, \vec{v} = \vec{0}$

- * obs: → se puede construir a partir del prod. interno
- $\langle \vec{v}, \vec{v} \rangle = \| \vec{v} \|^2$

Métrica: "distancia entre dos puntos"

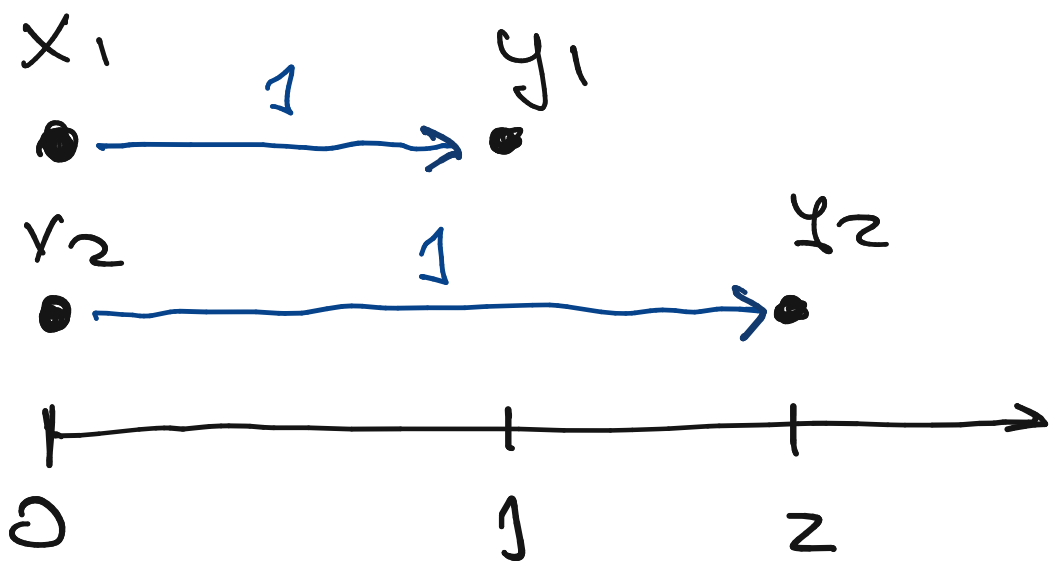
$d: V \times V \rightarrow \mathbb{R}$

- * propiedades:
- positiva: $d(x, y) \geq 0$
 - definida: $d(x, y) = 0 \iff x = y$
 - simétrica: $d(x, y) = d(y, x)$
 - triangular: $d(x, z) \leq d(x, y) + d(y, z)$

* $d(x, y) = \| x - y \|$

Ejemplo:
$$d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

- * obs:
- métrica ✓
 - norma ✗



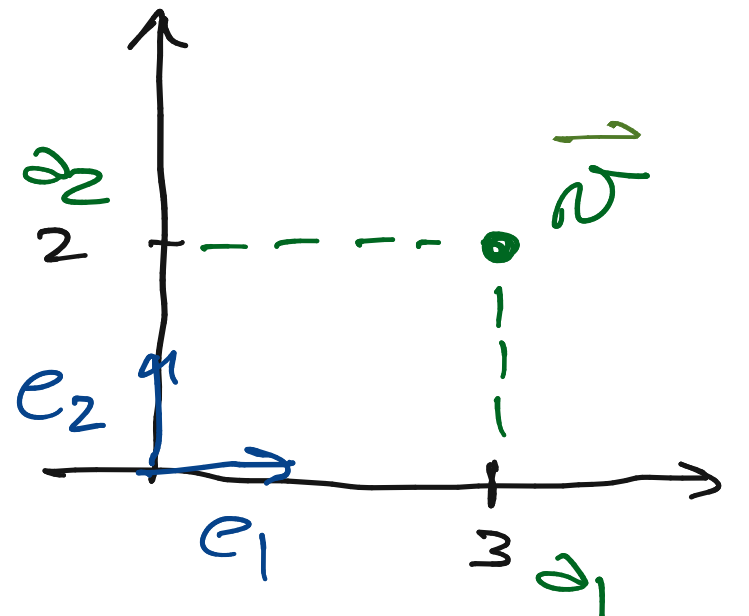
Bases

"conjunto de vectores sencillos que me permitan representar un E.V."

Reconstrucción

* Síntesis, generación

$$\vec{v} = \underbrace{a_1}_{\text{cantidades}} \cdot \underbrace{\vec{e}_1}_{\text{ingredientes}} + \underbrace{a_2}_{\text{cantidades}} \cdot \underbrace{\vec{e}_2}_{\text{ingredientes}}$$



Proyección

* Análisis, descomposición

$$\vec{v} = (a_1, a_2)_B$$

$$\begin{aligned} a_1 &= \langle \vec{v}, \vec{e}_1 \rangle \\ a_2 &= \langle \vec{v}, \vec{e}_2 \rangle \end{aligned}$$

$$\begin{aligned} \langle \vec{v}, \vec{e}_1 \rangle &= \langle a_1 \vec{e}_1 + a_2 \vec{e}_2, \vec{e}_1 \rangle = \langle a_1 \vec{e}_1, \vec{e}_1 \rangle + \langle a_2 \vec{e}_2, \vec{e}_1 \rangle \\ &= a_1 \underbrace{\langle \vec{e}_1, \vec{e}_1 \rangle}_{=1} + a_2 \underbrace{\langle \vec{e}_2, \vec{e}_1 \rangle}_{=0} = \boxed{a_1} \end{aligned}$$

Ejemplo:

$$\vec{v} = (4, 5)$$

$$\vec{e}_1 = (1, 0)$$

$$\vec{e}_2 = (0, 1)$$

$$a_1 = \langle (4, 5), (1, 0) \rangle$$

$$= 4 \cdot 1 + 5 \cdot 0 = \boxed{4}$$

Ejemplo: "proyección"

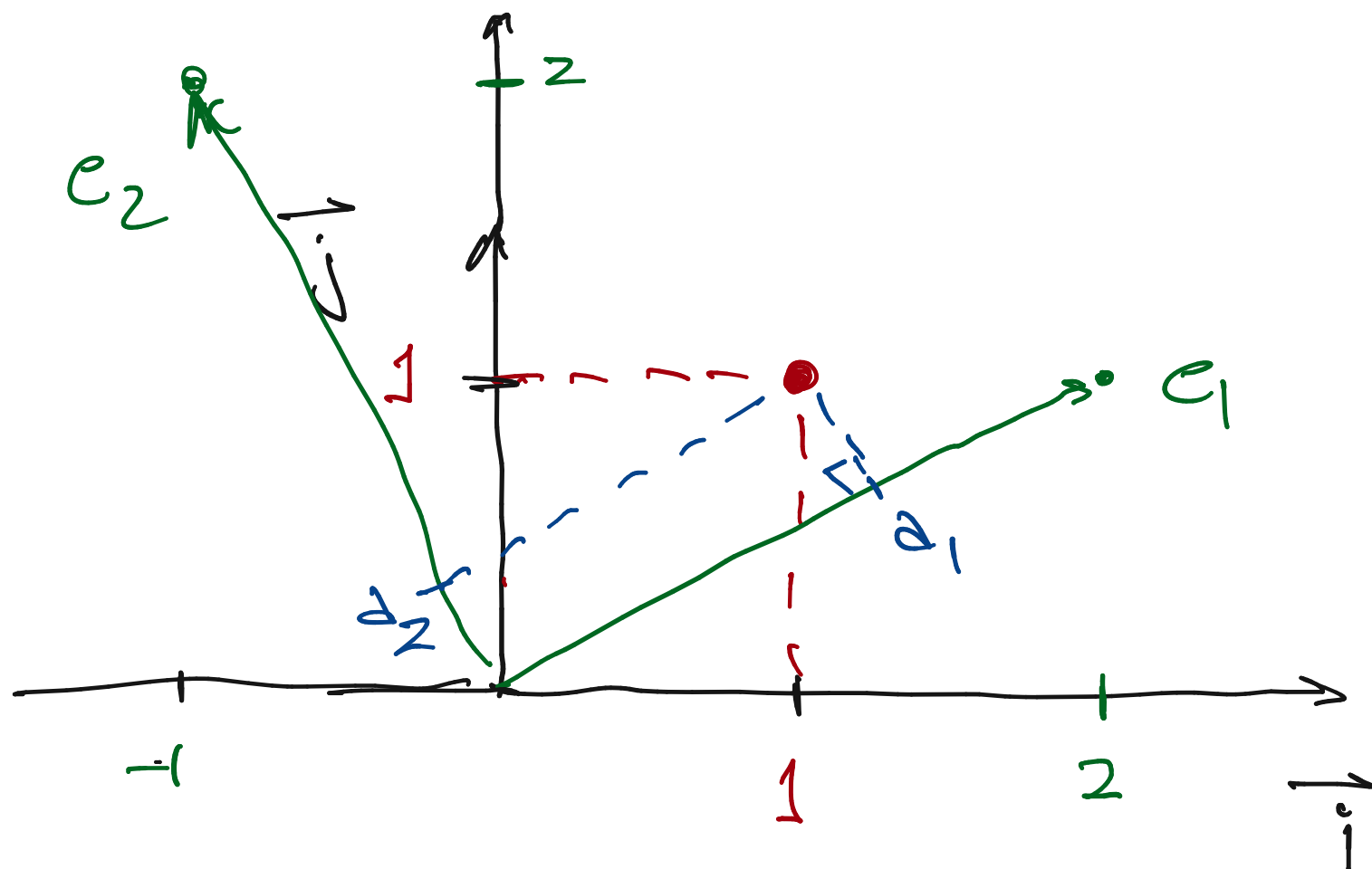
$$v = (1, 1)$$

$$\tilde{e}_1 = (2, 1)$$

$$\tilde{e}_2 = (-1, 2)$$

$$e_1 = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right)$$

$$e_2 = \left(\frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right)$$



$$d_1 = \langle v, e_1 \rangle = 1 \cdot \frac{2}{\sqrt{5}} + 1 \cdot \frac{1}{\sqrt{5}} = \frac{3}{\sqrt{5}} \approx 1.34$$

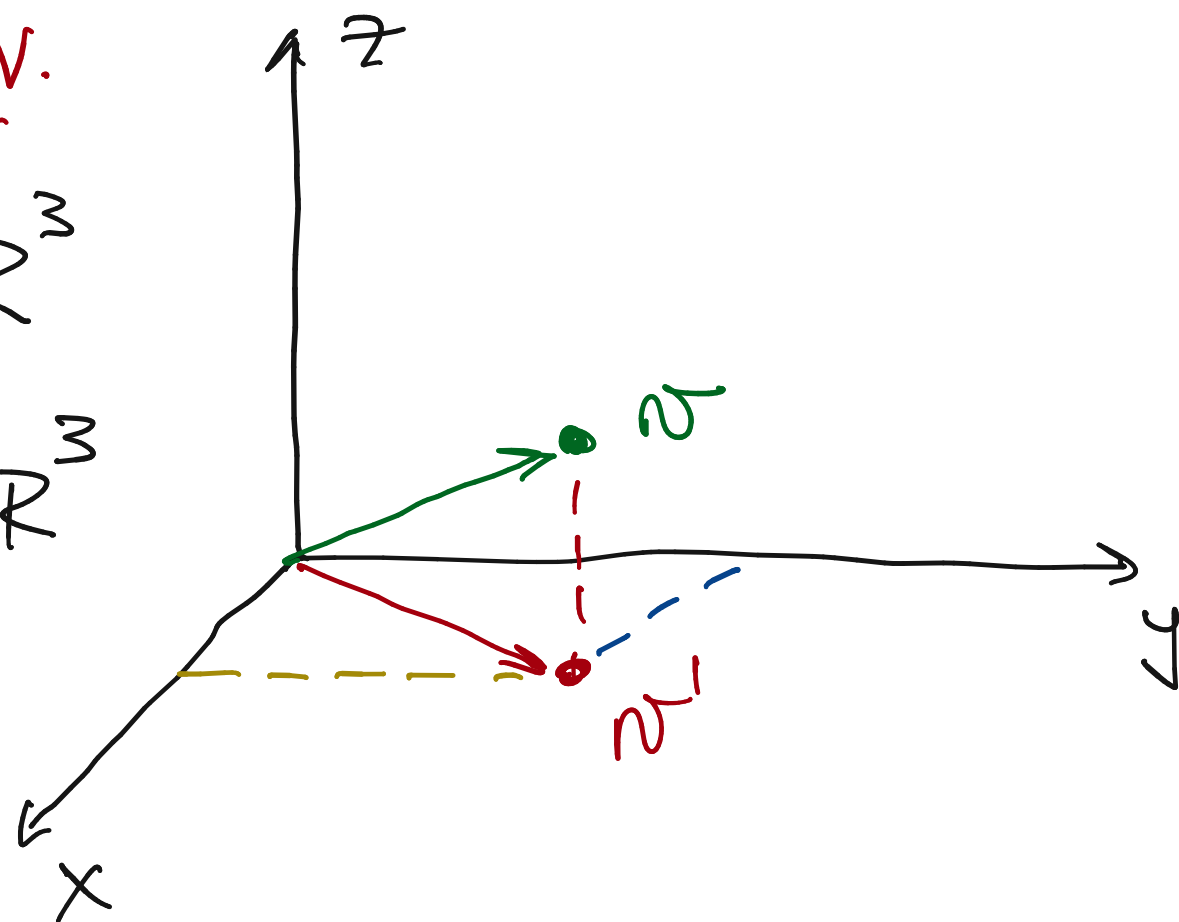
$$d_2 = \langle v, e_2 \rangle = 1 \cdot \frac{(-1)}{\sqrt{5}} + 1 \cdot \frac{2}{\sqrt{5}} = \frac{1}{\sqrt{5}} \approx 0.45$$

Aproximación en E.V.

$$v = (1, 4, 0.1) \in \mathbb{R}^3$$

$$v' \approx (1, 4, 0) \in \mathbb{R}^3$$

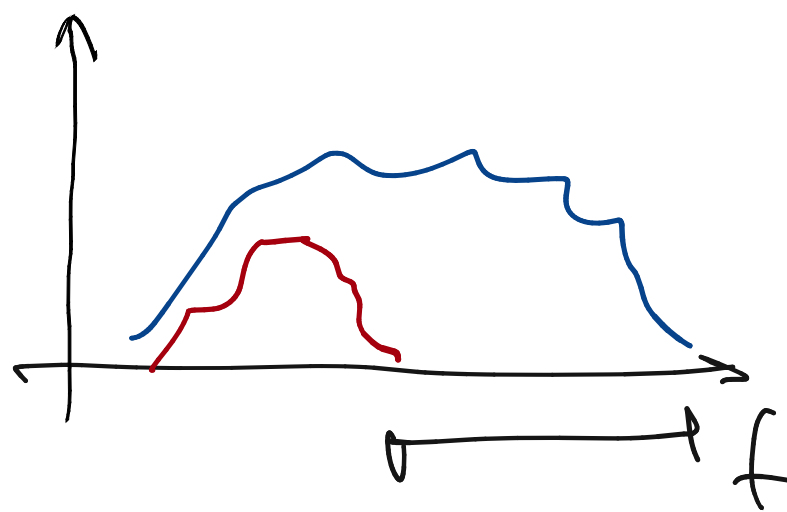
$$v'' = (1, 4) \in \mathbb{R}^2$$



$$E = \frac{\|v - v'\|}{\|v\|} = \frac{\|(0, 0, 0.1)\|}{\|(1, 4, 0.1)\|}$$

$$= \frac{0.1}{0.7} \approx 0.02 \rightarrow \boxed{E = 2\%}$$

Obs: * Necesito una forma de medir funciones.



Obs: * v' es la proyección de v en $\mathbb{R}^2$

CLASE 3.5

Señales

Espacios de funciones y aproximación continua

APROXIMACIÓN CONTINUA

Dado $f \in \mathcal{F}$, encontrar
 $\tilde{f} \in C'$ / $d(f, \tilde{f}) \approx 0$

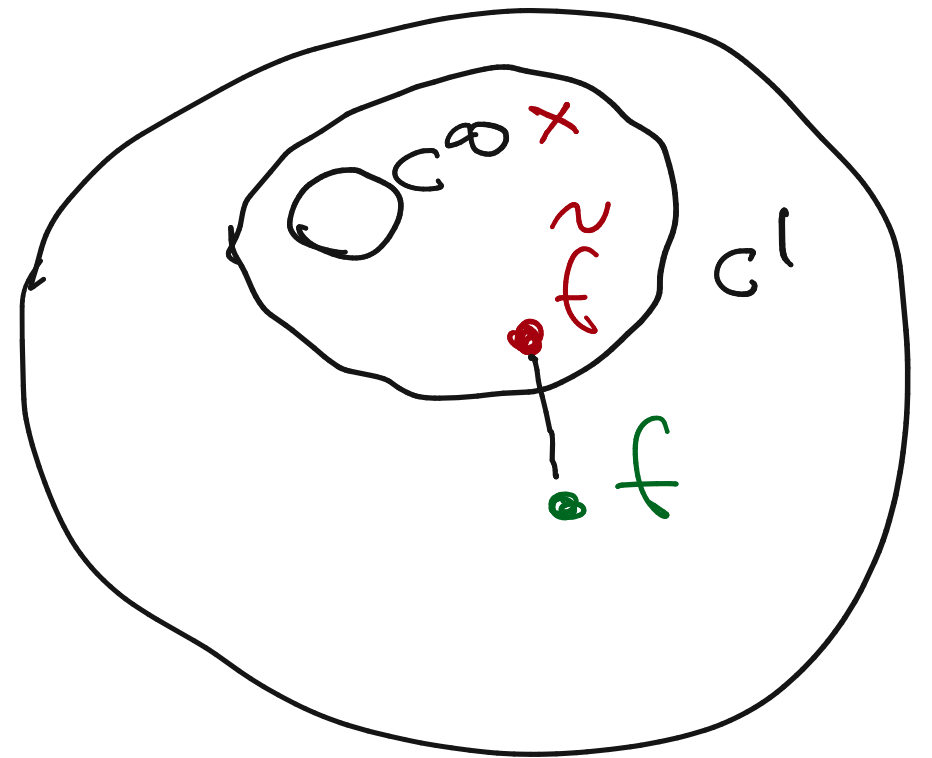
Espacios de funciones

Espacio vectorial: (V, E)

$$V = \{ f(x) \in \mathcal{F} / f: \mathbb{R} \rightarrow \mathbb{R} \}$$

$$E = \mathbb{R}$$

obs: necesito una forma de "medir" funciones



$\mathcal{F}$

métricas alternativas

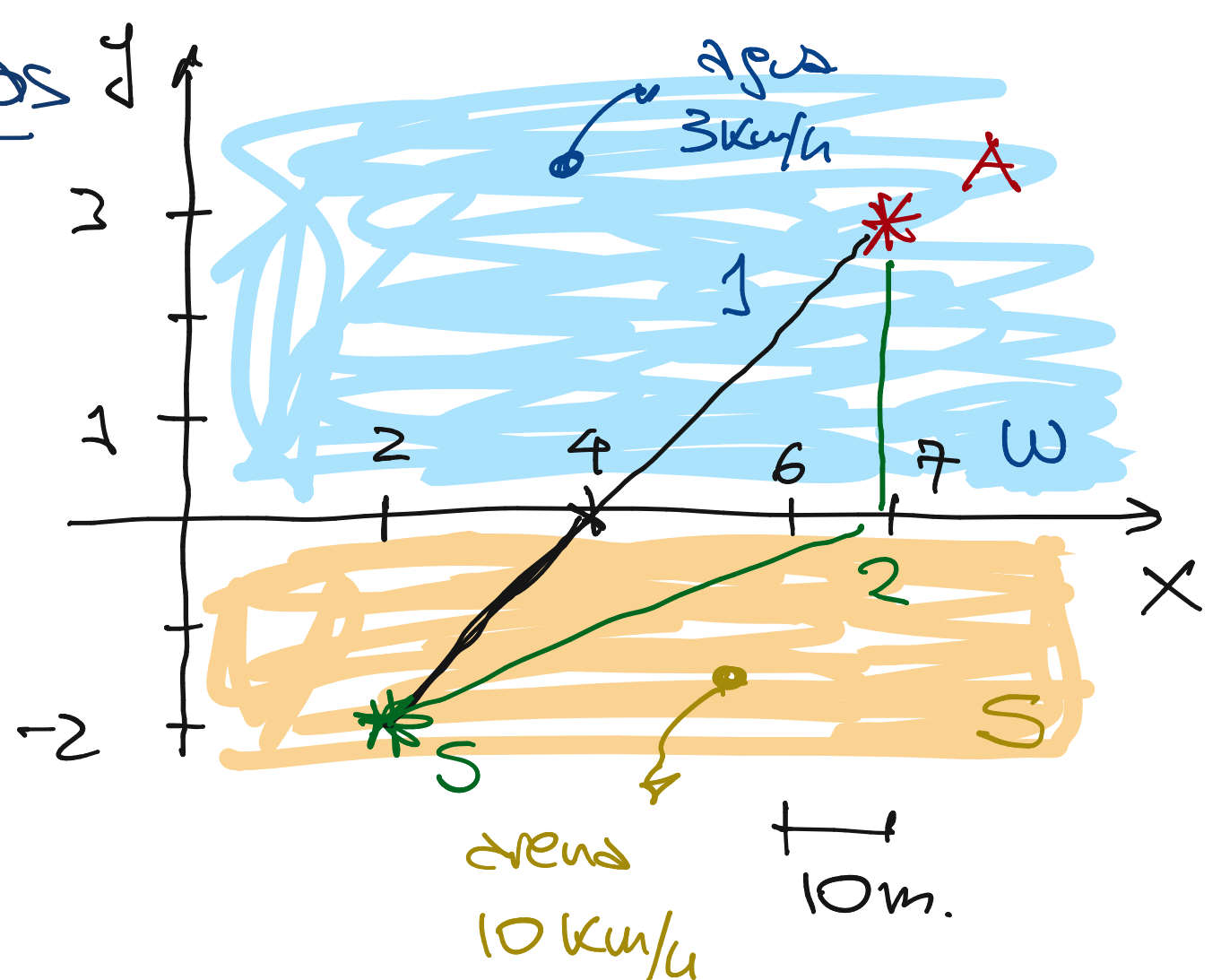
$$t_1 = \frac{d_1^s}{v_s} + \frac{d_1^w}{v_w}$$

$$v_s = 10 \text{ km/h} = 2.78 \text{ m/s}$$

$$v_w = 3 \text{ km/h} = 0.83 \text{ m/s}$$

$$t_1 = 61.08 \text{ s}$$

$$t_2 = 53.38 \text{ s}$$



Geodésica

* geodésica: camino más corto entre dos puntos

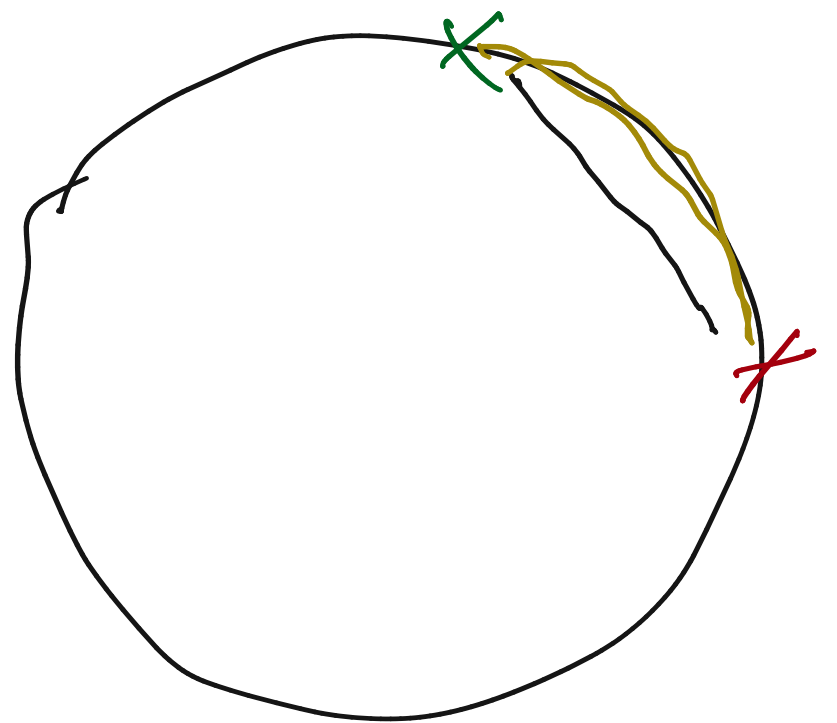
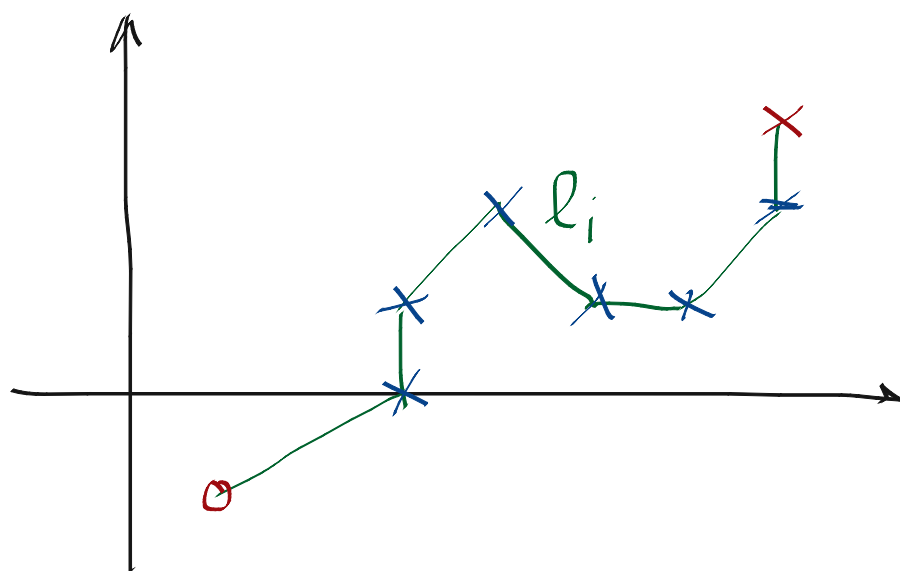
↳ dada una

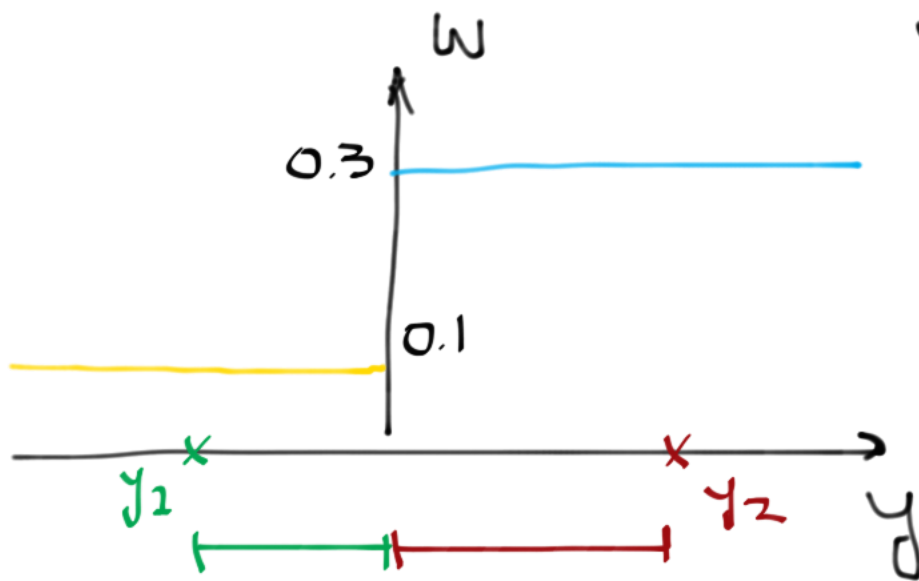
* distancia: es el largo de una geodésica métrica

Medidas

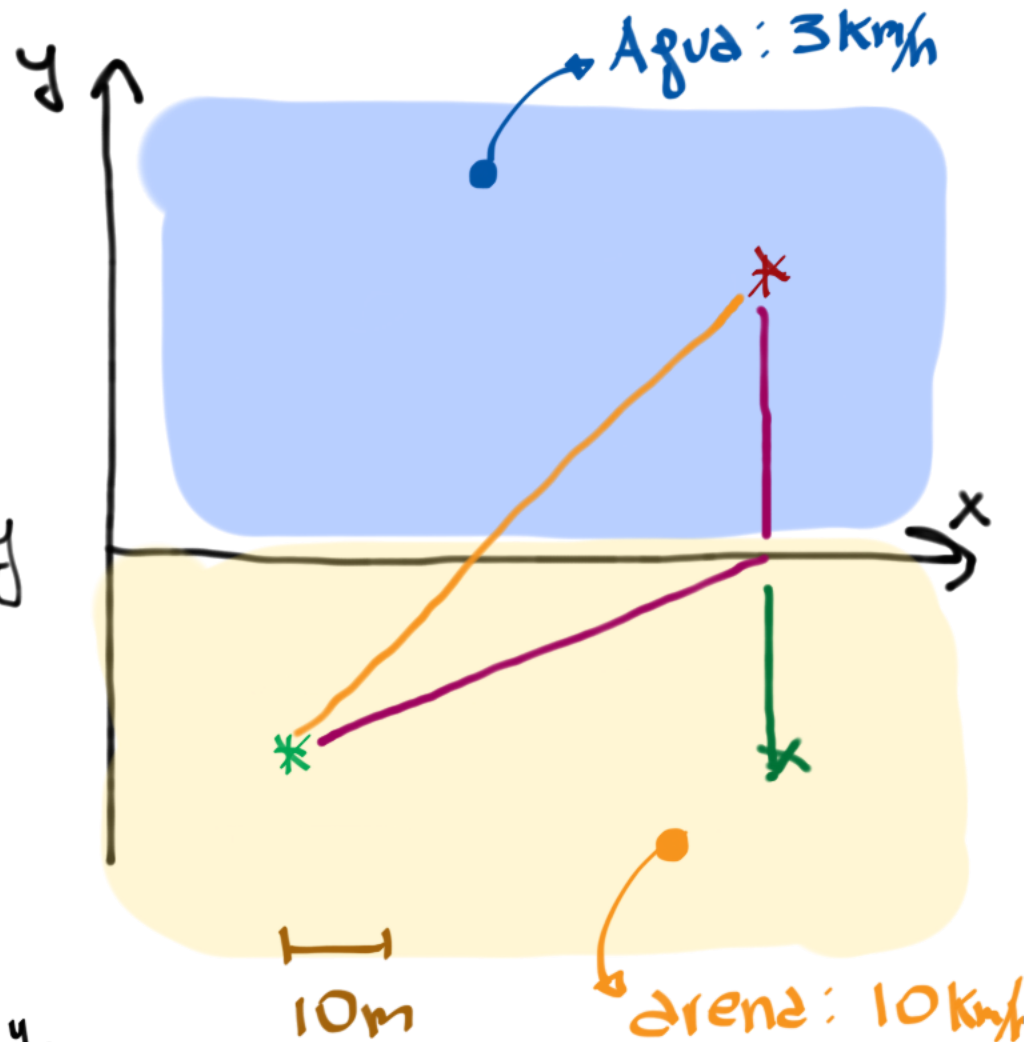
$$L = \sum_{i=0}^{N-1} l_i$$

$$L = \oint 1$$





$$d(y_2, y_1) = |y_2 - y_1|$$



$$d(y_2, y_1) = \int_{y_1}^{y_2} 1 \cdot dy = y \Big|_{y_1}^{y_2} = \boxed{y_2 - y_1} \checkmark$$

$$d(y_2, y_1) = \int_{y_1}^{y_2} \omega(y) \cdot dy = \int_{y_1}^0 0.1 \cdot dy + \int_0^{y_2} 0.3 \cdot dy$$

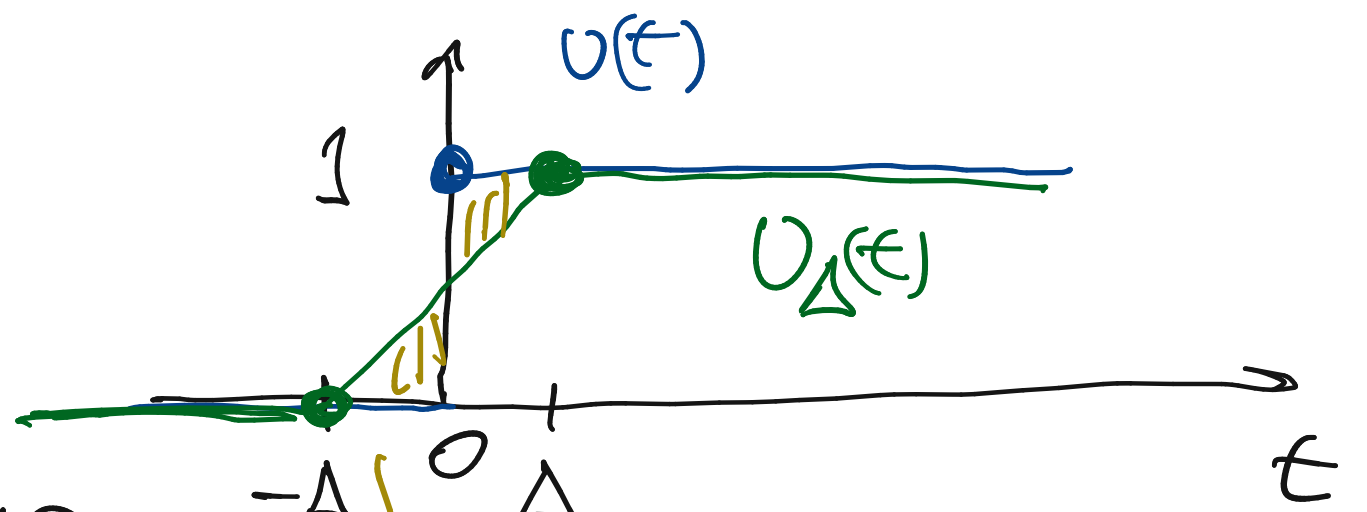
$$= 0.1(0 - y_1) + 0.3(y_2 - 0) = \boxed{0.3 y_2 - 0.1 y_1} =$$

$$= 0.3 \cdot 30 - 0.1(-20) = 11$$



Aproximación continua del escalón

$$U_{\Delta}(t) = \begin{cases} 1 & t \geq \Delta \\ \frac{1}{2\Delta}(t + \Delta) & -\Delta \leq t < \Delta \\ 0 & t < -\Delta \end{cases}$$



Aproxima bien si $\Delta \sim 0$

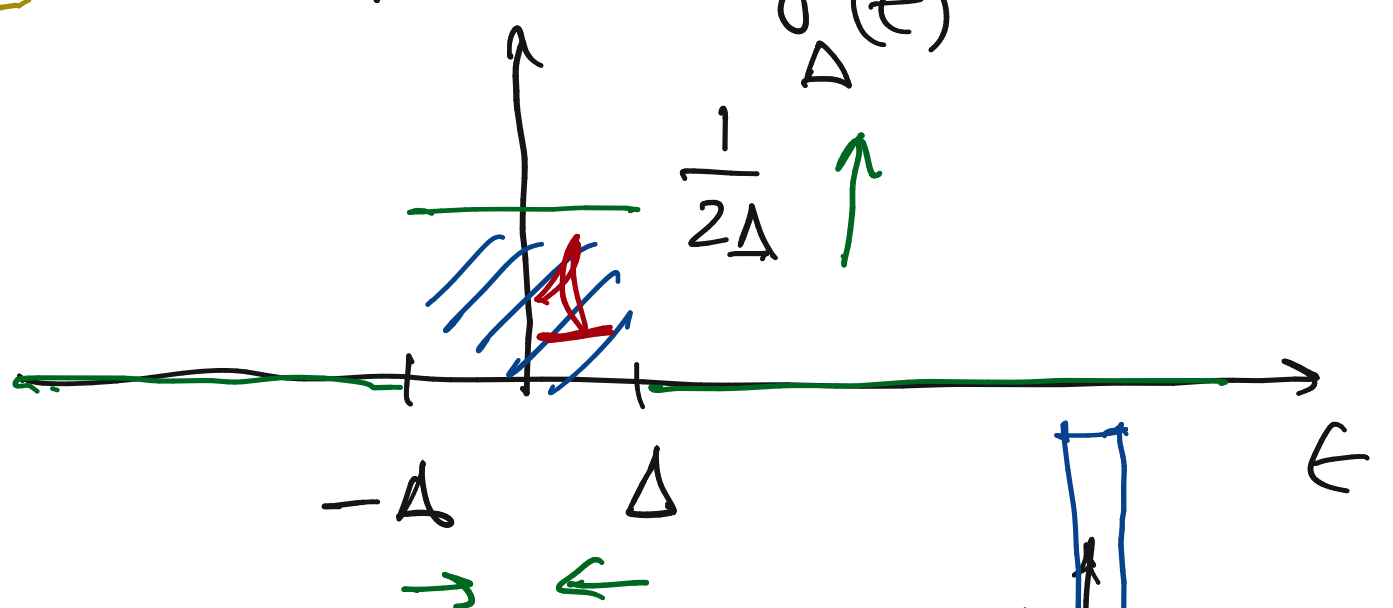
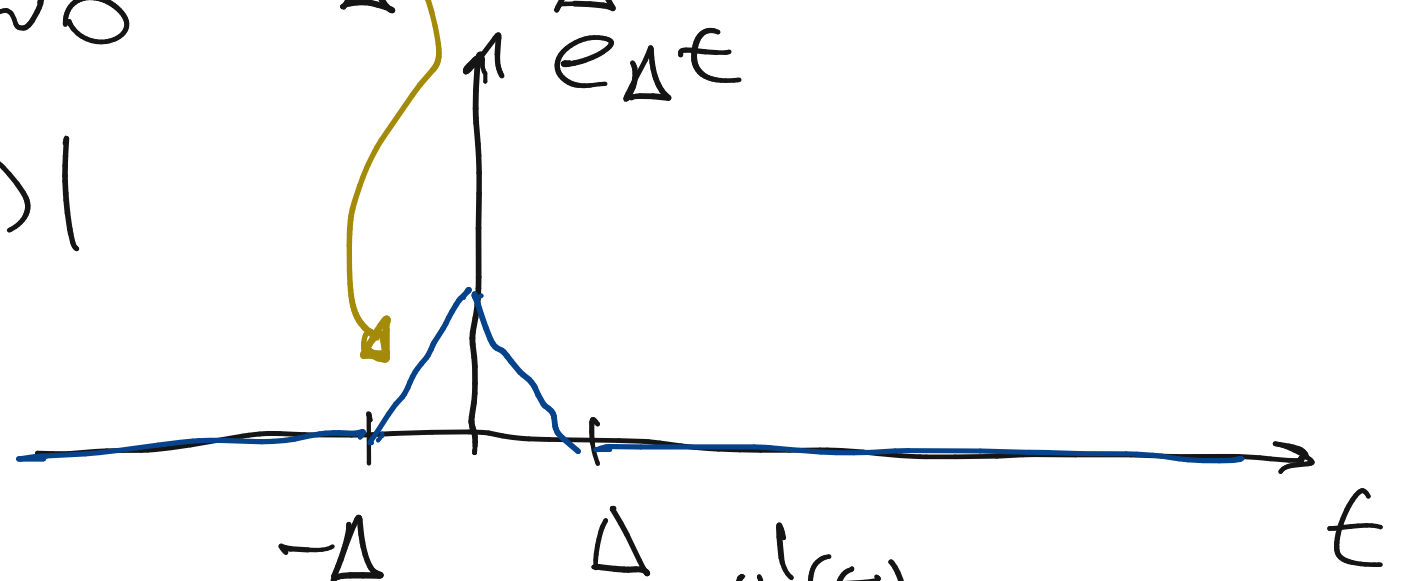
$$e_{\Delta}(t) = |U(t) - U_{\Delta}(t)|$$

$$e_{\Delta}(t) \rightarrow 0 \quad \Delta \rightarrow 0$$

cero en
casi todo
punto.

$$U(t) \notin C^0$$

$$U_{\Delta}(t) \in C^0$$



$$U'_{\Delta}(t) = \begin{cases} \frac{1}{2\Delta} & t \in [-\Delta, \Delta] \\ 0 & \text{e.o.c.} \end{cases}$$

* def: $\delta_{\Delta}(t) = U'_{\Delta}(t) \notin C^0$

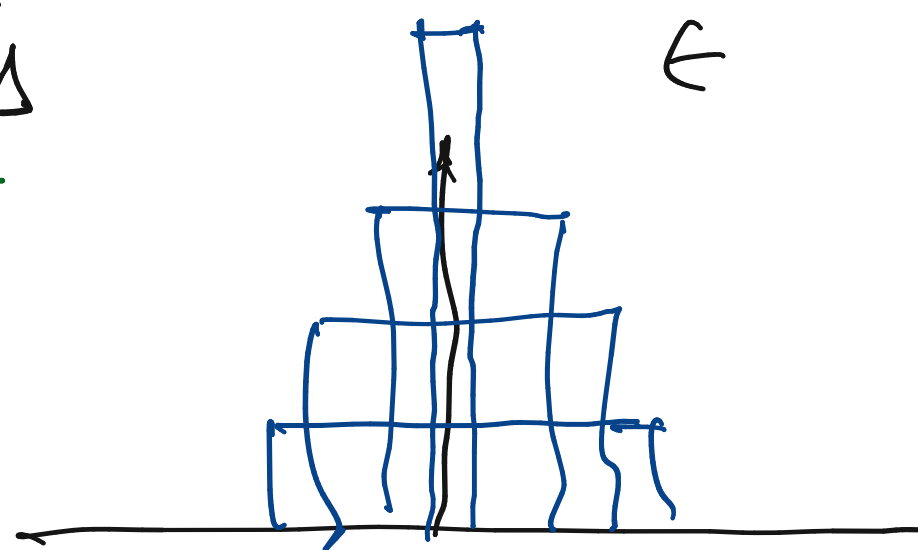
Def: "impulso unitario"

$$\delta(t) = \lim_{\Delta \rightarrow 0} \delta_{\Delta}(t)$$

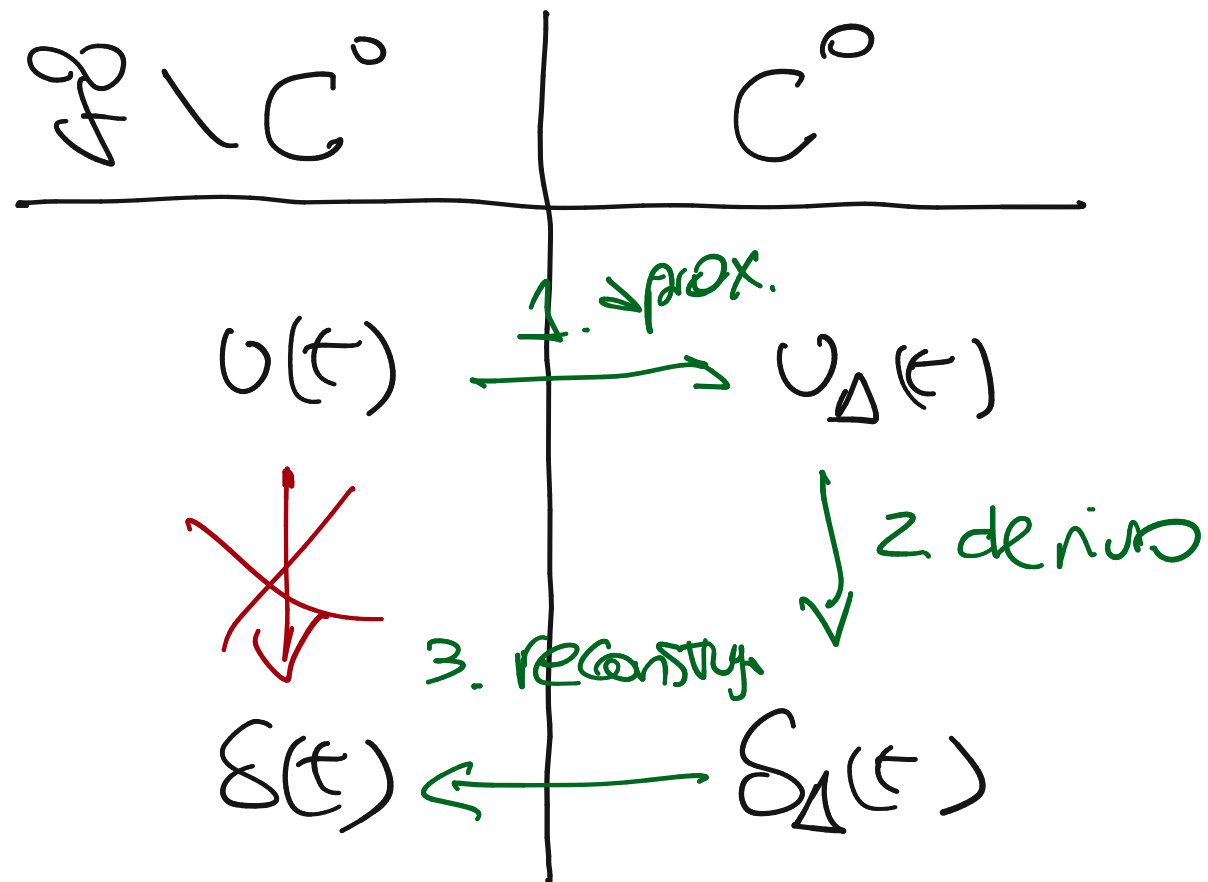
obs: * no tiene una fórmula explícita

$$* E_{\infty} = \int_{-\infty}^{\infty} |\delta(t)|^2 dt = \lim_{\Delta \rightarrow 0} \int_{-\infty}^{\infty} |\delta_{\Delta}(t)|^2 dt$$

$$E_{\infty} = 1$$



derivada generalizada



CLASE 4

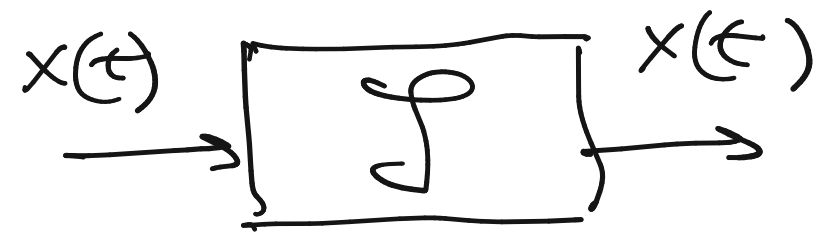
Sistemas

Modelado

SISTEMAS

Def: "sistema"

$$x, y \in (V, E)$$



* transformación: $\rightarrow x(t) \xrightarrow{f} y(t)$

$\rightarrow y(t) = f\{ \underbrace{x(t)} \}$

$\rightarrow f: V \rightarrow V$

Temario:

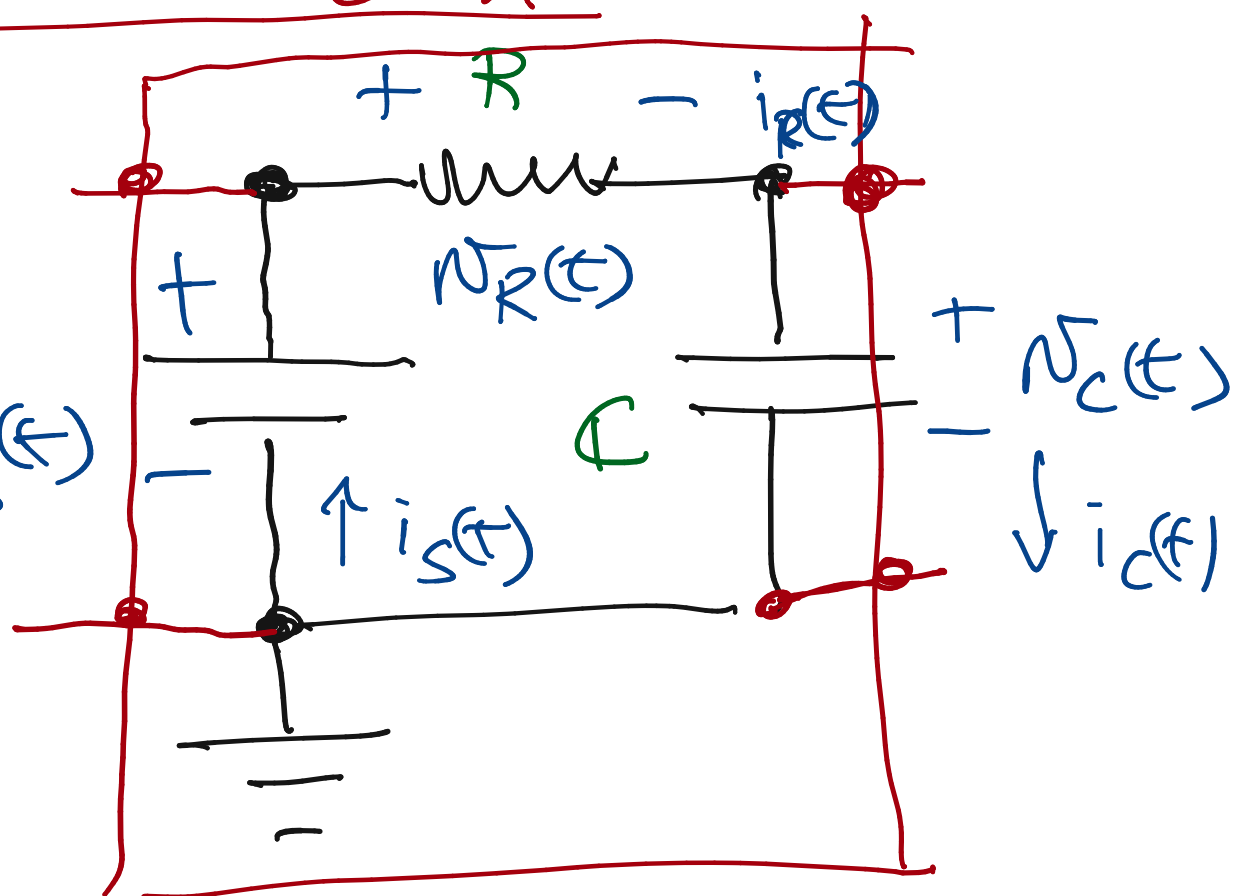
- * sistemas continuos y discretos
- $\rightarrow$ concepto sistema. Rep. matemática
- $\rightarrow$ Componentes de un sistema.
- $\rightarrow$ Ejemplos (biológicos, mecánicos, eléctricos)
- $\rightarrow$ Interconexión y escala.
- $\rightarrow$ Propiedades

EJEMPLOS DE SISTEMAS

sistema eléctrico

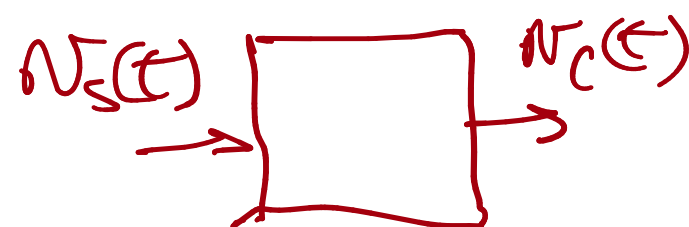
modelado:

- * identificar señales de interés
- * Componentes
- * leyes: $\rightarrow$ componentes
 $\rightarrow$ integrativas



señales $v_C, v_S, v_R, i_C, i_R, i_S$

- * entradas: $v_S(t)$
 - * salidas: $v_C(t)$
- } elección



Componentes

* resistencia: $v_R(t) = R \cdot i_R(t)$ ①

* Condensador: $i_C(t) = C \cdot \frac{d v_C(t)}{d t}$ ②

Leyes "leyes Kirchhoff"

* Mallas (conservación de energía): $v_S(t) - v_R(t) - v_C(t) = 0$ ③

* Nodos (conservación de carga): $i_S(t) = i_R(t) = i_C(t) = i(t)$ ④

Ley del sistema

③ $\rightarrow v_R(t) + v_C(t) - v_S(t) = 0 \xRightarrow{①} R \cdot i_R(t) + v_C(t) - v_S(t) = 0$

④ $\rightarrow R i(t) + v_C(t) - v_S(t) = 0 \xRightarrow{②} R \cdot C \cdot \frac{d v_C(t)}{d t} + v_C(t) - v_S(t) = 0$

Ley del sistema

$$R.C. \frac{dN_c(t)}{dt} + \underbrace{N_c(t)}_{\text{salida}} - \underbrace{N_s(t)}_{\text{entrada}} = 0$$

- * obs:
- Ecuación diferencial ordinaria
 - de primer orden
 - no homogénea
 - coeficientes constantes.

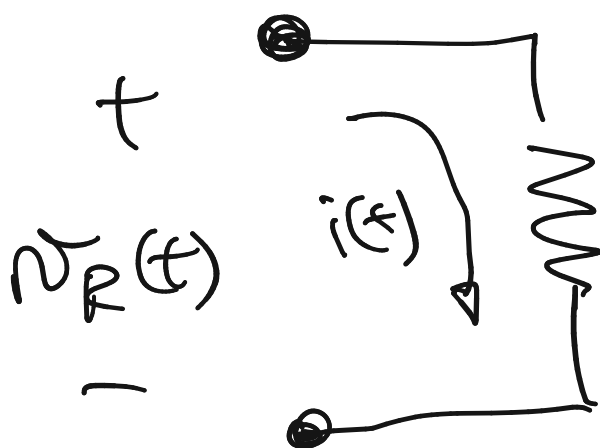
forma general

$$a \cdot y'(t) + b y(t) + c \cdot x(t) + d = 0.$$

- * obs:
- combinación lineal de la entrada y sus derivadas
 - idem para la salida

obs:

"escala"



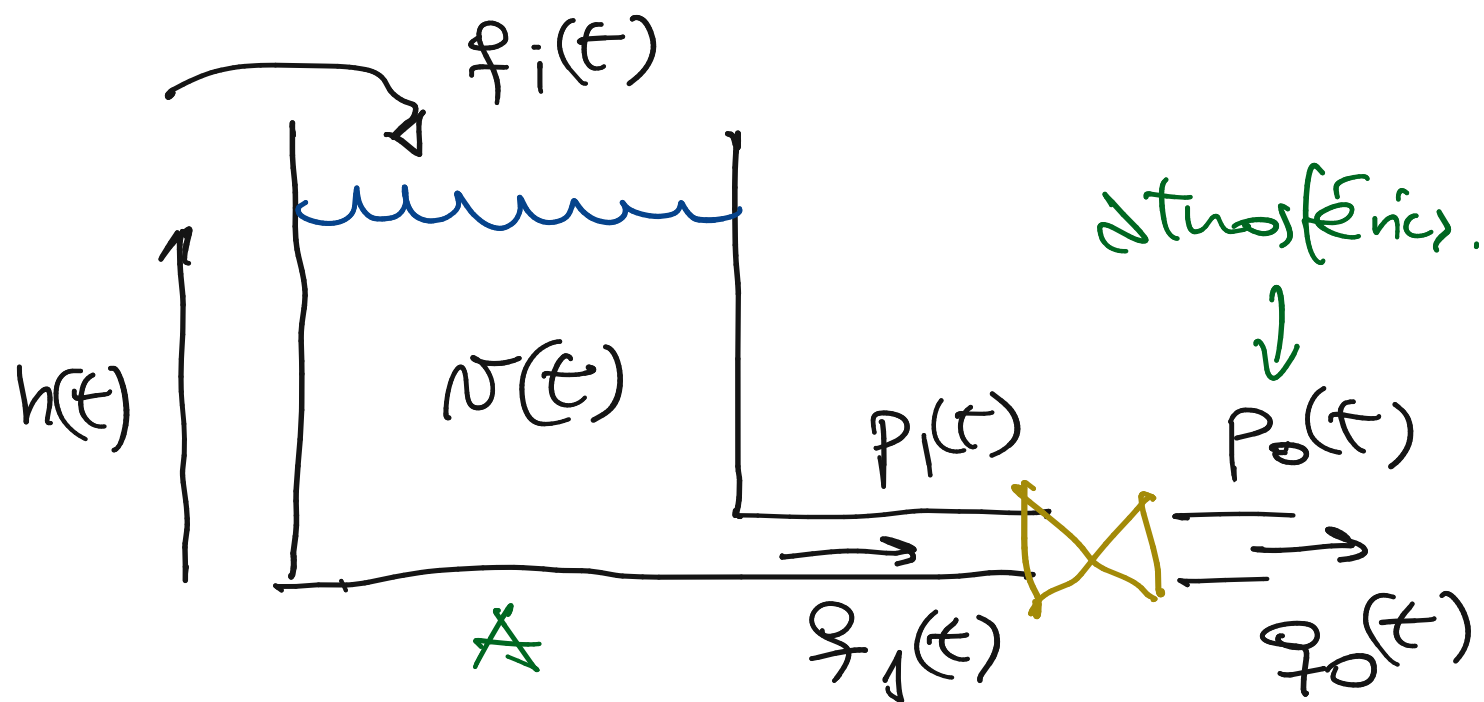
$$N_R(t) = R \cdot i_R(t)$$

Sistema Hidráulico

$q(t)$: caudal (l/s)

Señales:

$h, q_i, q_o, q_1, v(t)$



Componentes

* tanque: $\rightarrow A \cdot h(t) = v(t)$

$$\rightarrow p_1(t) = \frac{F}{A} = \frac{m \cdot g}{A} = \frac{\rho \cdot v(t) \cdot g}{A} \quad \text{①}$$

$$p_1(t) = \frac{\rho \cdot g \cdot A \cdot h(t)}{A}$$

$$\rightarrow q_i(t) - q_1(t) = \frac{d v(t)}{dt} \quad \text{②}$$

① y ② $\Rightarrow$

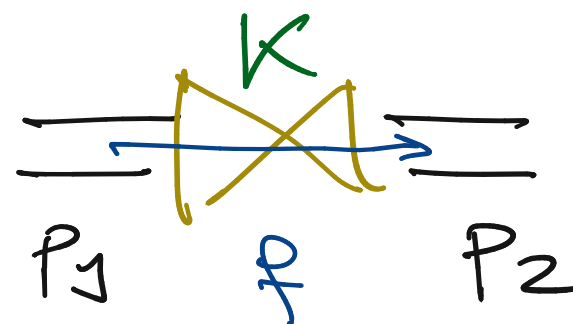
$$q_i(t) - q_1(t) = \frac{A}{\rho \cdot g} \cdot \frac{d p_1(t)}{dt}$$

* válvula:

$$p_2 = p_o = 0$$

$$p_1 = p_1(t)$$

$$q_1(t) = K \cdot p_1(t)$$



$$q = K \cdot (p_1 - p_2)$$

Ley del sistema

$$q_i(t) - K \cdot p_1(t) = \frac{A}{\rho \cdot g} \cdot \frac{d p_1(t)}{dt}$$

Ley del sistema

$$\underbrace{\frac{A}{s.g}}_C \cdot \underbrace{\frac{d p(t)}{d t}}_{\text{salida}} + k \cdot p(t) = \underbrace{q_i(t)}_{\text{entrada}}$$

* EDO, lineal, coef. ctes, primer orden, no homog.

Analogía

→ tanque: almacena energía potencial gravitatoria

→ válvula: freno o permite flujo.

$$C = \frac{A}{s.g}$$
$$R = \frac{1}{k}$$

$$C \cdot p'(t) + \frac{1}{R} \cdot p(t) = q_i(t)$$

$$RC \cdot p'(t) + p(t) = \underline{\underline{R \cdot q_i(t)}}$$

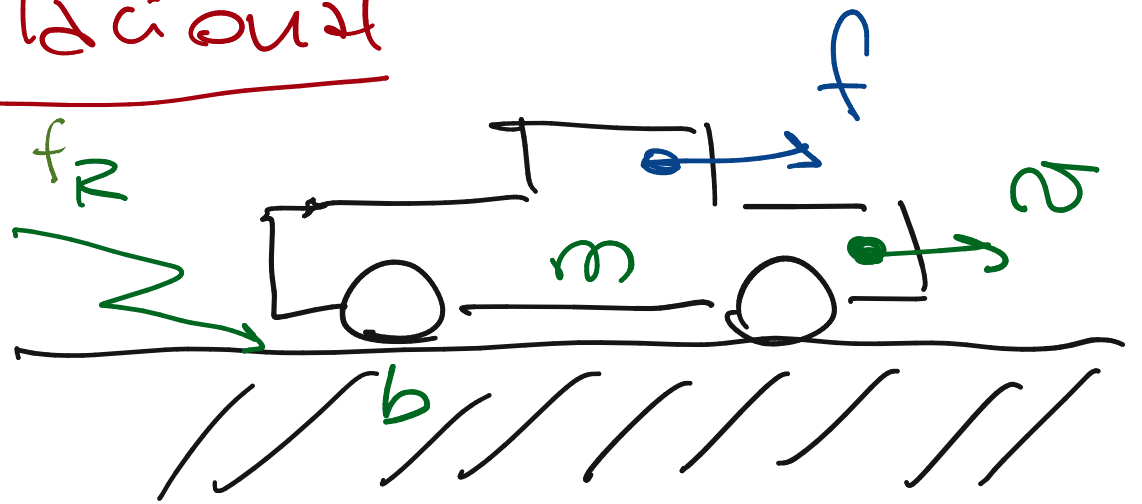
Circuito hidráulico

* señales: → presión (voltaje): potencial
→ flujo (corriente):

* componentes: → tanque: $q(t) = C \cdot p'(t)$ capacitor
→ válvula: $q(t) = \frac{1}{R} \cdot p(t)$ resistencia
→ caño ideal: $R=0$ cable ideal

Sistema mecánico traslacional

- * señales: → entrada: $f(t)$
→ salida: $v(t)$



* Componentes

- rozamiento dinámico (viscoso): $f_R(t) = b \cdot v(t)$

$$\begin{aligned} \text{prede ser } & x(t) \\ & a(t) = v'(t) \\ & v(t) = x'(t) \end{aligned}$$

elección!!

* Leyes:

- Newton: $\sum F = m \cdot a$

$$f(t) - f_R(t) = m \cdot a(t)$$

Ley del sistema

$$m \cdot \underline{v'(t)} + b \cdot \underline{v(t)} = \underline{f(t)}$$

EDO, lineal, coef. ctes, 1º orden, no H.

- * Analogía: → masa: almacena energía (inercia) cinética
→ rozamiento: disipa energía. R

Sistema masa-resorte

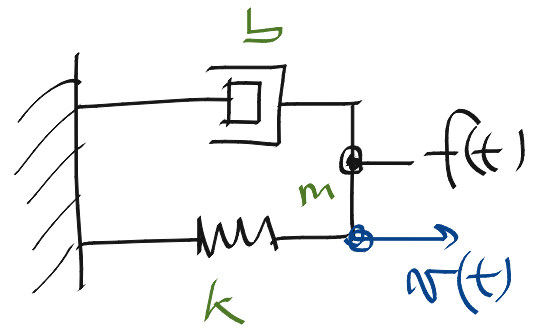
$$f_r(t) = b \cdot v(t)$$

$$f_e(t) = -k \cdot x(t)$$

$$f(t) - k \cdot x(t) - b \cdot v(t) = 0$$

$$b \cdot x'(t) + k \cdot x(t) = -f(t)$$

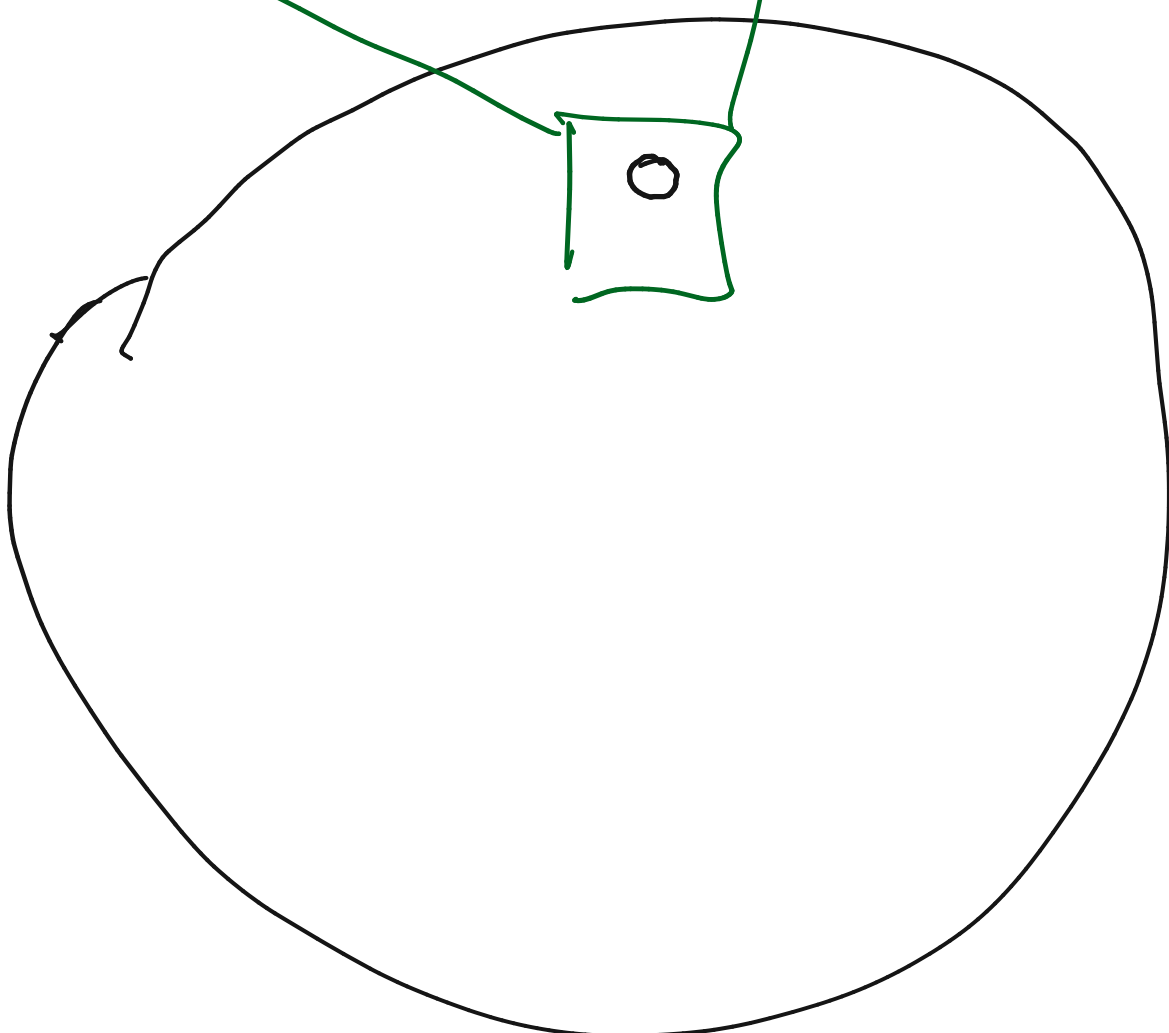
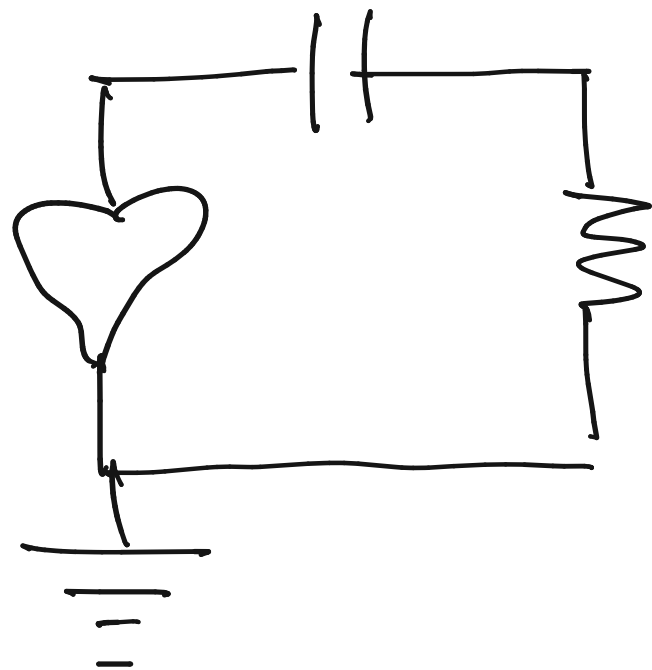
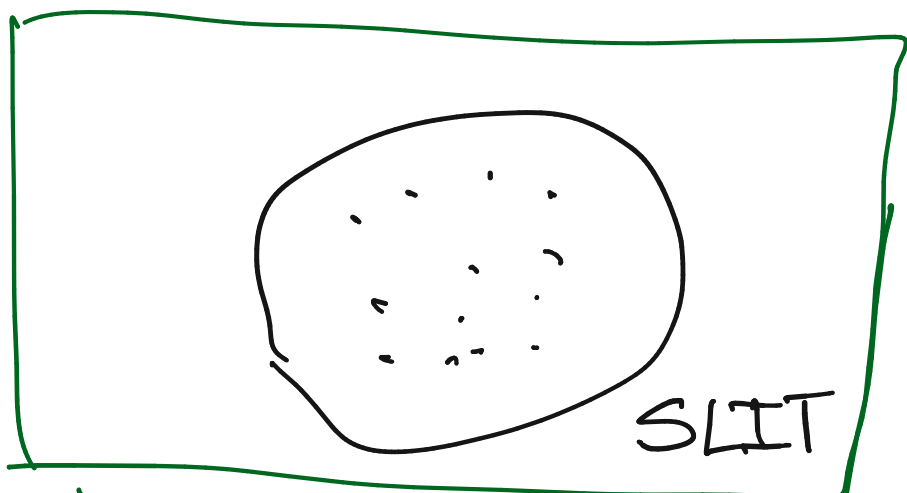
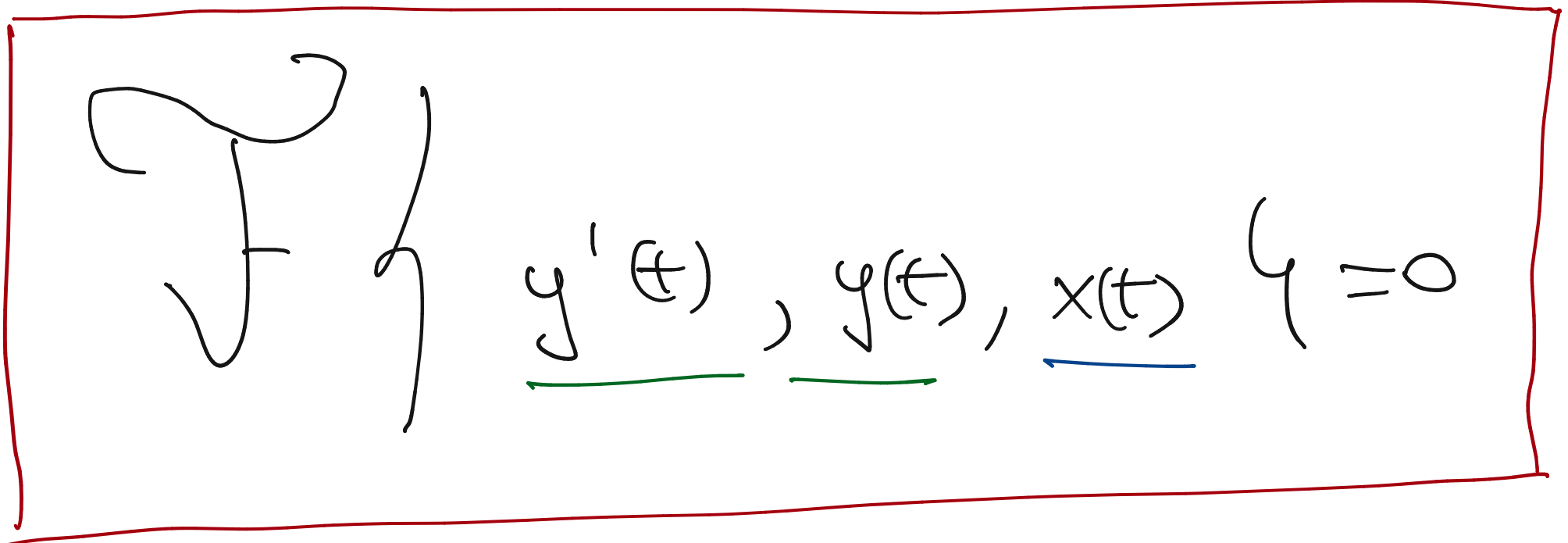
salidas



entrada

Resumen

- * Una gran clase de problemas se modela de la misma manera.
- * modelos biológicos
- * Fisiología C. y Modelos y Simulación



g

CLASE 5

Sistemas

Propiedades

① Propiedades

② Espacios de funciones

PROPIEDADES

Memoria (con o sin):

sin: "La salida solo depende de la entrada actual"

ej: $y(t) = 2 \cdot x(t)$
 $t=0$
 $x(0)=3$ $\Rightarrow 6 = 2 \cdot 3$

ej: "resistencia", $R = 100 \Omega$

$$i(t) = 0.01 v(t)$$

ej: "media móvil" (promediador)

$$y[n] = \frac{x[n] + x[n-1] + x[n-2]}{3}$$

→ pasado

ej: "capacitor"

$y(t)$: voltaje

$x(t)$: corriente

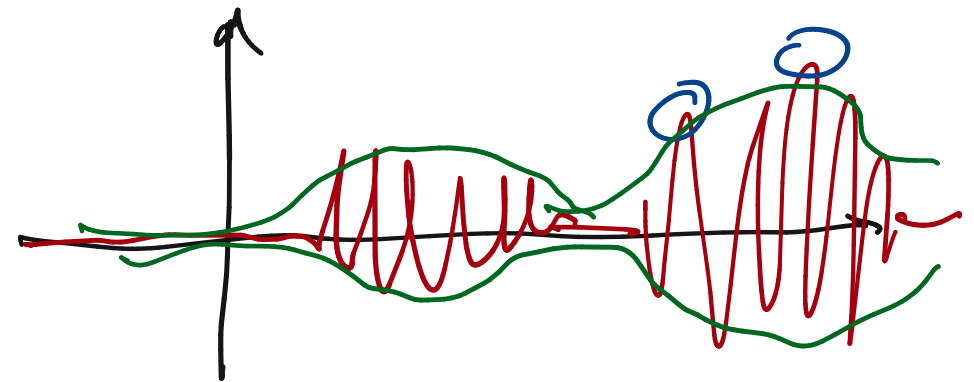
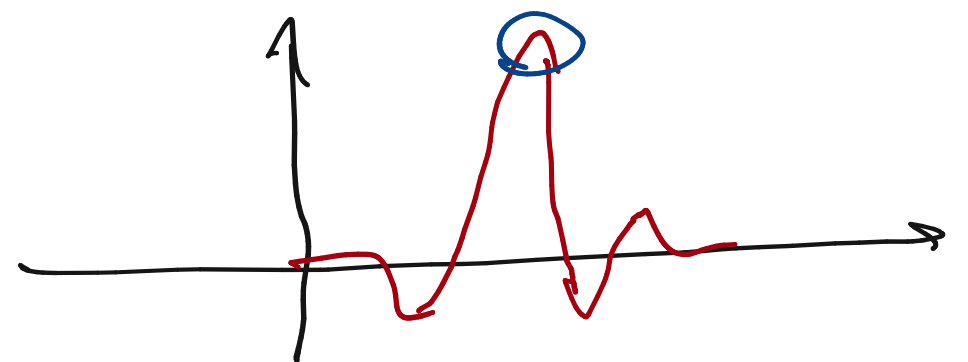
$$y(t) = \int_{-\infty}^t x(\tau) \cdot d\tau$$

↓
acumula carga

Determinismo

- * si la entrada es fija, la salida es fija
- * por oposición a los sistemas aleatorios
- * hay sistemas o señales que son aleatorios (ruleta)
- * hay sistemas o señales que son muy complejos y difícil de modelar exactamente.

- EMG
- sonidos atonales
- clima



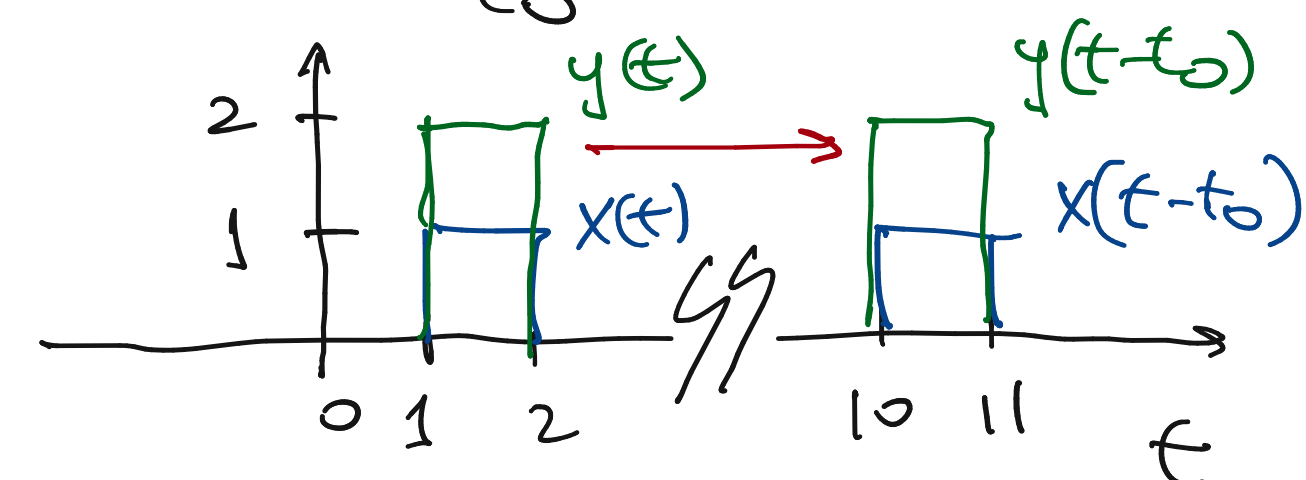
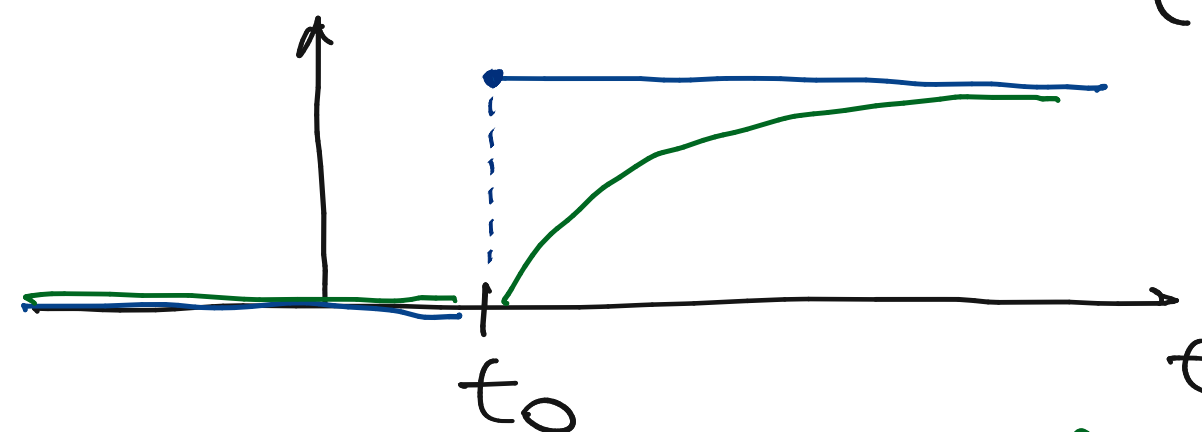
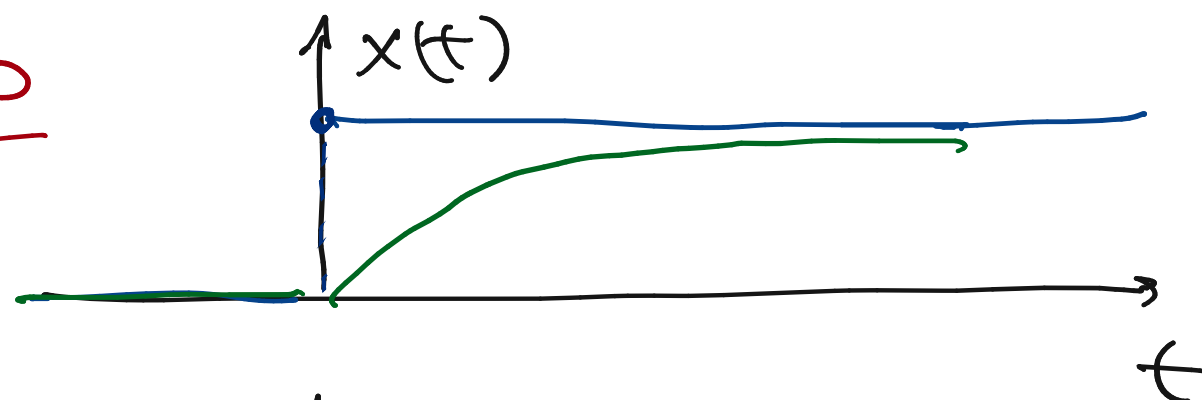
Estructura / cantidad

- * cantidad de entradas y salidas
 - 1 entrada y 1 salida

Invarianza en el tiempo

$$x(t) \xrightarrow{f} y(t)$$

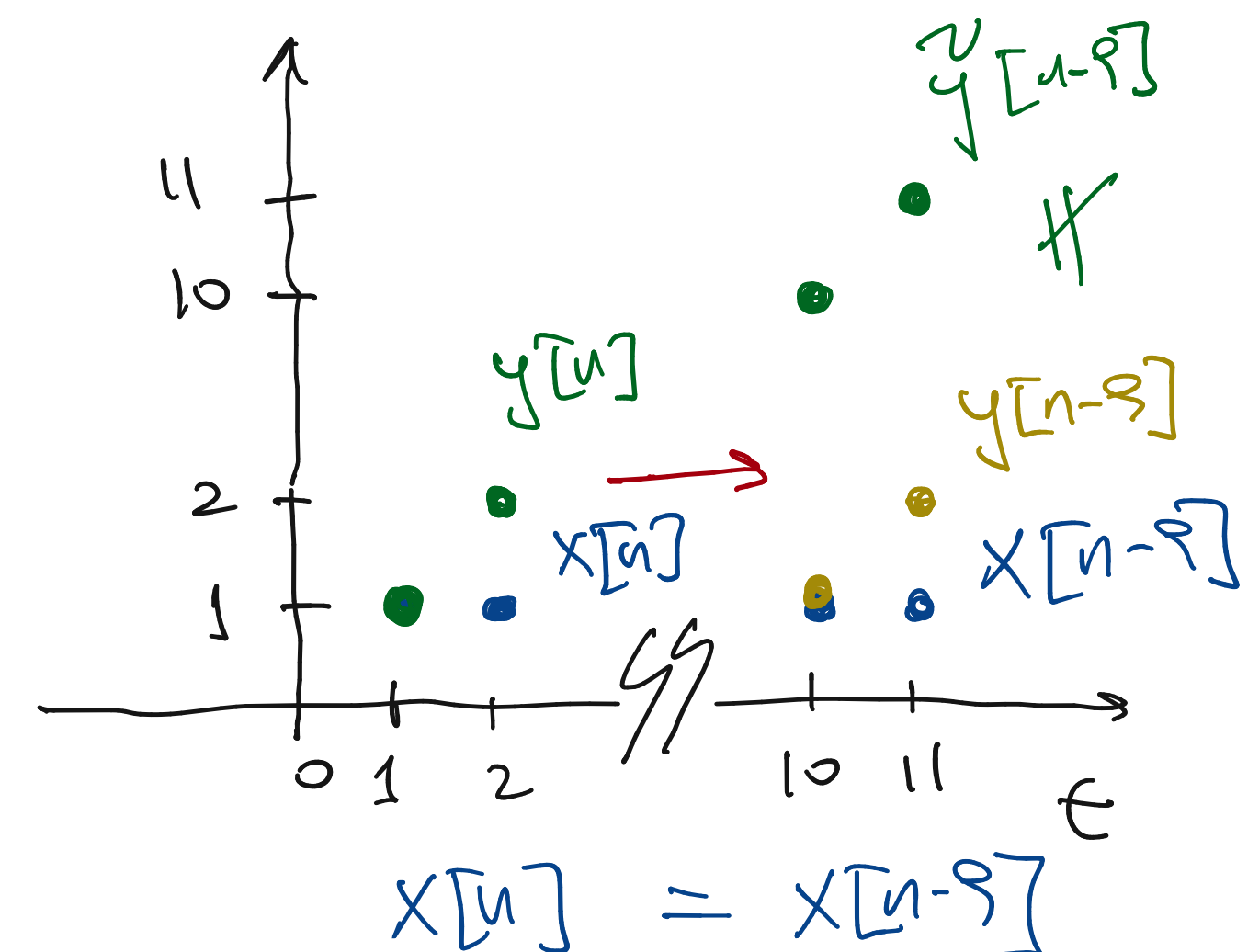
$$x(t-t_0) \xrightarrow{f} ? y(t-t_0)$$



ej: $y(t) = 2 \cdot x(t)$ ✓

$x(t-9), t_0 = 9$

ej: $y[n] = n \cdot x[n]$ ✗



$$x[n] = x[n-9]$$

Invertibilidad

ej:

$$y(t) = z \cdot x(t)$$



$$y(t) = \int \underline{x(t)} dt$$

$$x(t) = \frac{1}{z} \cdot y(t)$$



ej:

"derivador discreto"

$$y[n] = x[n] - x[n-1]$$

$$\longleftrightarrow x[n] = \sum_{k=-\infty}^n y[k]$$



ej:

*

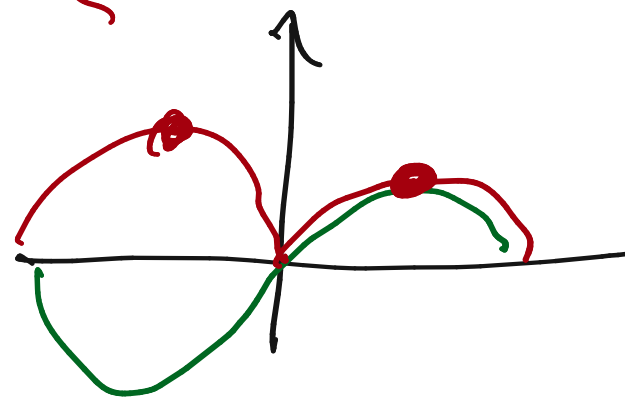
$$y[n] = 0$$

X

*

$$y[n] = x^2[n]$$

~~"inyectividad"~~



Causalidad

la salida en un momento dado depende solo del pasado

ej:

*

$$y[n] = x[\underline{n}] - x[\underline{n-1}]$$



*

$$y[n] = x[\underline{n+1}] - x[n]$$

X

obs:

→

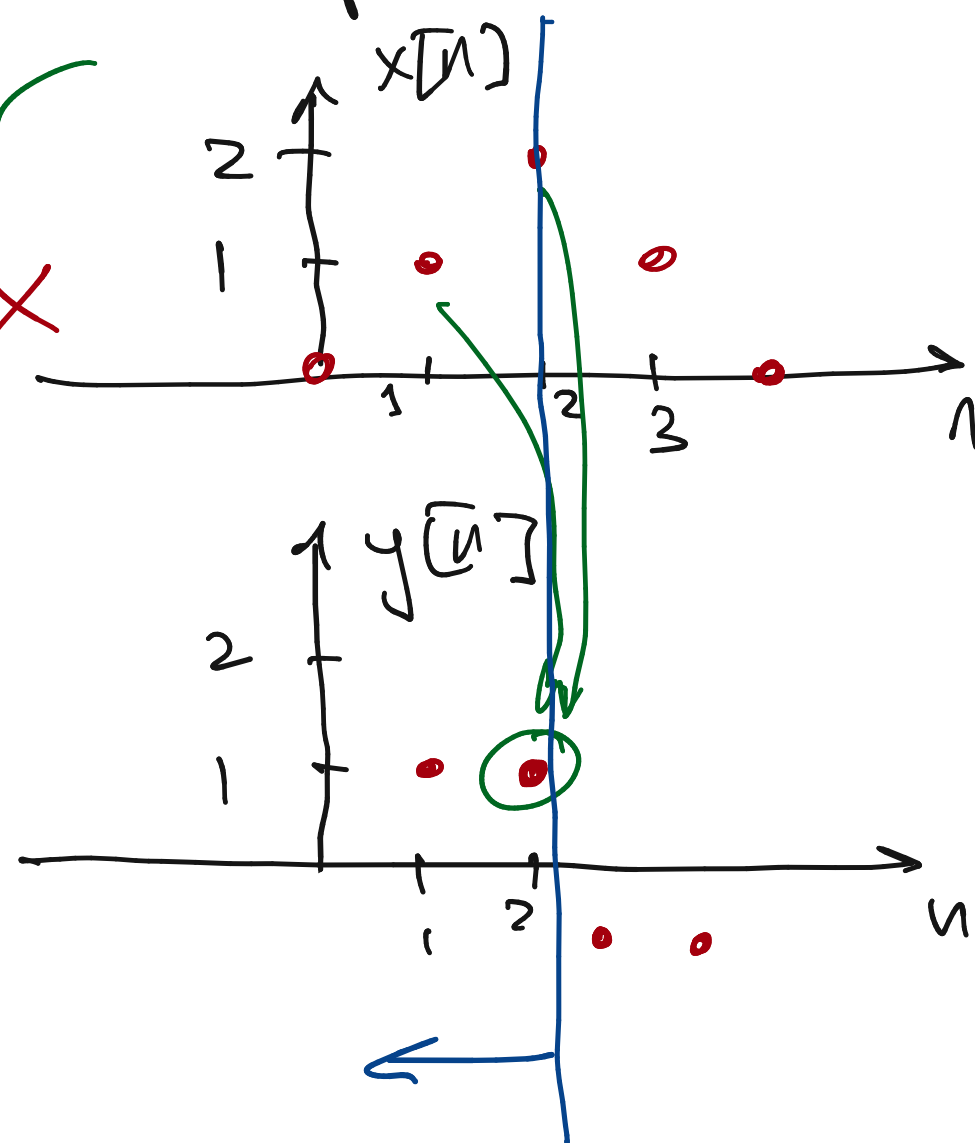
señal (tiempo real)



→

señal grabado

X



Linealidad

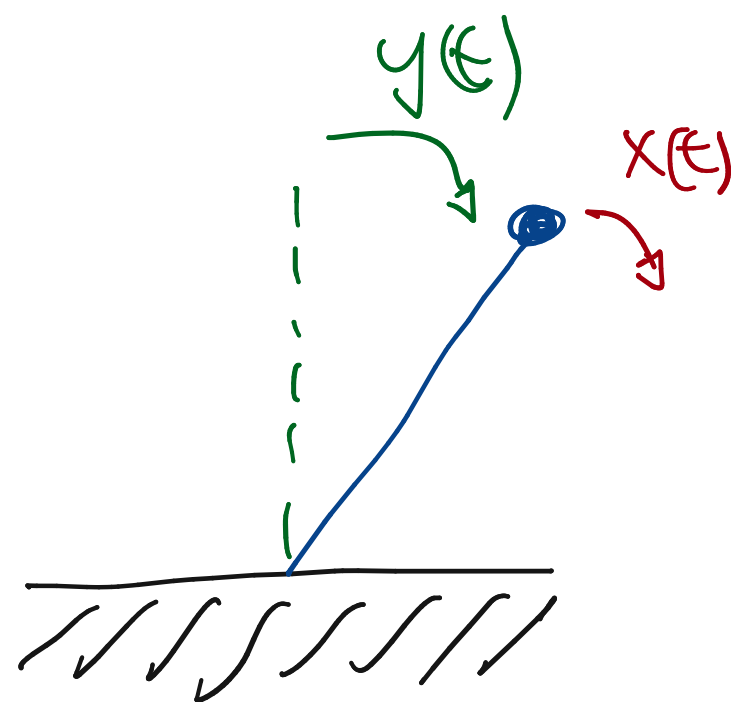
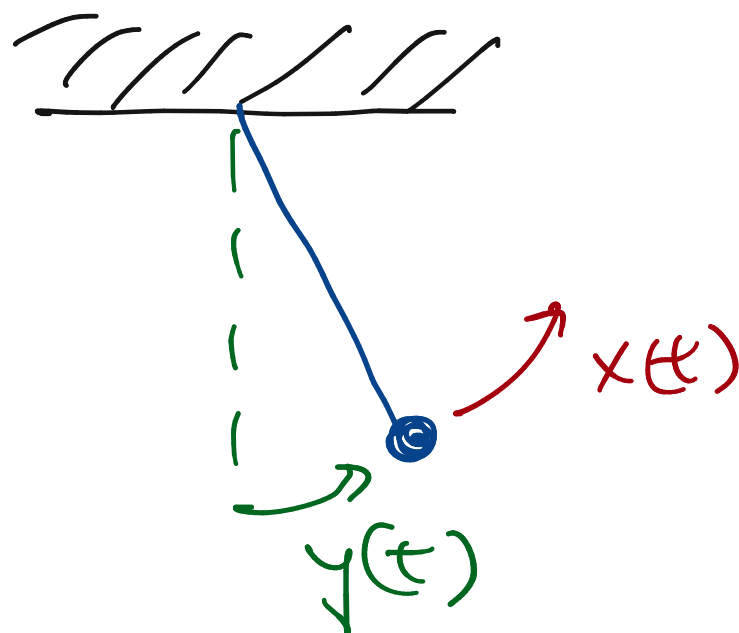
$$\text{sean } \begin{array}{ccc} x_1(t) & \xrightarrow{f} & y_1(t) \\ x_2(t) & \xrightarrow{f} & y_2(t) \end{array}$$

$$a_1 \cdot x_1(t) + a_2 \cdot x_2(t) \xrightarrow{f} ? \quad a_1 \cdot y_1(t) + a_2 \cdot y_2(t)$$

$$f \{ a_1 x_1(t) + a_2 x_2(t) \} = a_1 \cdot f \{ x_1(t) \} + a_2 \cdot f \{ x_2(t) \}$$

Estabilidad

→ entrada pequeña
no diverge



def: → BIBO estable (EASA)

→ entrada acotada (Bounded Input)

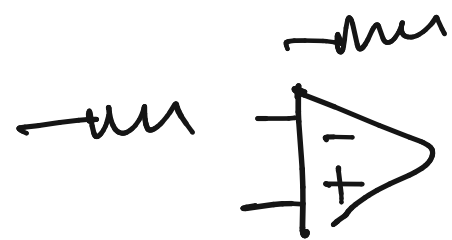
salida acotada (Bounded Output)

ej: "trailer"

ver en youtube

Resumen

$$y(t) = 2 \cdot x(t)$$



Sea $x_3(t) = x_1(t) + x_2(t)$

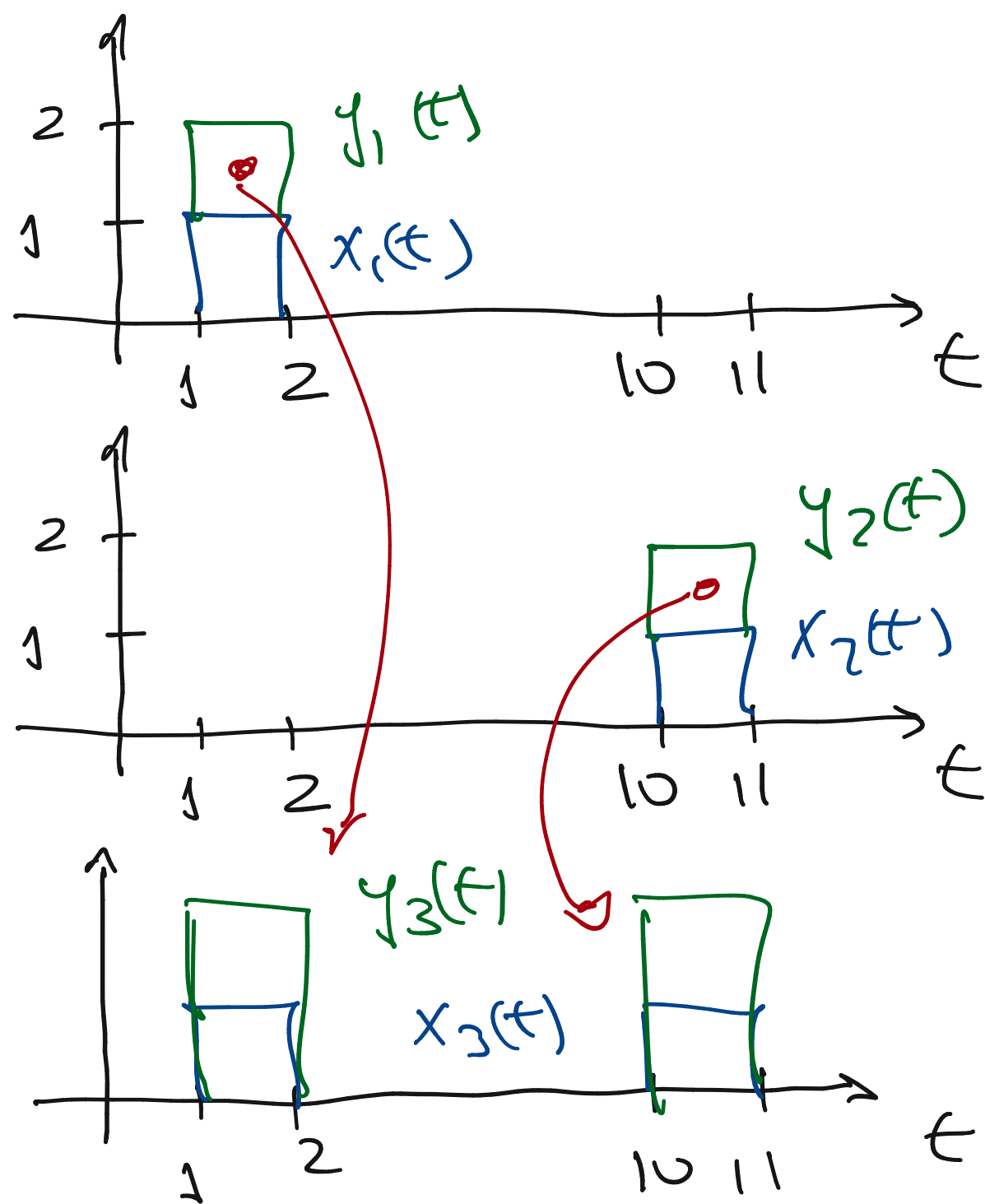
$$y_3(t) = 2 \cdot x_3(t) = 2 \cdot (x_1(t) + x_2(t))$$

$$= \underbrace{2 x_1(t)}_{y_1(t)} + \underbrace{2 x_2(t)}_{y_2(t)} = y_1(t) + y_2(t)$$

→ Lineal

→ Ahora cuentas

→ Descomposición
y
superposición



ESPACIOS DE FUNCIONES

Def:

$$(V, \in, +, < >)$$

$$V: C^\infty(\mathbb{R} \rightarrow \mathbb{R})$$

$$E: \mathbb{R}$$

funciones reales de una variable

obs: dimensión infinita!!

Operations

* producto por un escalar:

$$\forall \lambda \in E, \quad (\lambda \cdot f)(t) = \lambda \cdot f(t)$$

* producto interno $\langle f(t), g(t) \rangle = \int_D f(t) \cdot g^*(t) \cdot dt$

* norm: $\|f(t)\|^2 = \langle f(t), f(t) \rangle = \int_D |f(t)|^2 \cdot dt$
↪ energy

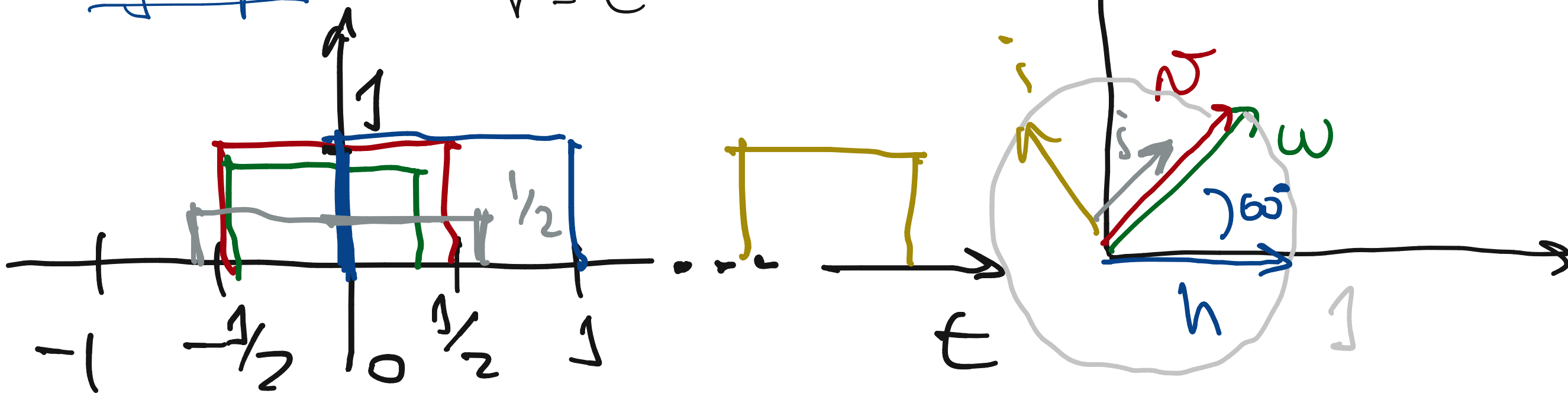
obs: * general: $D = (-\infty, \infty)$

* periódica: $D = (-\pi, \pi)$

Ejemplo

$$V = C^0$$

$$V = \mathbb{R}^2$$



$$1) \langle f(t), g(t) \rangle = \int_{-\infty}^{\infty} f(t) g(t) dt = 1 \quad 1) \langle \sigma, \omega \rangle = 1$$

$$2) \langle f(t), h(t) \rangle = \int_{-\infty}^{\infty} f(t) h(t) dt = \frac{1}{2} \quad 2) \langle \sigma, h \rangle = \frac{1}{2}$$

$$3) \langle f(t), i(t) \rangle = 0$$

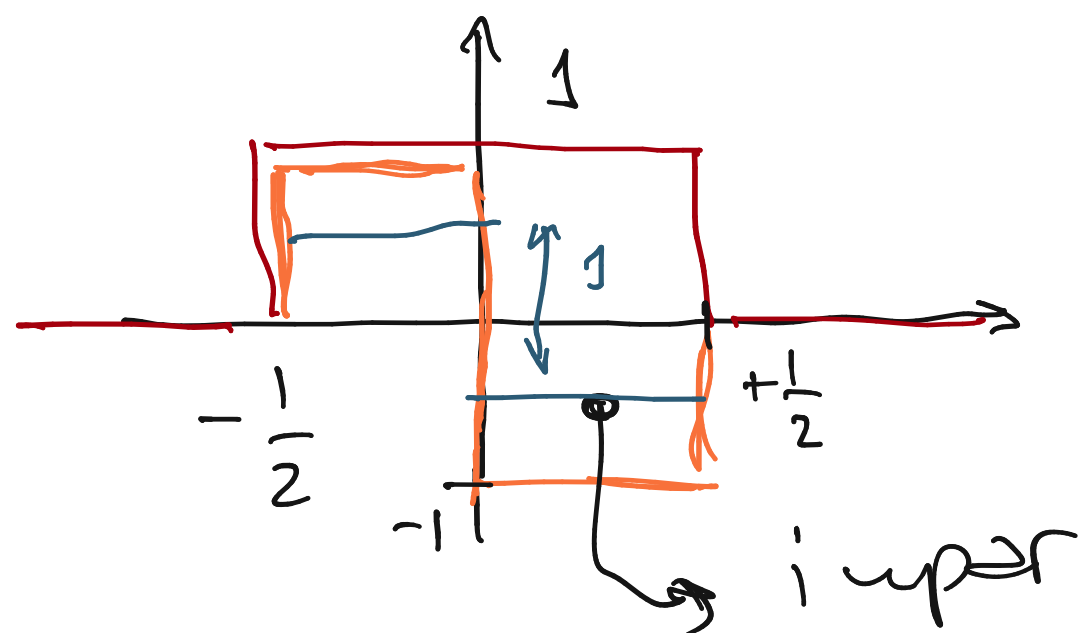
$$3) \langle \sigma, i \rangle = 0$$

$$4) \langle f(t), j(t) \rangle = \frac{1}{2}$$

$$4) \langle \sigma, j \rangle = 0$$

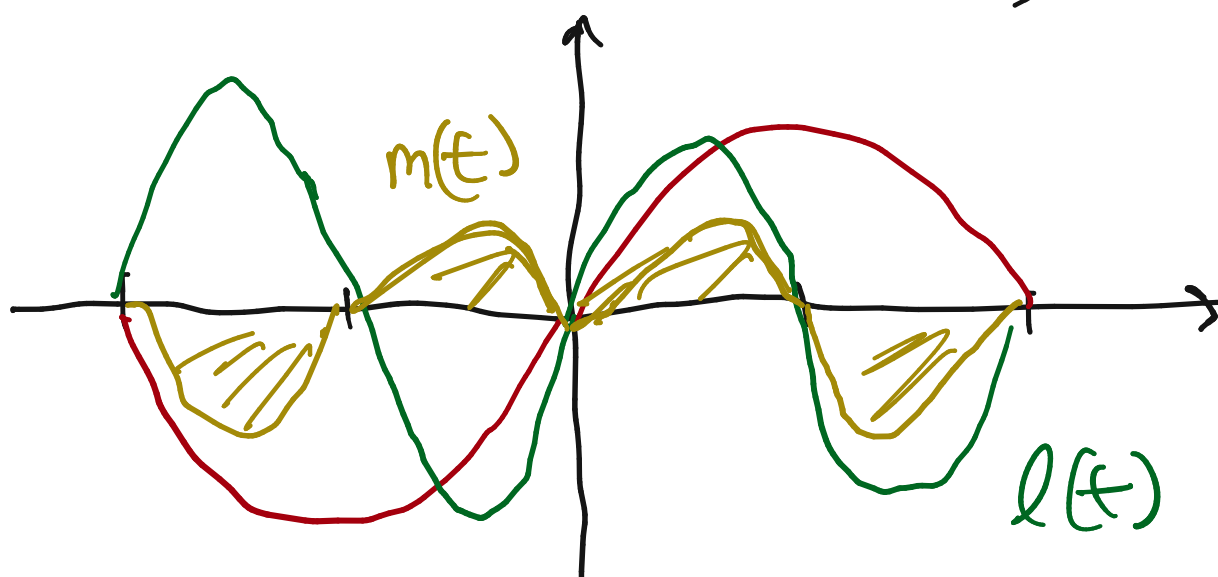
$$5) \langle f(t), k(t) \rangle = 0$$

diferente velocidad.



$$6) \langle f(t), l(t) \rangle = 0$$

$$\langle f(t), m(t) \rangle = 0$$



Exponenciales como base de C^∞

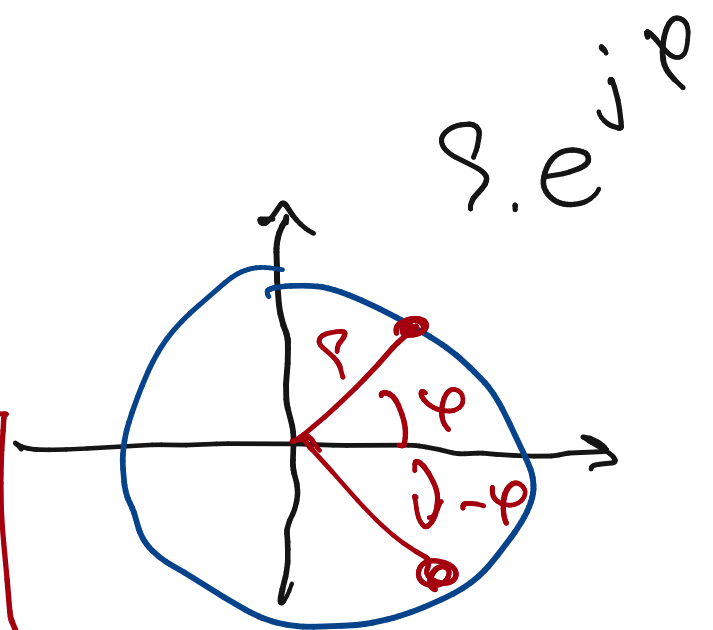
* ¿Cuánto mide un exponencial? $x(t) = e^{j2t}$

$$\rightarrow \|e^{j2t}\|^2 = \langle e^{j2t}, e^{j2t} \rangle$$

$$D = (-\pi, \pi)$$

$$\stackrel{D}{=} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j2t} \cdot e^{-j2t} \cdot dt$$

$$= \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} 1 \cdot dt = \frac{1}{2\pi} \cdot 2\pi = 1$$



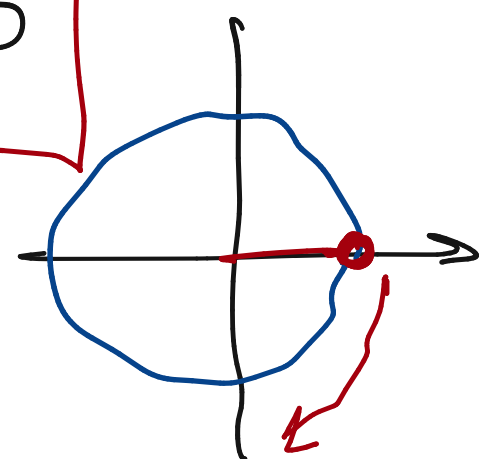
$e^{j\omega_0 t}$ velocidad oscilación

* $\left. \begin{array}{l} x_1(t) = e^{j2\pi t} \\ x_2(t) = e^{j4\pi t} \end{array} \right\} \begin{array}{l} = \text{amplitud} \\ \neq \text{velocidad (doble)} \end{array}$

$$\langle e^{j2t}, e^{j4t} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j2t} \cdot e^{-j4t} \cdot dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-j2t} \cdot dt = \frac{1}{2\pi} \cdot \frac{e^{-j2t}}{-j2} \Big|_{-\pi}^{\pi}$$

$$= -\frac{1}{4\pi j} \cdot \left(\underbrace{e^{-j2 \cdot \pi}}_{=1} - \underbrace{e^{-j2 \cdot (-\pi)}}_{=1} \right) = 0$$



Base de C^∞

* el conjunto $B = \{ e_1, \dots, e_k, \dots \}$
exponenciales es una base ortogonal de C^∞

* $\langle e_i, e_i \rangle = 1$

* $\langle e_i, e_j \rangle = 0 \quad i \neq j$

CLASE 6

Sistemas

SLIT discretos

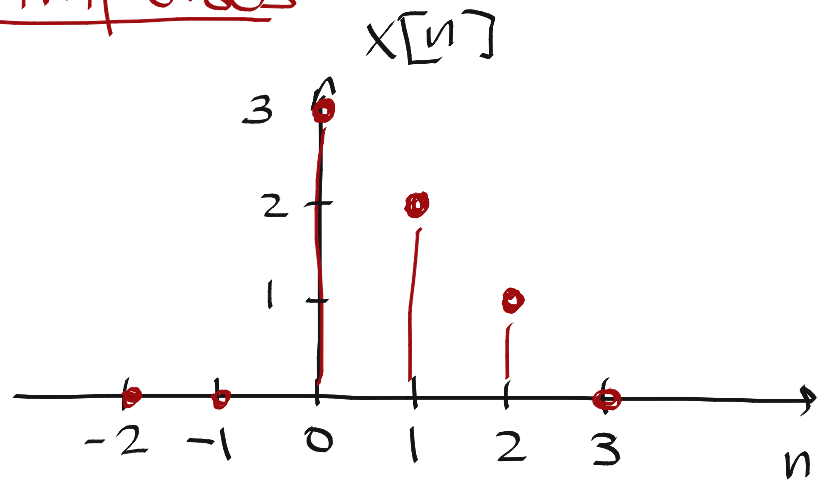
Descomposición en impulsos

① $x_1[n] = 3 \cdot \delta[n]$

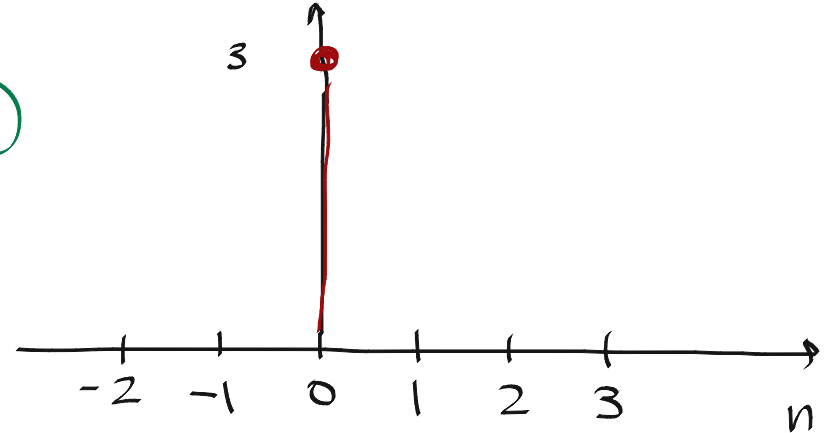
+ ② $x_2[n] = 2 \cdot \delta[n-1]$

③ $x_3[n] = 1 \cdot \delta[n-2]$ =

$x[n] = 3 \cdot \delta[n] + 2 \cdot \delta[n-1] + \delta[n-2]$

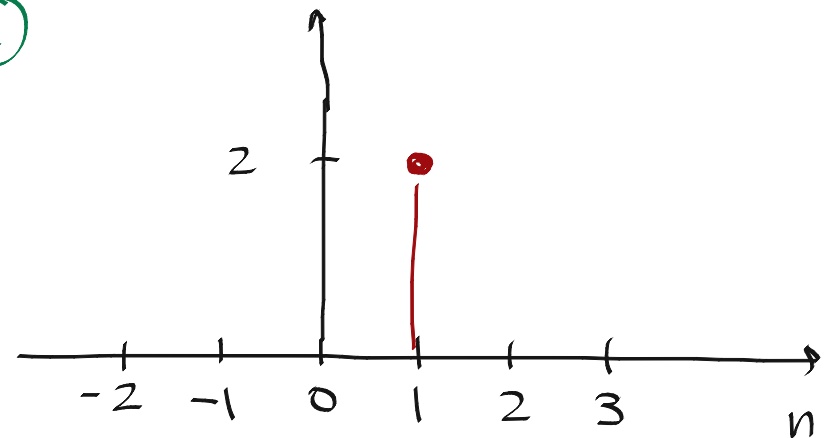


①

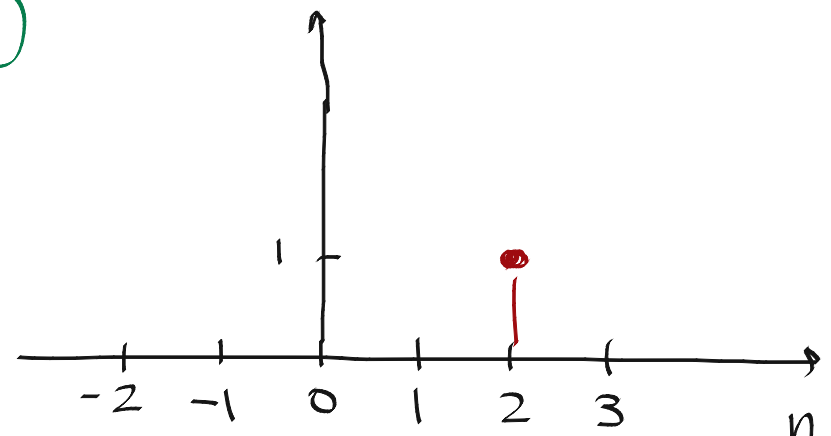


+

②



③



Estabilidad:

Dada $|x[n]| < B_x < \infty$

$$\textcircled{H} \text{ S estable} \Leftrightarrow \textcircled{I} \quad S = \sum_{k=-\infty}^{\infty} |x[k]| < \infty$$

Directo

$$\text{Supongo que } \sum_{k=-\infty}^{\infty} |x[k]| = \infty$$

voy a demostrar que $\exists x[n]$, $|x[n]| < B_x < \infty$ / $\sum |x[k]| = \infty$

$$\text{sea } x[n] = \begin{cases} \frac{h^*[-n]}{|h[n]|} & h[n] \neq 0 \\ 0 & h[n] = 0 \end{cases}$$

$$|x[n]| = \begin{cases} \frac{|h^*[-n]|}{|h[-n]|} = 1 \\ 0 \end{cases} \Rightarrow \boxed{|x[n]| \leq 1}$$

$$y[n] = \sum_{k=-\infty}^{\infty} \frac{h^*[k]}{|h[k]|} \cdot h[n-k] \Rightarrow y[0] = \sum_{k=-\infty}^{\infty} \frac{h^*[-k] \cdot h[k]}{|h[k]|}$$

$$y[0] = \sum_{k=-\infty}^{\infty} |h[k]| = \infty \text{ Abs!!}$$

CLASE 7

Sistemas

SLIT discretos ejemplos

EJEMPLOS DE SISTEMAS

Representaciones

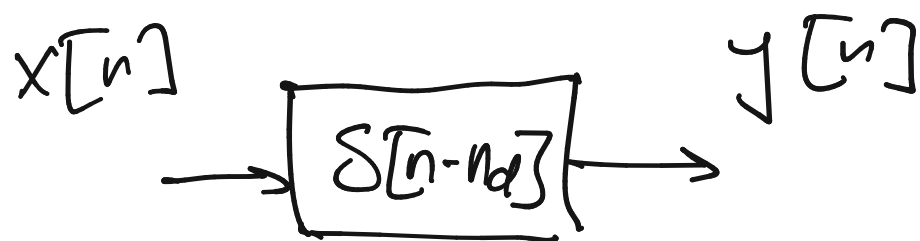
* ec. transformación: $y[n] = \mathcal{F}\{x[n]\}$

* ec. diferenciales: $\rightarrow \mathcal{F}\{y'(t), y(t), x(t)\} = 0$

$\rightarrow \mathcal{F}\{y[n+1], y[n], x[n]\} = 0$

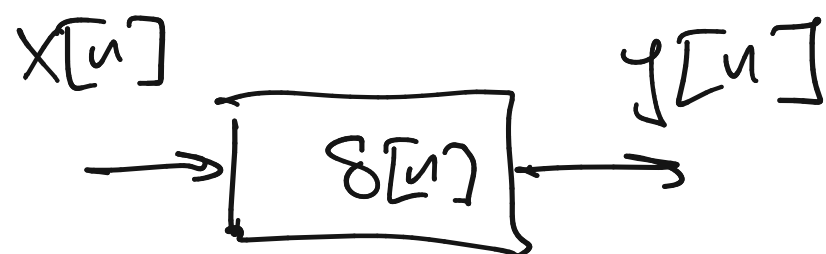
* resp. impulso: $h[n] = \dots$

Diagrama bloques



Retardo ideal

obs: caso $n_d = 0$



obs:

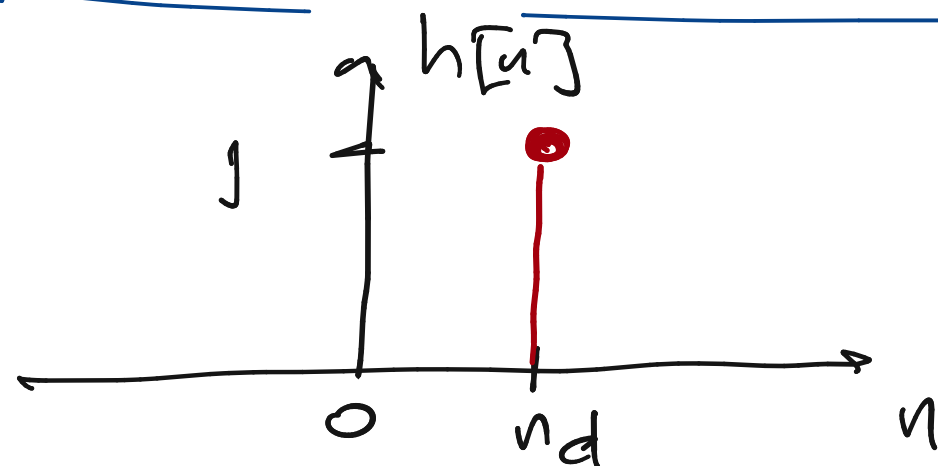
$$y[n] = x[n - n_d]$$

$$y[n] = \delta[n - n_d] * x[n]$$

$\Rightarrow$

$$\delta[n - n_d] * x[n] = x[n - n_d]$$

Representación gráfica de $h[n]$



$\delta[n]$

$$y[n] = h[n] * x[n]$$

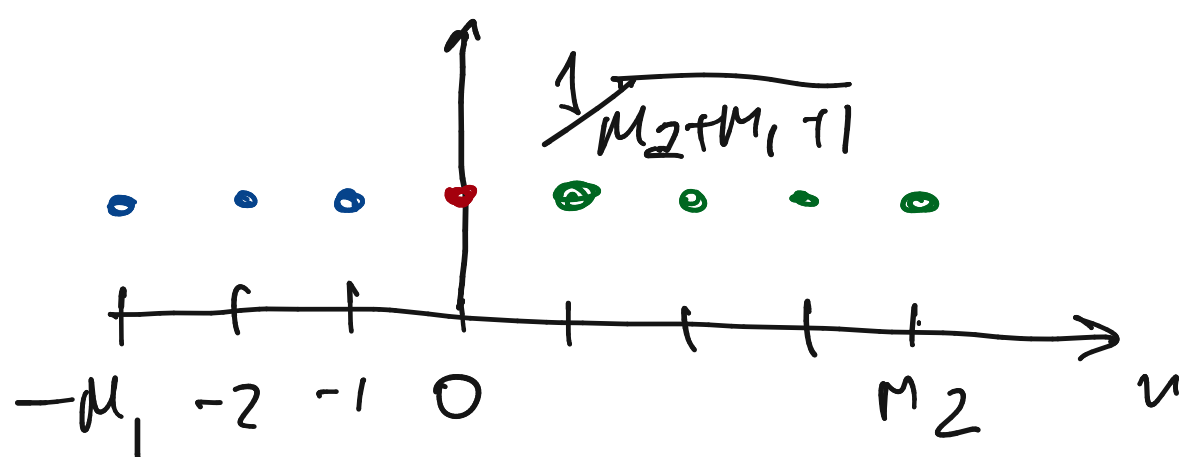
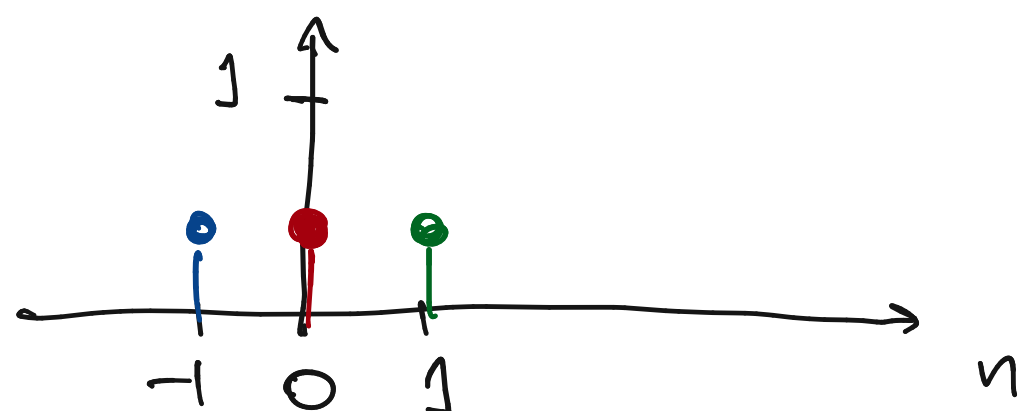
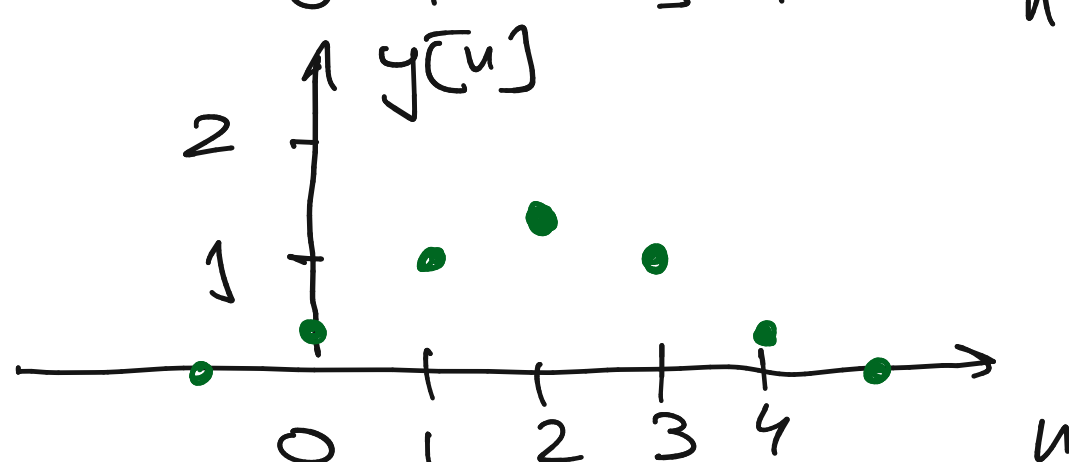
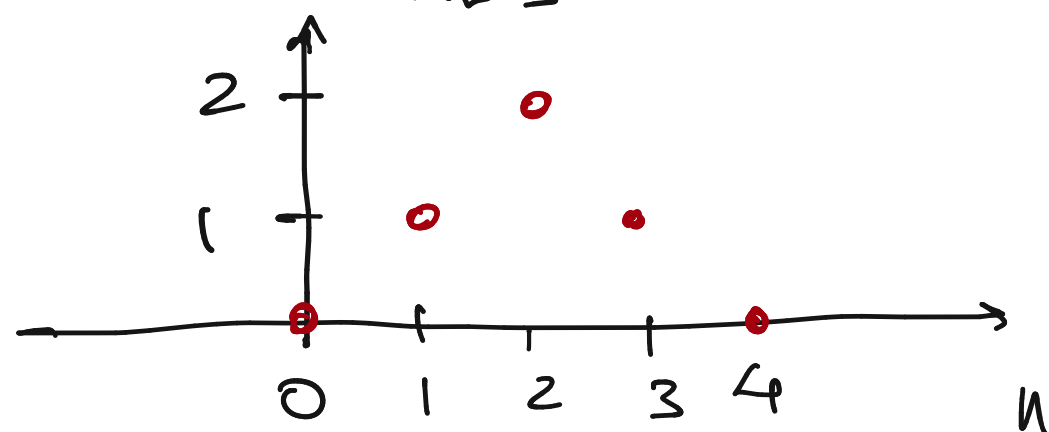
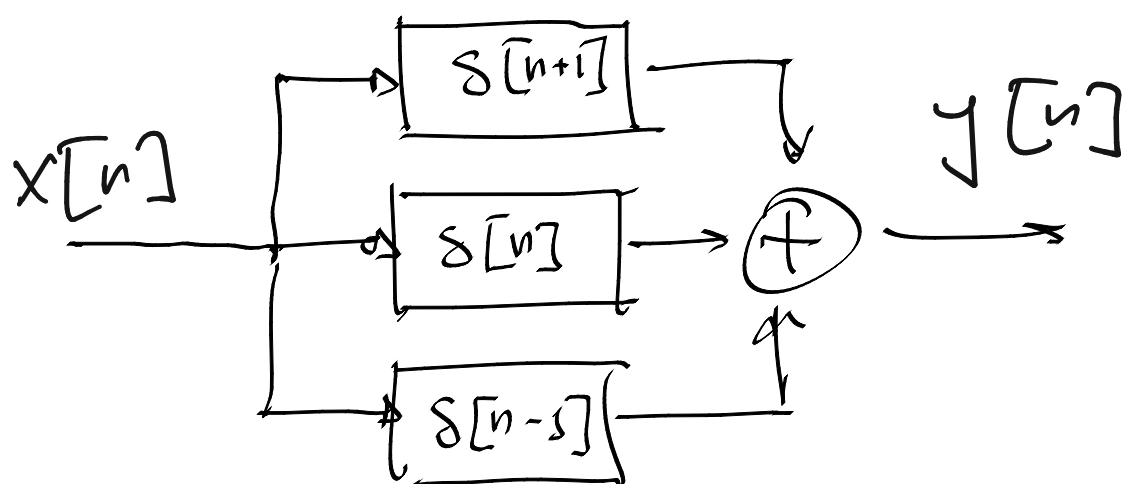
$$y[n] = x[n] \quad \text{"identidad"}$$

Media móvil

$$y[n] = \frac{x[n+1] + x[n] + x[n-1]}{3}, \quad M_1 = M_2 = 1$$

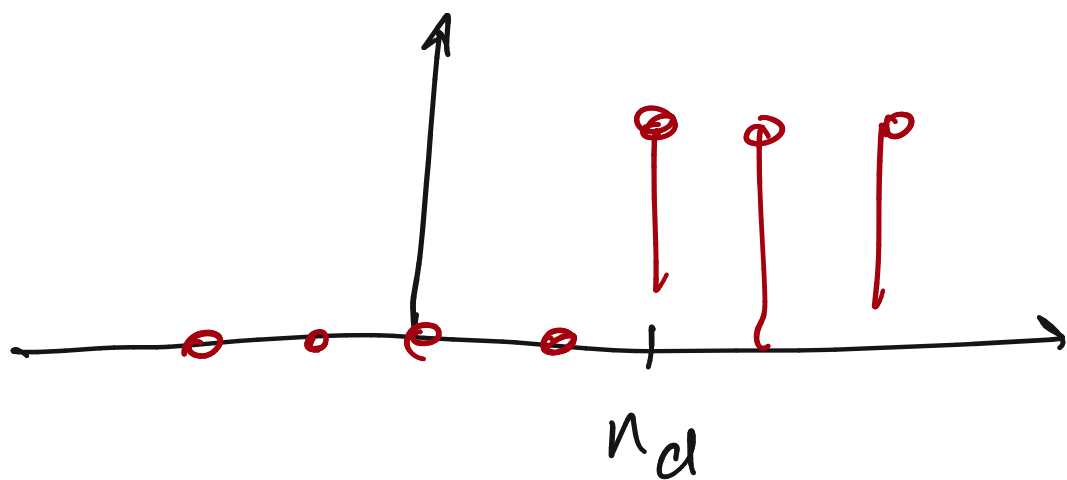
$$x[n] = s[n] \Rightarrow h[n] = \underbrace{\frac{1}{3} \cdot s[n+1]}_{\text{futuro}} + \underbrace{\frac{1}{3} \cdot s[n]}_{\text{presente}} + \underbrace{\frac{1}{3} \cdot s[n-1]}_{\text{pasado}}$$

Diagrama bloques



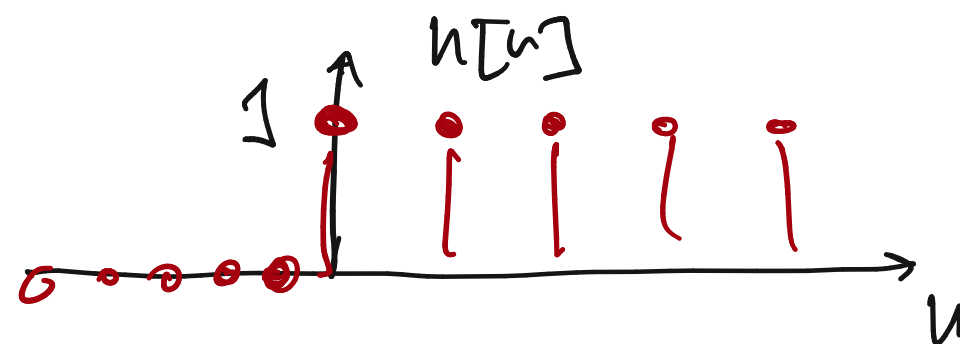
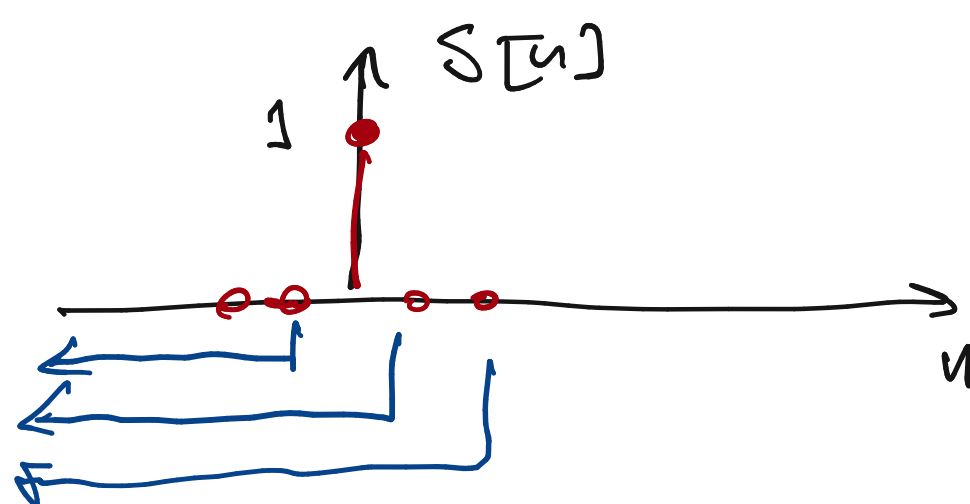
Acumulador

$$h[n - n_d]$$

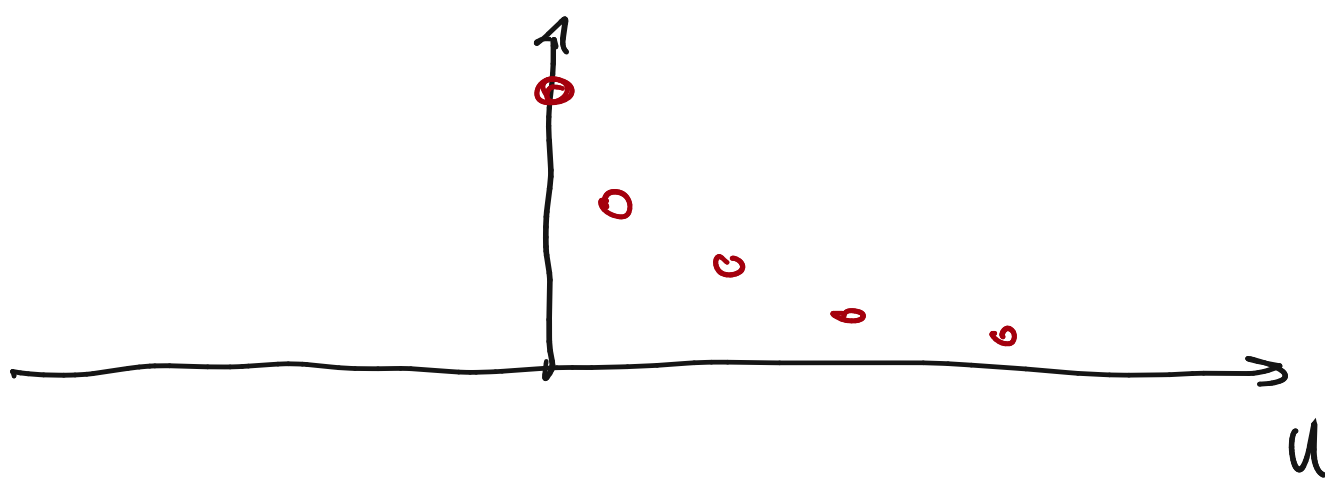


$$n_d > 0$$

causal



obs:



Para ser estable

$$h[n] \rightarrow 0$$

$$n \rightarrow \infty$$

$$y[n] = \sum_{k=-\infty}^{\infty} \underline{h[n-k]} \cdot x[k]$$

Ejemplo IIR

* ec. diferencias: $\rightarrow$ 1^{er} orden ($y[n-1]$)
cantidad de retardos no
la salida
 $\rightarrow$ coefs. cts. (2)

$$h[n] = \begin{cases} a^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

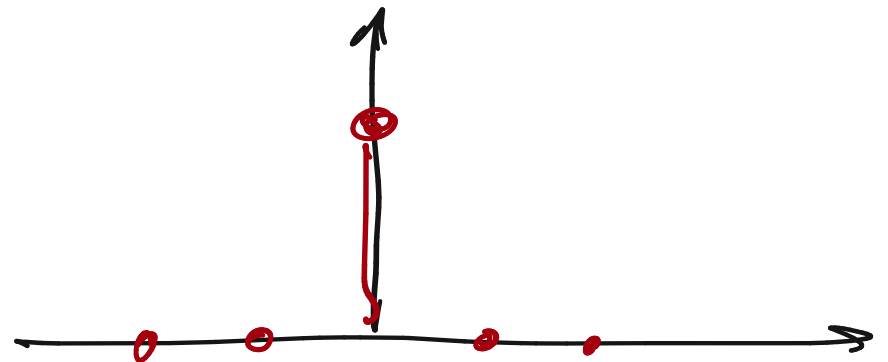
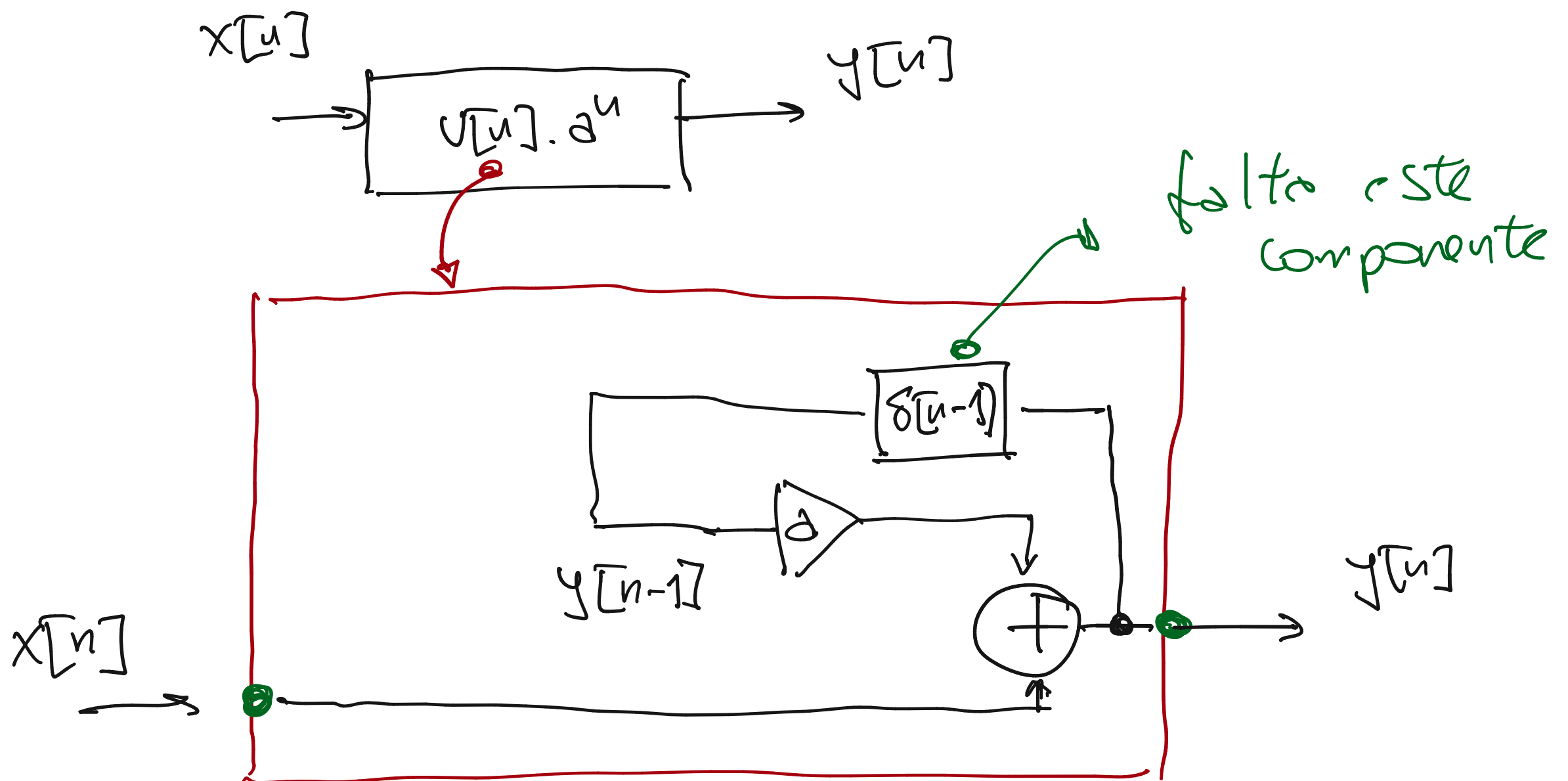
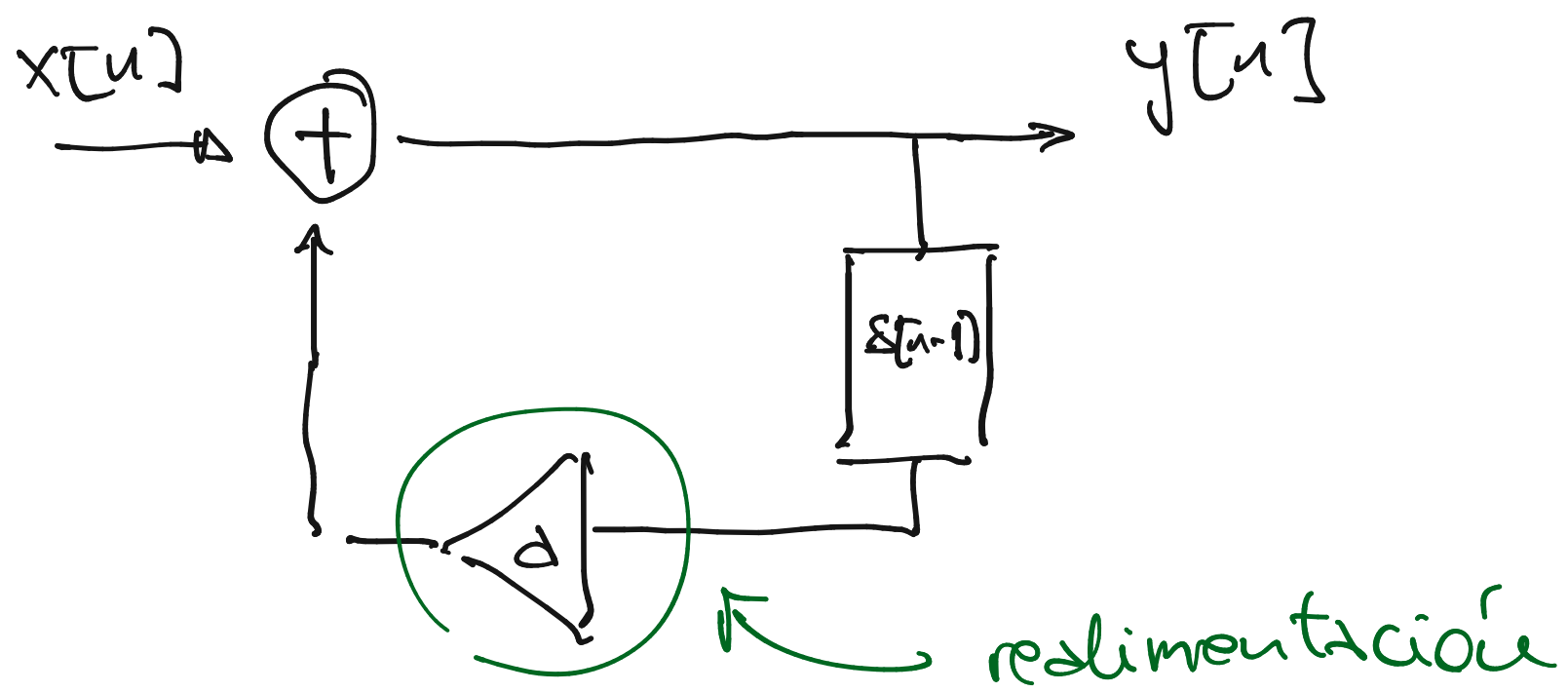


Diagrama de bloques





- Obs: *
- * sistema realimentado
 - * los elementos del diagrama se "parecen" a componentes de un circuito

SISTEMAS DEFINIDOS POR EC. EN DIFERENCIAS

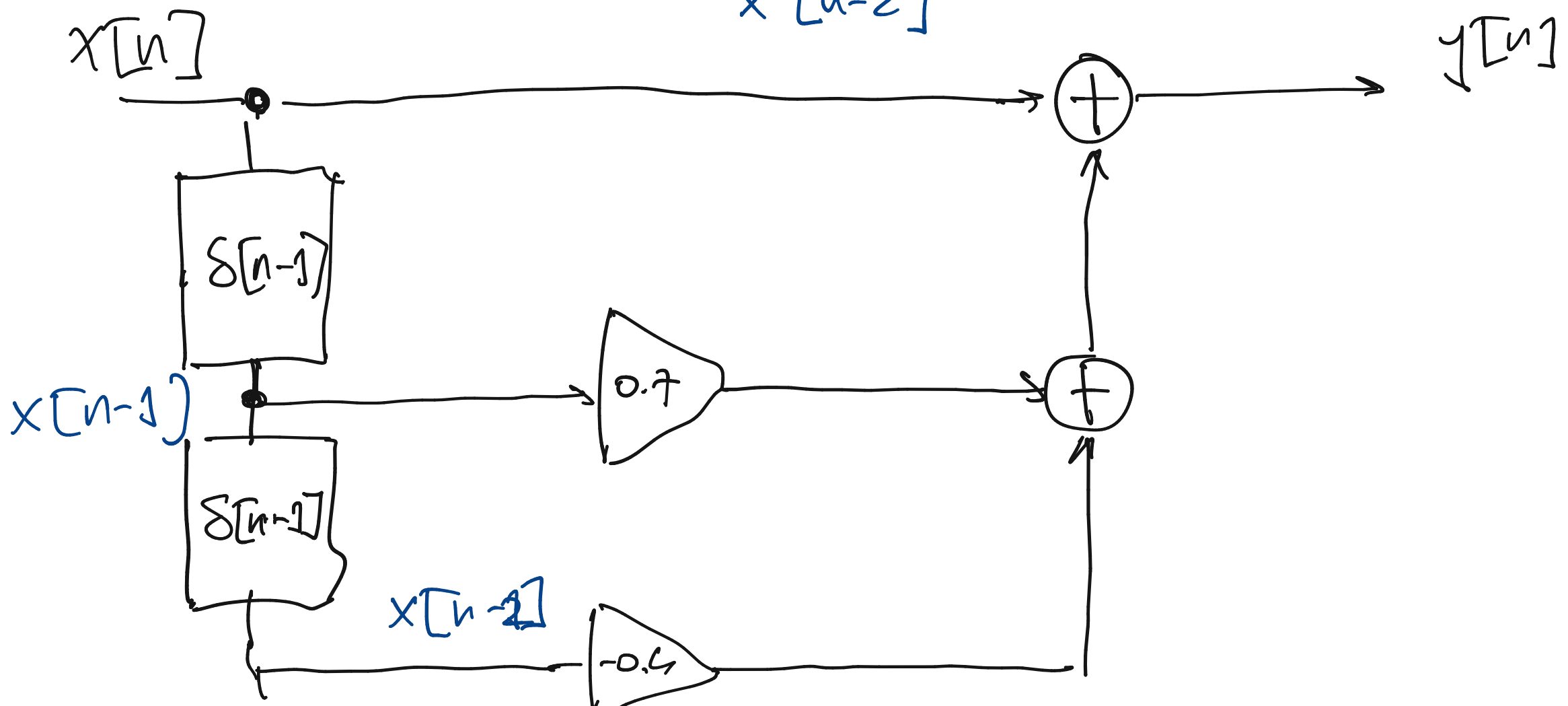
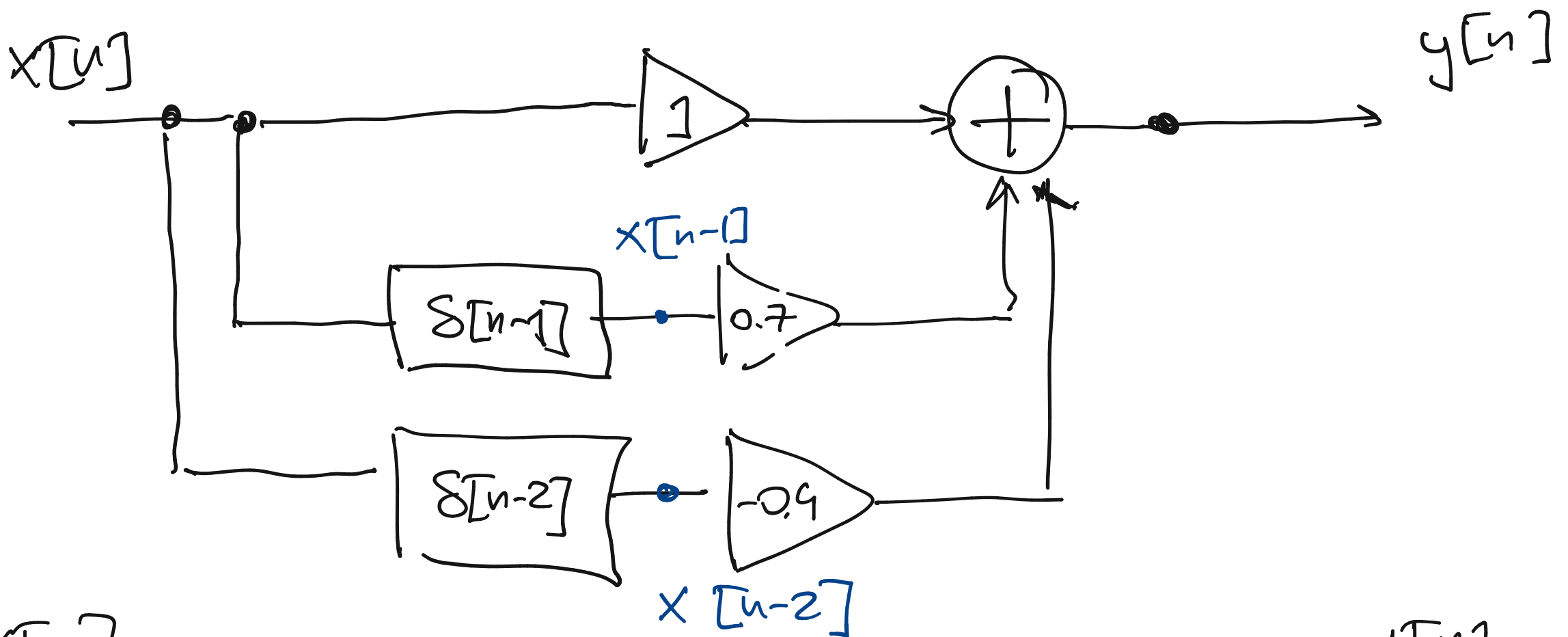
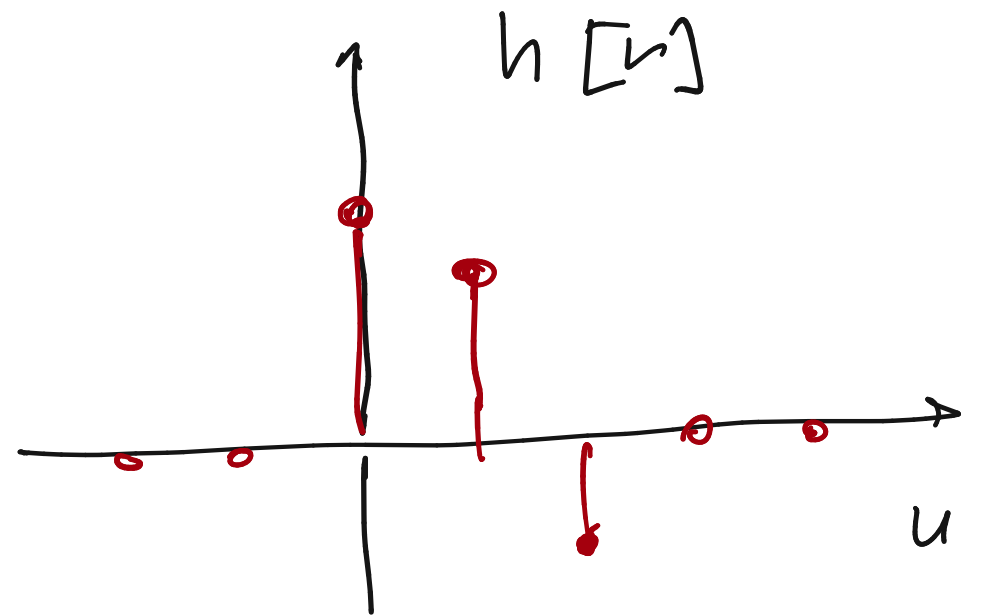
[ver slides]
p. 65

Ec. diferencias

$$\underbrace{1}_{b_0} \cdot y[n] - \underbrace{a}_a \cdot y[n-1] = \underbrace{1}_{b_0} \cdot x[n]$$

Ejemplo: * pre-alimentación: $a_0 = 1$, $a_j = 0 \quad \forall j \geq 1$
 (2 ct) * re-alimentación: $b_0 = 1$, $b_1 = 0.7$
 $b_2 = 0.4$, $b_i = 0 \quad \forall i \geq 3$

$$h[n] = \delta[n] + 0.7 \cdot \delta[n-1] - 0.4 \delta[n-2]$$



AC unzulässig

$$y[10] = x[10] + x[9] + \dots$$

$$y^{[11]} = x^{[11]} + \underbrace{x^{[10]} + x^{[9]} + \dots}$$

$$y[n] = y[n-1] + x[n] \quad y[0]$$

$$y[n] = a y[n-1] + x[n] \quad , \quad a=1 \Rightarrow \text{instable.} !!$$

Media móvil

$$M_2 = 5, \quad M_1 = 0.$$

$$y[10] = \frac{x[10] + x[9] + \dots + x[6] + x[5]}{2}$$

$$y[11] = \frac{x[11] + x[10]^6 + \dots + x[7] + x[6]}{6}$$

$$y[n] = y[n-1] + \frac{1}{6}x[n] - \frac{1}{6}x[n-4]$$

obs: * 6 weeks vs 3.

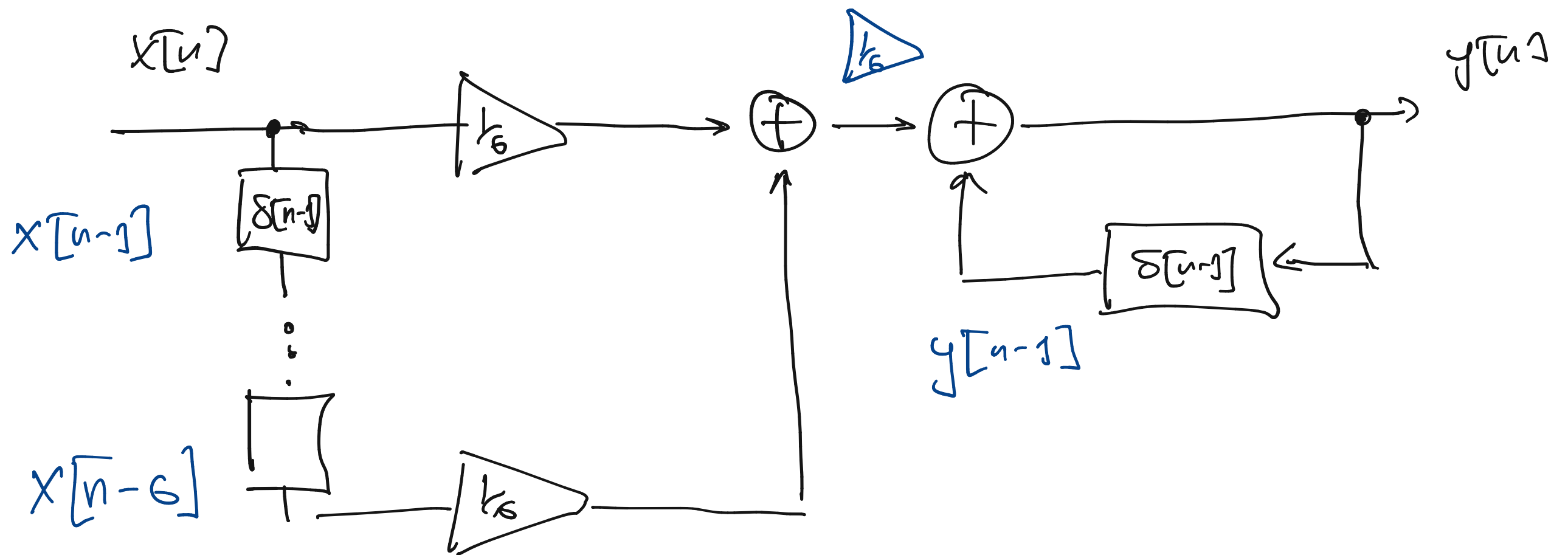
* efficient computational.

* se dio : $\rightarrow$ 1 conversión $\rightarrow 2^1$

→ 1s → 44 kHz

→ total: $\frac{1}{2} M$.

* diagrama de bloques



CONDICIONES INICIALES NO NULAS

Linealidad

$$y[n] = a y[n-1] + x[n]$$

$$\underline{y[-1] = 1}$$

$$x[n] = \delta[n]$$

$$x[n] = 2 \cdot \delta[n]$$

$$y[0] = a \cdot 1 + 1$$

$$y[1] = a(a+1)$$

$$y[0] = a \cdot 1 + 2$$

$$y[1] = a(a+2)$$

CLASE 8

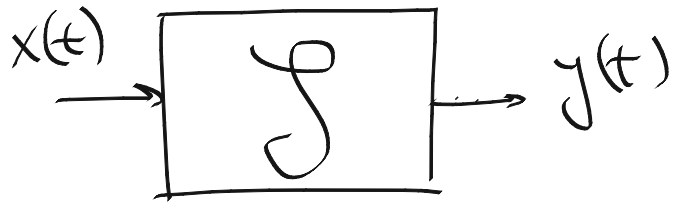
Sistemas

SLIT en tiempo continuo

Definición

$$y(t) = \mathcal{F}\{x(t)\}$$

$$x(t) \xrightarrow{\mathcal{F}} y(t)$$



Propiedades:

- * Determinístico
 - * Cantidad y tipo de entradas y salidas
 - * Memoria
 - * Invertibilidad
 - * Linealidad
 - * Invarianza en el tiempo
 - * Causalidad
 - * Estabilidad
- } SLITs

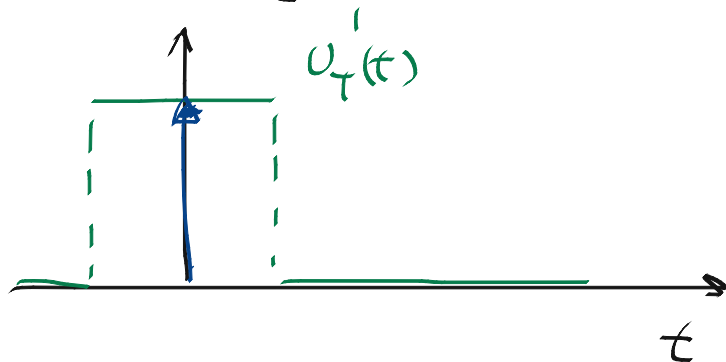
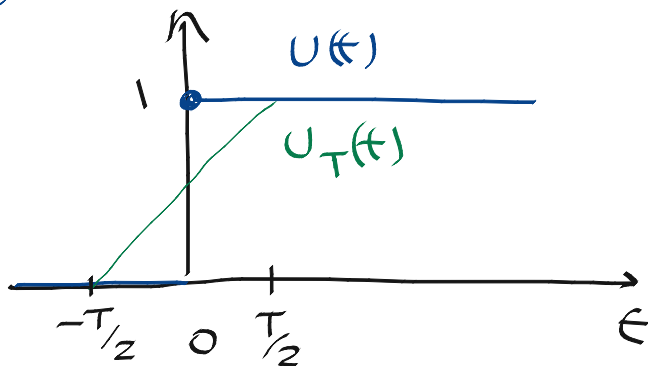
SLITS CONTINUOS

Def. funciones singulares

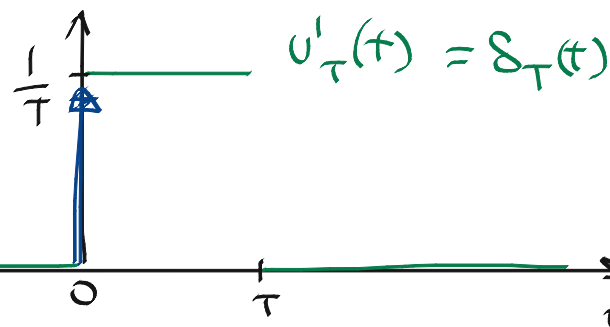
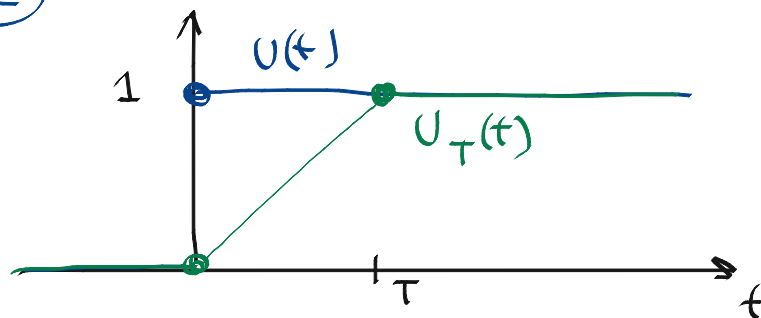
que sabemos de la delta? $\rightarrow \delta(t) = u'(t)$

$$\rightarrow \int_{-\infty}^{\infty} \delta(t) = 1$$

①



②



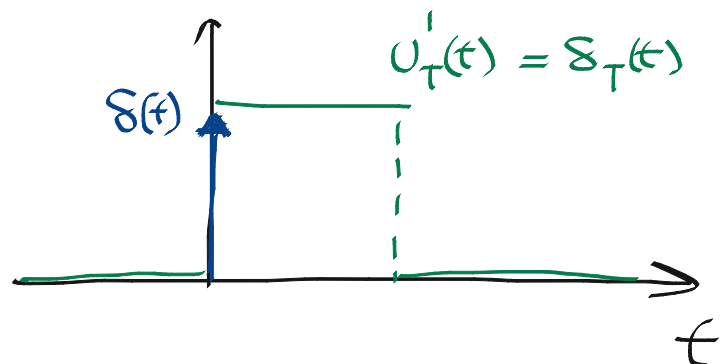
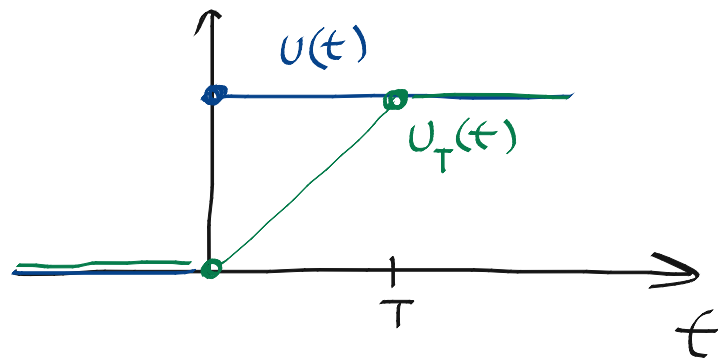
obs: la aprox. de la delta no es única.

	$\mathcal{D}$	C^0
escalón	$u(t)$	$u_T(t)$
	$\xrightarrow{\text{aproxima}}$	
	$\downarrow \text{derivo}$	$\downarrow \text{derivo}$
impulso	$\delta(t)$	$\delta_T(t)$
	$\xleftarrow{\text{reconstruye}}$	

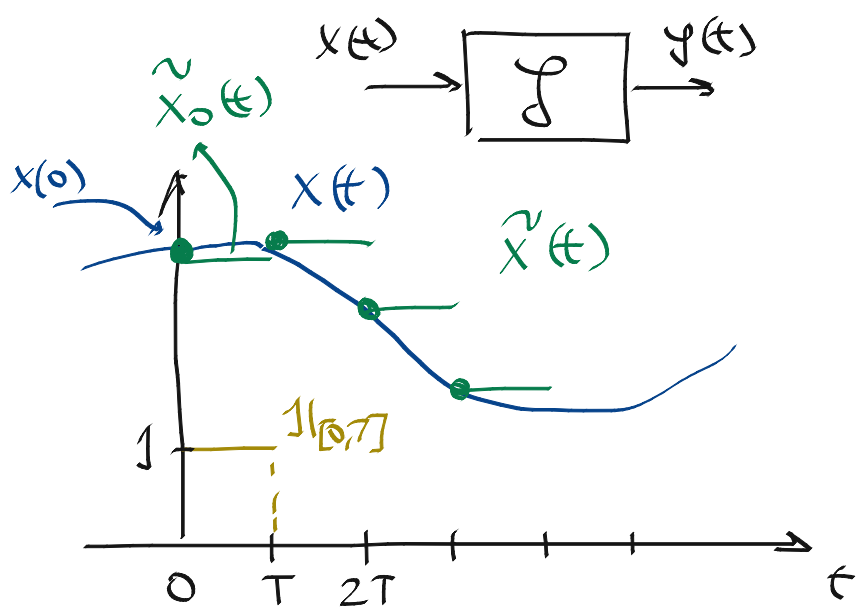
Definición alternativa

$$h_T(t) = \begin{cases} 1 & t < -T \\ t/T & 0 \leq t < T \\ 0 & 0 < t \end{cases}$$

$$\delta_T(t) = \begin{cases} \frac{1}{T} & t \in [0, T] \\ 0 & \text{e.o.c.} \end{cases}$$



Descomposición



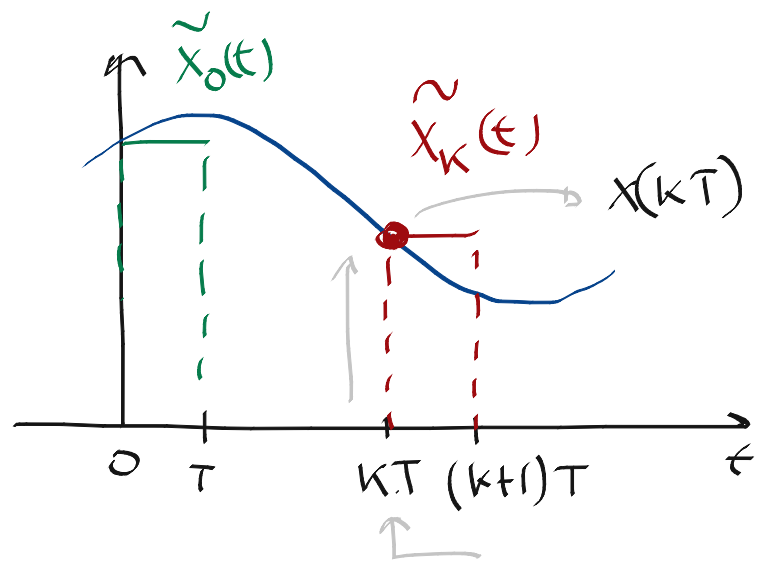
1) Aproximar $x(t)$ en $[0, T]$
 $\mathbb{1}_{[0, T]} = \begin{cases} 1 & t \in [0, T] \\ 0 & \text{e.o.c.} \end{cases}$

$$\tilde{x}_0(t) = \begin{cases} x(0) & t \in [0, T] \\ 0 & \text{e.o.c.} \end{cases}$$

$$\tilde{x}_0(t) = x(0) \cdot \mathbb{1}_{[0, T]} \\ = x(0) \cdot T \cdot \delta_T(t)$$

2) Aprox. global

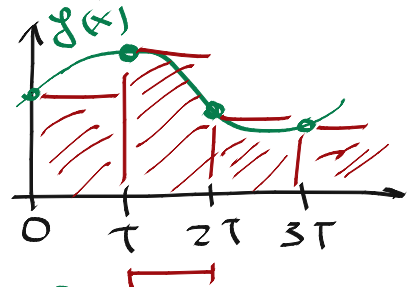
$$\tilde{x}_k(t) = \underbrace{x(kT)}_h \cdot T \cdot \delta_T(t - kT)$$



$$\tilde{x}_T(t) = \sum_{k=-\infty}^{\infty} \tilde{x}_k(t) = \sum_{k=-\infty}^{\infty} x(kT) \cdot T \cdot \delta_T(t - kT)$$

$$\lim_{T \rightarrow 0} \tilde{x}_T(t) = x(t) \Rightarrow$$

$$\lim_{T \rightarrow 0} \tilde{x}_T(t) = \lim_{T \rightarrow 0} \sum_{k=-\infty}^{\infty} \underbrace{x(kT)}_{g(kT)} \cdot \delta(t - kT) \cdot T = \int_{-\infty}^{\infty} g(\tau) \cdot d\tau$$

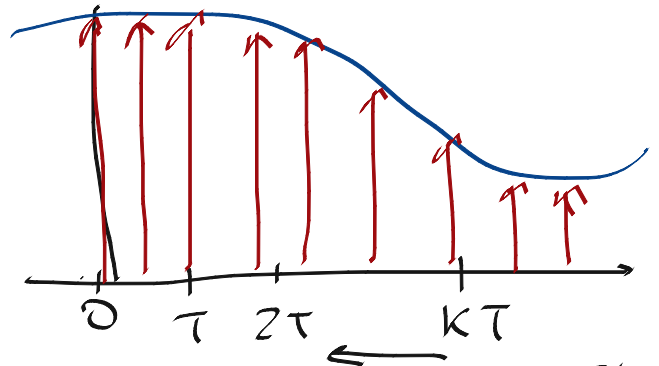


$$\lim_{T \rightarrow 0} \tilde{x}_T(t) = \int_{-\infty}^{\infty} x(\tau) \cdot \delta(t - \tau) \cdot d\tau$$

$$x(t) = \int_{-\infty}^{\infty} x(\tau) \cdot \delta(t - \tau) d\tau$$

Superposición

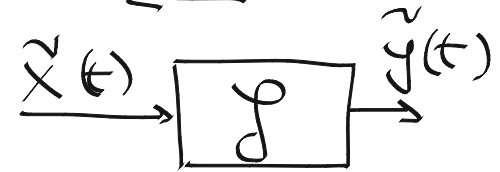
$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} x(kT) \cdot \delta(t - kT)$$



$$\tilde{y}(t) = \mathcal{F} \left\{ \sum_{k=-\infty}^{\infty} \underbrace{x(kT)}_{\text{cte}} \cdot \underbrace{\delta(t - kT)}_{\text{función}} \right\}$$

linealidad
↓

$$= \sum_{k=-\infty}^{\infty} x(kT) \cdot \underbrace{\mathcal{F} \{ \delta(t - kT) \}}_{h_k(t)}$$



$$= \sum_{k=-\infty}^{\infty} x(kT) \cdot \underbrace{h_k(t)}_{h(t - kT)} \xrightarrow{\text{inv. tiempo}} \boxed{\sum_{k=-\infty}^{\infty} x(kT) \cdot h(t - kT)}$$

$$y(t) = \lim_{T \rightarrow 0} \tilde{y}(t) = \boxed{\int_{-\infty}^{\infty} x(\tau) \cdot \underline{h(t - \tau)} \cdot d\tau = y(t)}$$

$T \rightarrow 0, kT \rightarrow \tau$

integral de convolución

$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n - k]}$$

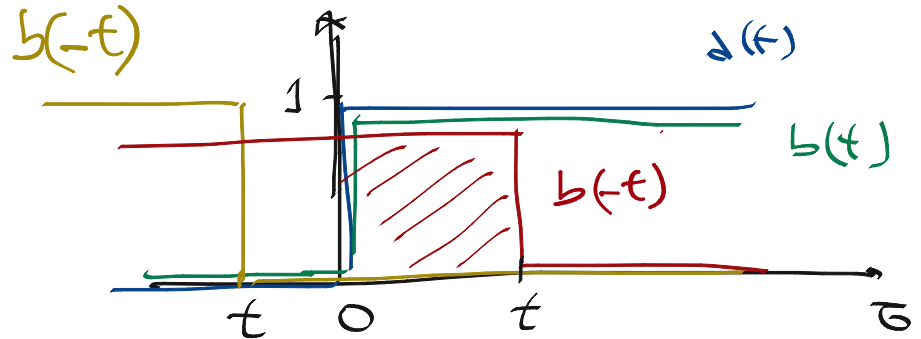
suma de convolución

$$\boxed{y(t) = x(t) * h(t)}$$

convolución

Ejemplo: $a(t) = u(t)$, $b(t) = u(t) \rightarrow c(t) = a(t) * b(t)$

[mostrar slides
con construcción
discreta]



① $c(t) = \int_{-\infty}^{\infty} \underline{u(\tau)} \cdot \underline{u(t-\tau)} d\tau \neq \boxed{0}$, $t < 0$

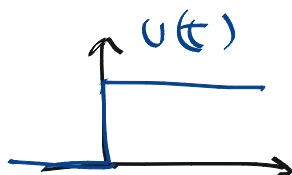
② $c(t) = \int_0^t 1 \cdot 1 \cdot d\tau \neq \boxed{t}$, $t \geq 0$

$c(t) = \begin{cases} 0 & t < 0 \\ t & t \geq 0 \end{cases} \Rightarrow \boxed{c(t) = u(t) \cdot t}$

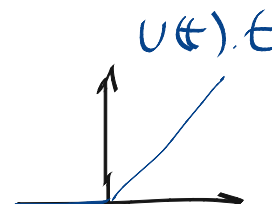
ejemplo: "interpretado como sistema"

$y(t) = x(t) * h(t)$

$x(t) = u(t)$, $h(t) = u(t)$



"1"



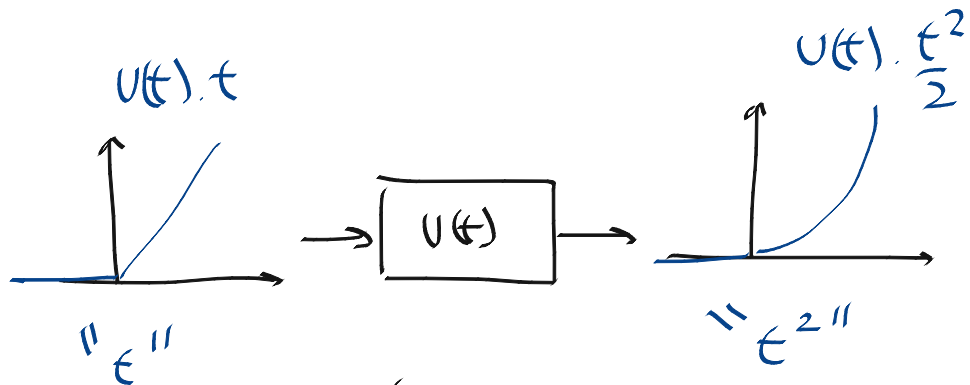
"t"

obs: sistema "integrador"

Ejemplo:

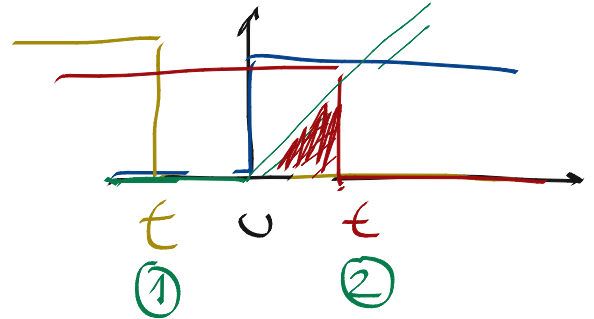
$$h(t) = u(t)$$

$$x(t) = u(t) \cdot t$$



$$\textcircled{2} \quad y(t) = \int_0^t 1 \cdot \tau \, d\tau = \left. \frac{\tau^2}{2} \right|_0^t$$

$$y(t) = \frac{t^2}{2}, \quad t > 0$$



$$y(t) = u(t) \cdot \frac{t^2}{2}$$

Ejemplo 2.6 (98)

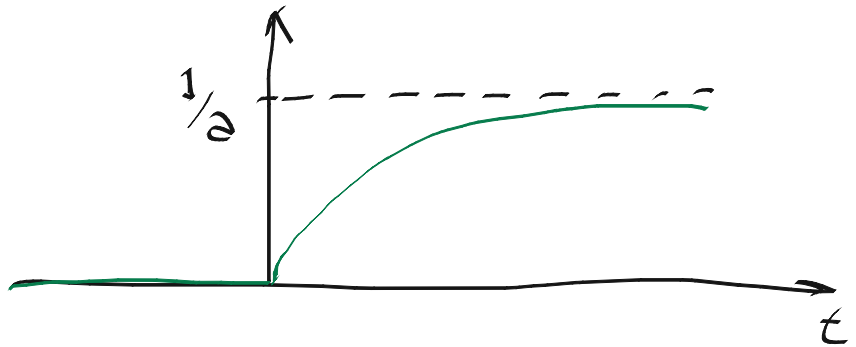
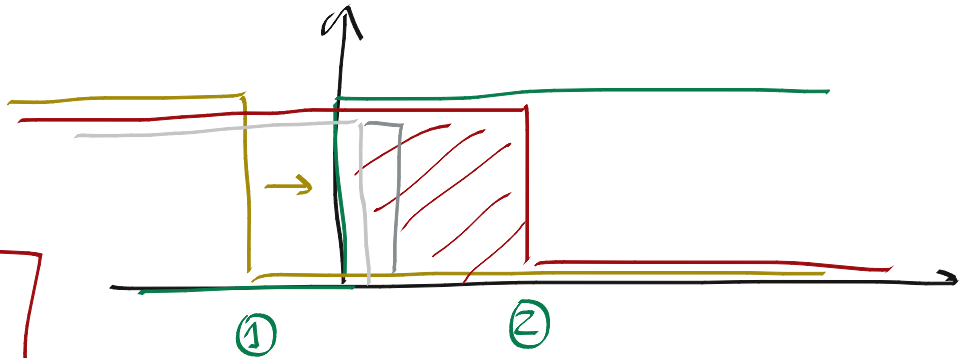
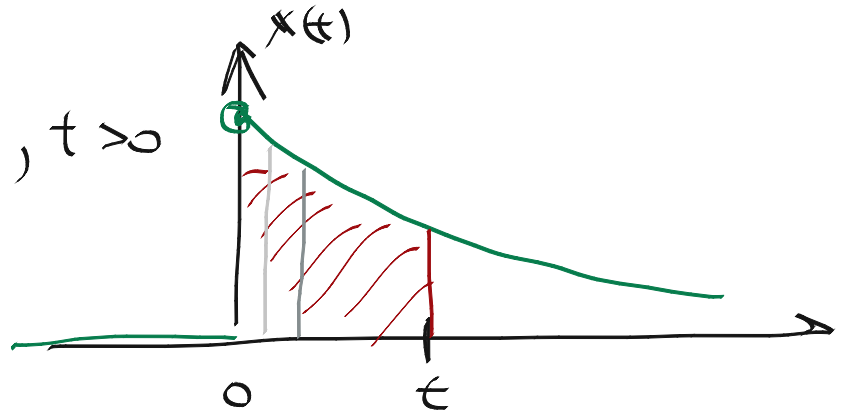
$$h(t) = u(t), \quad x(t) = u(t) \cdot e^{-at}, \quad a > 0$$

$$\textcircled{2} \quad y(t) = \int_0^t e^{-a\tau} \cdot 1 \cdot d\tau, \quad t > 0$$

$$y(t) = \left. \frac{e^{-a\tau}}{-a} \right|_0^t$$

$$y(t) = \frac{1 - e^{-at}}{a}$$

$$y(t) = u(t) \cdot \left(\frac{1 - e^{-at}}{a} \right)$$



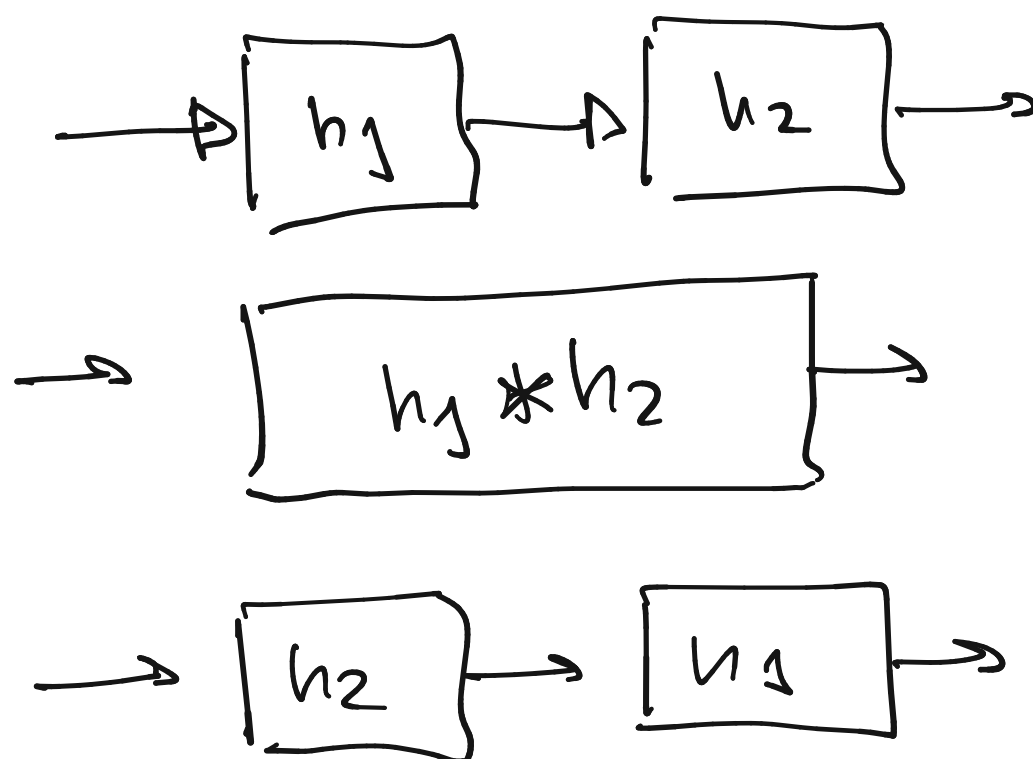
Propiedades

operación

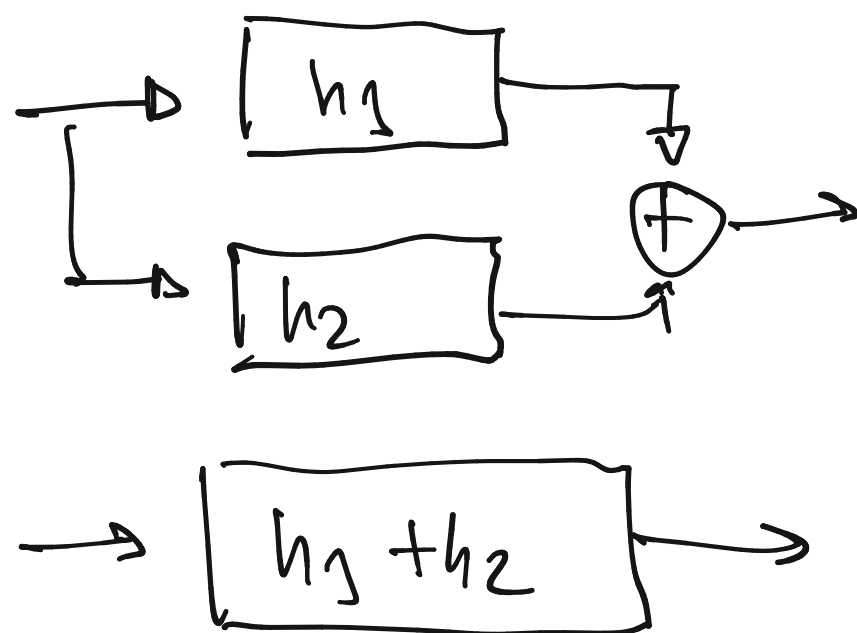
- * identidad: $x(t) * \delta(t) = x(t)$
- * asociativa: $(x(t) * y(t)) * z(t) = x(t) * (y(t) * z(t))$
- * distributiva.

interconexión

* Serie:



* paralelo



Estabilidad

$$|x(t)| < B_x < \infty, \forall t \in \mathbb{R}$$

Ⓗ

$$|y(t)| \leq B_y < \infty, \forall t$$



Ⓙ

$$S = \int_{-\infty}^{\infty} |h(t)| dt = \|h(t)\|_1 \leq \infty$$

Causalidad

$$\text{S causal} \Leftrightarrow h(t) = 0, \forall t < 0$$

Ejemplo 2.7 (modificado)

$$T_1 < T_2$$

$$x(t) = \begin{cases} 1 & t \in [0, T_1] \\ 0 & \text{e.o.c.} \end{cases}$$

$$h(t) = \begin{cases} 1 & t \in [0, T_2] \\ 0 & \text{e.o.c.} \end{cases}$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau) \cdot d\tau$$

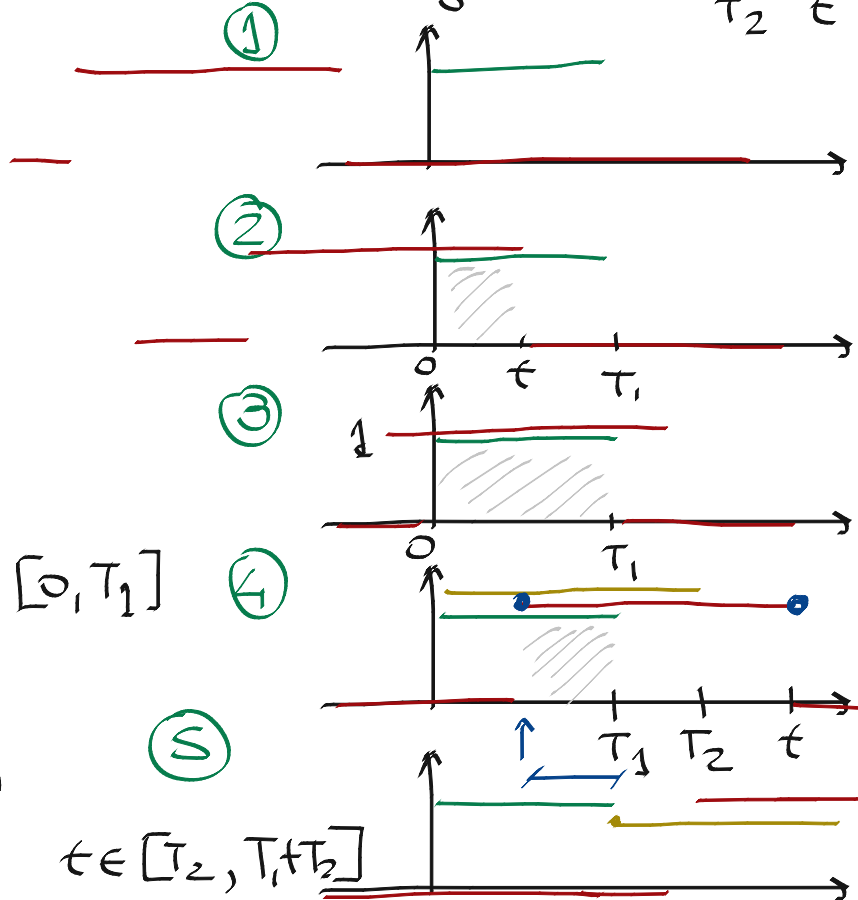
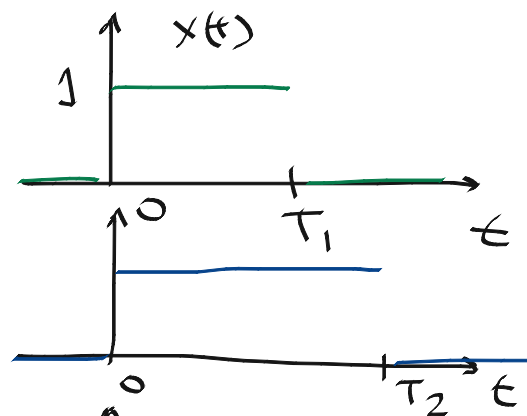
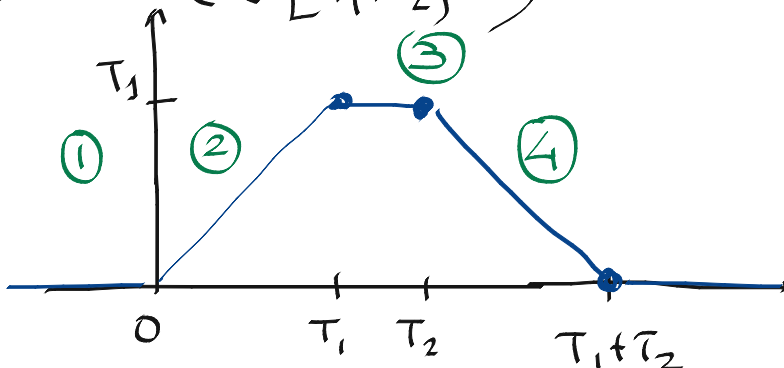
① $y(t) = 0$, $t < 0$

② $y(t) = \int_0^t 1 \cdot 1 d\tau = t$, $t \in [0, T_1]$

③ $y(t) = T_1$, $t \in [T_1, T_2]$

④ $y(t) = T_1 - (t - T_2) = T_1 + T_2 - t$, $t \in [T_2, T_1 + T_2]$

⑤ $y(t) = 0$, $t \in [T_1 + T_2, \infty)$



ejemplo 2.7 (sin cuantización)

$$x(t) = u(t - T_1) - u(t) = u(t) * \delta(t - T_1) - u(t) * \delta(t)$$

$$x(t) = u(t) * [\delta(t - T_1) - \delta(t)]$$

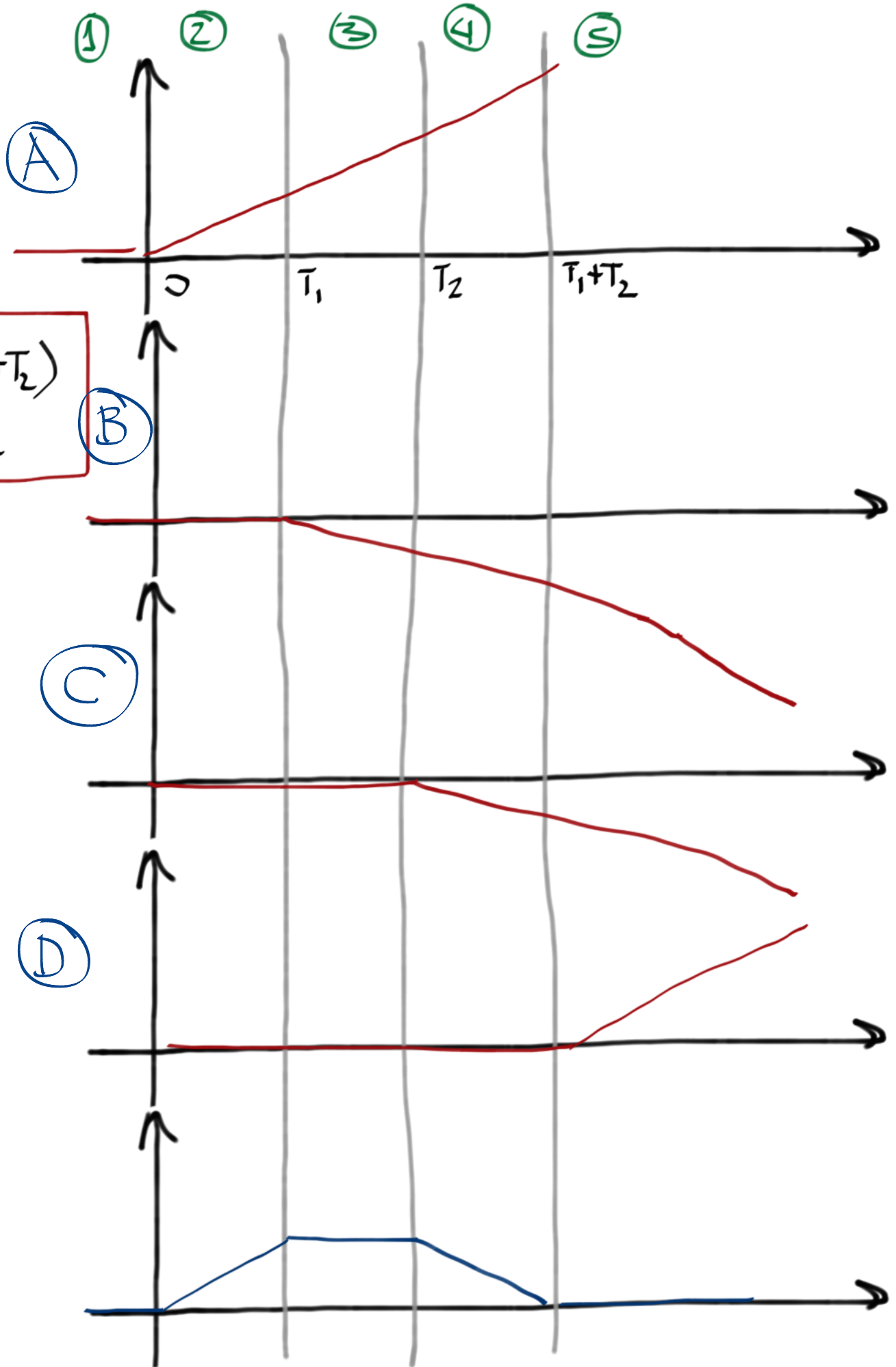
$$\begin{aligned} x(t) * x(t) &= \left[u(t) * [\delta(t - T_1) - \delta(t)] \right] * \left[u(t) * [\delta(t - T_2) - \delta(t)] \right] \\ &= u(t) * u(t) * \left[\delta(t - T_1) * \delta(t - T_2) - \delta(t - T_1) * \delta(t) - \delta(t) * \delta(t - T_2) \right. \\ &\quad \left. - \delta(t) * \delta(t) \right] \end{aligned}$$

$$= u(t) * u(t) * \left[\underbrace{\delta(t - (T_1 + T_2))}_{\textcircled{D}} - \underbrace{\delta(t - T_1)}_{\textcircled{B}} - \underbrace{\delta(t - T_2)}_{\textcircled{C}} - \underbrace{\delta(t)}_{\textcircled{A}} \right]$$

$u(t) * u(t) = u(t) \cdot t$

 $\textcircled{A}$

$t < 0$
 $t \in (0, T_1)$
 $T_1, t \in (T_1, T_2)$
 $T_1 + T_2 - t, t \in (T_2, T_1 + T_2)$
 $0, t > T_1 + T_2$



CLASE 9

Sistemas

SLIT en tiempo continuo

Funciones Singulares

Obs: "¿que sabemos de la delta?"

$$\rightarrow \boxed{U'(t) = \delta(t)}$$

→ Límite de aprox. continuo

$$\rightarrow U_T(t) = \begin{cases} 0 & t \leq 0 \\ t/T & 0 < t \leq T \\ 1 & T < t \end{cases}$$

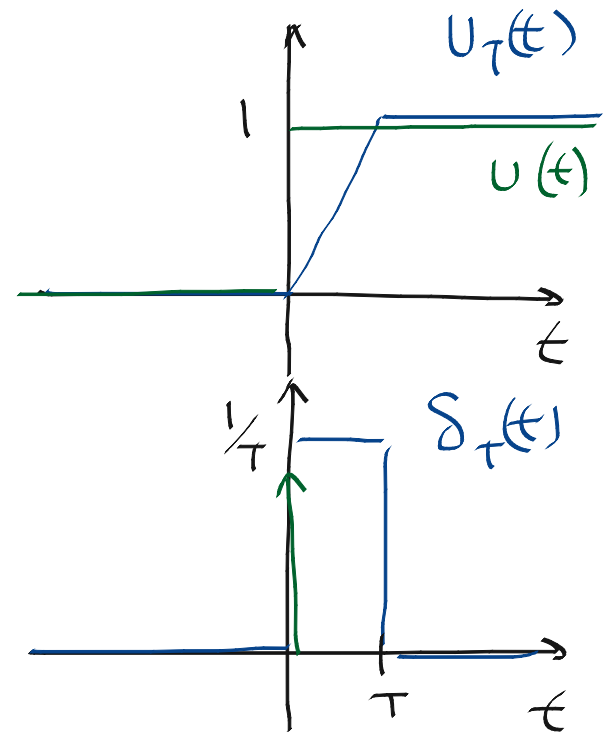
$$\rightarrow \delta_T(t) = \begin{cases} 1/T & 0 < t \leq T \\ 0 & \text{e.o.c} \end{cases}$$

$$\rightarrow \lim_{T \rightarrow \infty} U_T(t) = U(t)$$

$$\rightarrow \lim_{T \rightarrow 0} \delta_T(t) = \delta(t)$$

$$\rightarrow \int_{-\infty}^{\infty} \delta_T(t) dt = 1 \Rightarrow \int_{-\infty}^{\infty} \delta(t) dt = 1$$

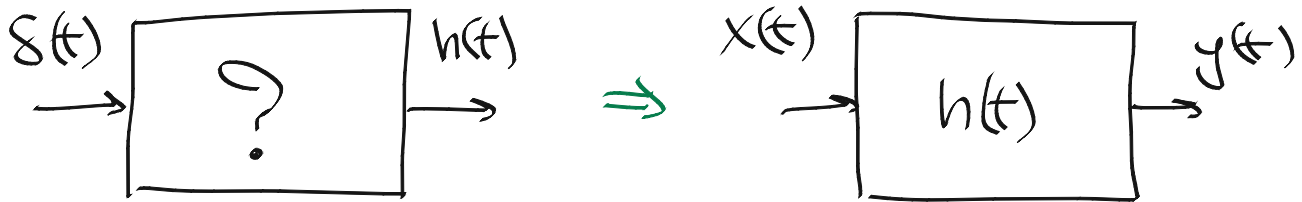
→ def. operativa: $\delta(t)$ tal que $\delta(t) = x(t) * \delta(t)$
 $\forall x(t) \in \mathcal{F}$
- "neutro de la convolución"



Propiedades adicionales

"¿se agregamos?"

obs: "caracteriza por completo un sistema"



$$y(t) = x(t) * h(t)$$

obs: "propiedad de muestreo"

$$\rightarrow \tilde{x}_0(t) \simeq x(t) \quad \text{en } [0, T]$$

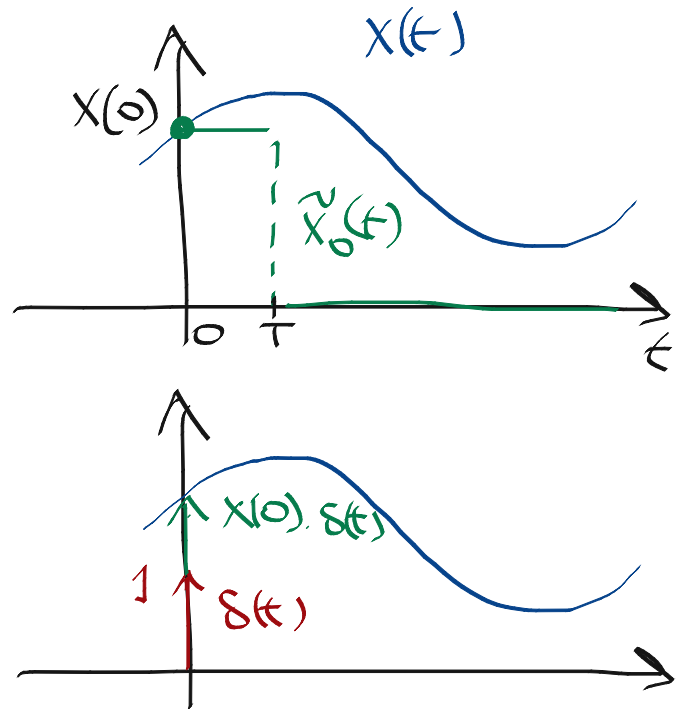
si $T \rightarrow 0$

$$\tilde{x}_0(t) = x(0) \cdot \delta_T(t) \cdot T$$

$$x(0) \cdot \delta_T(t) \cdot T \simeq x(t) \cdot \delta_T(t) \cdot T$$

$$\lim_{T \rightarrow 0} \underline{x(0) \cdot \delta_T(t)} = \lim_{T \rightarrow 0} \underline{x(t) \cdot \delta_T(t)}$$

$$x(0) \cdot \delta(t) = x(t) \cdot \delta(t)$$



propiedad de
selección o
muestreo

Respuesta al escalón unitario (2.38 p 115)

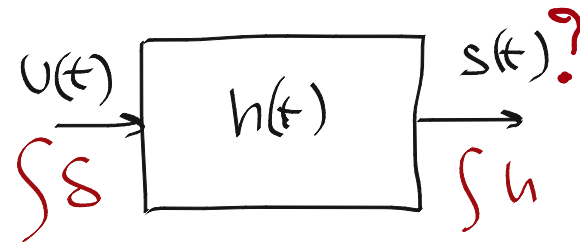
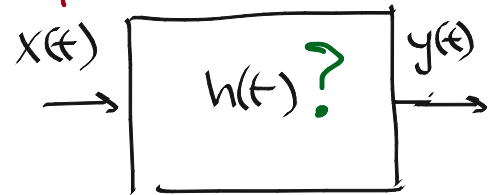
a)

$$s(t) = h(t) * u(t)$$

$$s(t) = \int_{-\infty}^{\infty} h(\tau) \cdot u(t-\tau) \cdot d\tau$$

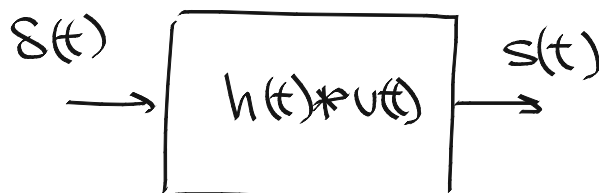
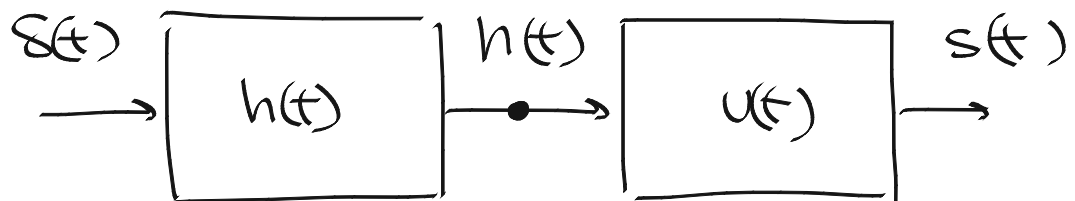
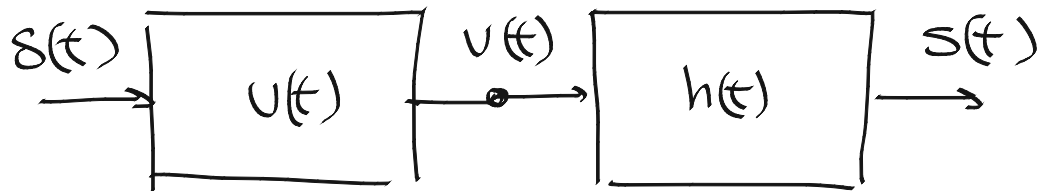
$$s(t) = \int_{-\infty}^t h(\tau) \cdot d\tau$$

$$s'(t) = h(t)$$



b) $u(t) * \delta(t) = u(t)$

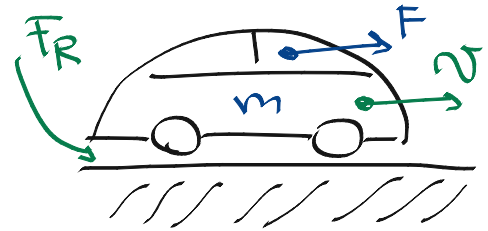
integrator
↓



obs: → En la práctica es más fácil generar escalones
→ El escalón tiene una utilidad en si mismo.

SISTEMAS BASADOS EN EC. DIF. ORD. LINEALES

modelado: * Señales: $\rightarrow$ entrada: $f(t)$
 $\rightarrow$ salida: $v(t)$



$$f(t) = m \cdot v'(t) + b \cdot v(t) \quad \text{EDO lineal, coef.s. cts.}$$

$$b = 2 \text{ kg/s} \quad m = 1 \text{ kg}$$

$$v'(t) + \frac{b}{m} v(t) = \frac{1}{m} f(t)$$

$$v'(t) + 2 \cdot v(t) = f(t)$$

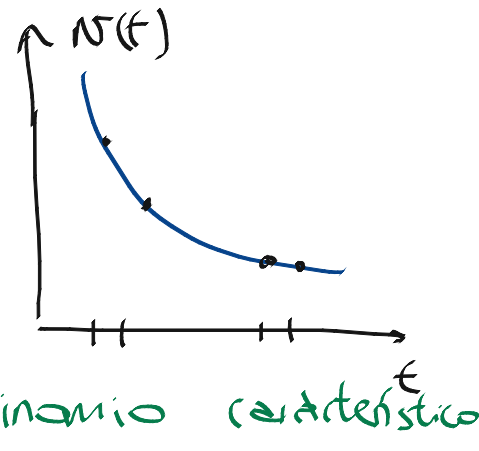
Ejemplo 2.14 (p. 118+28) $f(t) = u(t) \cdot k \cdot e^{3t}$

$$v(t) = v_h(t) + v_p(t)$$

a) homogénea: $f(t) = 0 \Rightarrow v'(t) + 2v(t) = 0$

supongo: $v_h(t) = A \cdot e^{rt}$ $\leftarrow$ exponenciales "fáciles"

$$\cancel{A \cdot r \cdot e^{rt}} + 2 \cdot \cancel{A \cdot e^{rt}} = 0 \Rightarrow \boxed{r + 2 = 0}$$



$\uparrow$ polinomio característico

$$\boxed{r = -2} \Rightarrow \boxed{v_h(t) = A \cdot e^{-2t}}$$

obs: * la homogénea se calcula una vez

* transformo una ec. dif. en una algebraica

b) particular: $N_p(t) = A(t) \cdot e^{-2t}$ ← Variación de ctes.

$$\overbrace{A'(t) \cdot e^{-2t}}^{N_p'(t)} - 2A(t) \cdot e^{-2t} + 2A(t) \cdot e^{-2t} = f(t)$$

$$A'(t) \cdot e^{-2t} = f(t) \Rightarrow A'(t) = \underbrace{e^{2t} \cdot f(t)}_{\text{integral}} \Rightarrow A(t) = \int e^{2\tau} f(\tau) d\tau$$

$$A(t) = \int U(\tau) \cdot K \cdot e^{3\tau} \cdot e^{2\tau} d\tau$$

$$A(t) = \frac{K}{s} e^{st} \Rightarrow N_p(t) = \frac{K}{s} e^{3t}$$

c) completa:

$$N(t) = A \cdot e^{-2t} + \frac{K}{s} e^{3t}$$

K=1 dato

$$N(t) = A \cdot e^{-2t} + \frac{1}{s} e^{3t} \quad \leftarrow \text{necesito c.c.i.i.}$$

$$N(0) = 0 \Rightarrow N(0) = A + \frac{1}{s} \Rightarrow A = -\frac{1}{s}$$

$$N(t) = -\frac{1}{s} e^{-2t} + \frac{1}{s} e^{3t}, \quad t \geq 0$$

$$N(t) = U(t) \cdot \frac{1}{s} (e^{3t} - e^{-2t})$$

← después
reposo
inicial.

c)

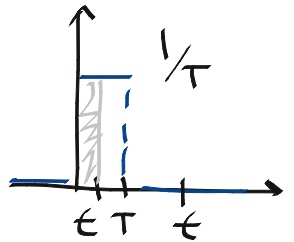
"respuesta al impulso" (otra particular)

$$f(t) = \delta(t) \Rightarrow A'(t) \cdot e^{-2t} = f(t) \Rightarrow A(t) = \int_{-\infty}^t f(\tau) \cdot e^{2\tau} d\tau$$

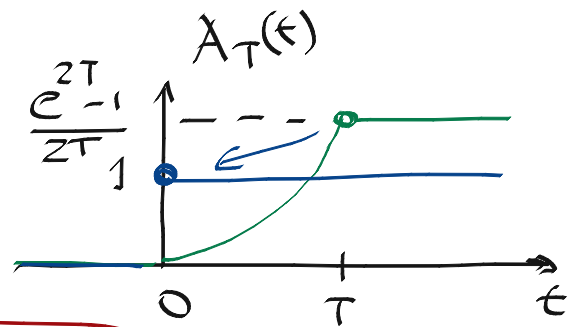
$$A(t) = \int_{-\infty}^t \delta(\tau) \cdot e^{2\tau} d\tau \leftarrow \text{aprox. continua.}$$

$$A_T(t) = \int_{-\infty}^t \delta_T(\tau) \cdot e^{2\tau} d\tau = \int_0^t \frac{1}{T} \cdot e^{2\tau} d\tau$$

$$A_T(t) = \frac{1}{2T} \cdot e^{2\tau} \Big|_0^t = \begin{cases} 0 & t \leq 0 \\ \frac{e^{2t} - 1}{2T} & t \in [0, T] \\ \frac{e^{2T} - 1}{2T} & t \geq T \end{cases} \quad (*)_1$$

(*)₁

$$\lim_{T \rightarrow 0} \frac{e^{2T} - 1}{2T} \stackrel{\text{L'Hopital}}{=} \lim_{T \rightarrow 0} \frac{2e^{2T}}{2} = 1$$



$$\lim_{T \rightarrow 0} A_T(t) = u(t)$$

$$N_p(t) = u(t) \cdot e^{-2t}$$

$$N(t) = A \cdot e^{-2t} \cdot u(t) + u(t) \cdot e^{-2t}$$

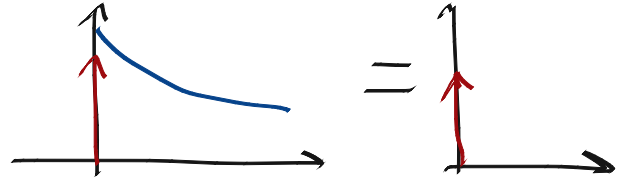
$$h(t) = u(t) \cdot (A+1) \cdot e^{-2t}$$

Verificación $N_p(t) = u(t) \cdot e^{-2t}$

$$\underbrace{u'(t) \cdot e^{-2t} - 2 \cdot u(t) \cdot e^{-2t}}_{N_p'} + \cancel{u(t) \cdot e^{-2t}} = \delta(t)$$

$$u'(t) \cdot e^{-2t} = \delta(t)$$

$$\delta(t) \cdot e^{-2t} = \delta(t)$$



$$\delta(t) \cdot e^0 = \delta(t)$$

$\delta(t) = \delta(t)$

✓

RESUMEN: Análisis en el dominio del tiempo

- * Def. señales y propiedades: clasificación + herramientas
- * Def. sistemas y propiedades: clasificación + herramientas
- * modelado de sistemas reales: gran familia que se describe por EDO lineales
- * SLIT: $\rightarrow$ se caracterizan por su respuesta al impulso $h(t)$
 $h[n]$
 $\rightarrow$ herramienta: convolución
- * Ejemplo integrador:
 - $\rightarrow$ señales involucradas
 - $\rightarrow$ modelado $\rightarrow$ EDO
 - $\rightarrow$ SLIT: calculamos $h(t)$ como solución de la EDO

Análisis espectral: ideas

- * es útil ver la respuesta de un SLIT a una combinación lineal de señales sencillas
 - $\rightarrow$ Este conjunto debe permitir construir una clase de amplia y útil.
 - $\rightarrow$ La respuesta del sistema a cada señal debe ser sencilla.
- * Vamos a hacerlo mismo pero con otra base.
 - $\rightarrow$ Exponenciales periódicas
 - $\rightarrow$ Cuanto más "pauca" es la base a las señales a representar, "mejor" es la representación

CLASE 10

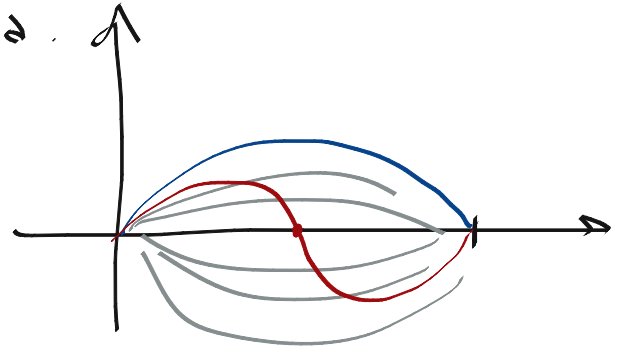
Series de fourier

Señales periódicas continuas

ANÁLISIS ESPECTRAL

Historia

- 1748: Euler, cuerda vibratoria.
- 1753: Bernoulli, todos los movimientos físicos de una cuerda son combinación lineal de los modos originales
- 1759: Lagrange, crítico por el caso discontinuo
- 1807: Fourier desarrolló la teoría (impresora) extendió este análisis al caso no periódico
- 1829: Dirichlet demostró la teoría



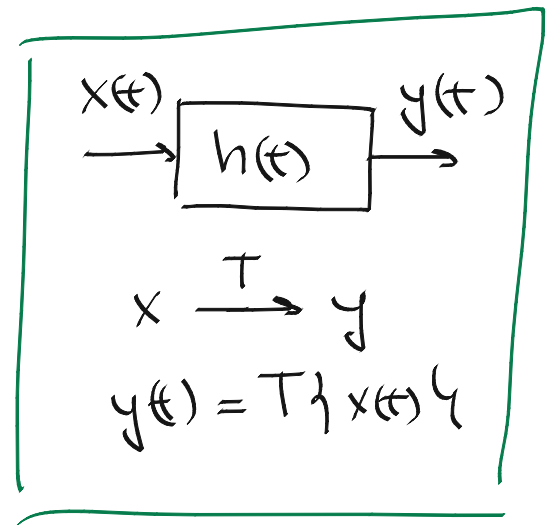
RESPUESTA DE UN SLIT A EXPONENCIALES COMPLEJAS

Exponencial simple

$$x(t) = e^{st}, s \in \mathbb{C}$$

$$y(t) = T\{e^{st}\} = h(t) * e^{st}$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau) \cdot e^{s(t-\tau)} d\tau$$



$$y(t) = \int_{-\infty}^{\infty} h(\tau) \cdot \underbrace{e^{st}}_{cte} \cdot e^{-s\tau} d\tau = e^{st} \underbrace{\int_{-\infty}^{\infty} h(\tau) \cdot e^{-s\tau} d\tau}_{\text{sólo depende de } s} \quad \textcircled{*}_1$$

defino $H(s) = \int_{-\infty}^{\infty} h(\tau) \cdot e^{-s\tau} d\tau \quad \textcircled{*}_2$

obs: * sólo depende de $h(t) \rightarrow$ del sistema

* el caso discreto es análogo.

$\textcircled{*}_1$ y $\textcircled{*}_2$

$$y(t) = \underbrace{H(s)}_{\text{no hay } t \text{ (escalar)}} \cdot \underbrace{e^{st}}_{\text{vector}}$$

no hay t
(escalar)

vector

$$\underbrace{e^{st}}_{\text{vector}} \xrightarrow{T} \underbrace{H(s)}_{\text{escalar}} \cdot \underbrace{e^{st}}_{\text{el mismo vector}}$$

obs: * los exponenciales

son vectores propios de los SLIT

$$\vec{v} \xrightarrow{T} \lambda \cdot \vec{v}$$

Suma de exponenciales

$$x(t) = a_1 \cdot e^{s_1 t} + a_2 e^{s_2 t} + a_3 \cdot e^{s_3 t}$$

linealidad

$$y(t) = T\{x(t)\} = a_1 \cdot \underbrace{T\{e^{s_1 t}\}}_{H(s_1) \cdot e^{s_1 t}} + a_2 T\{e^{s_2 t}\} + a_3 \cdot T\{e^{s_3 t}\}$$

$$y(t) = a_1 \cdot \underline{H(s_1)} \cdot e^{s_1 t} + a_2 \cdot \underline{H(s_2)} \cdot e^{s_2 t} + a_3 \cdot \underline{H(s_3)} \cdot e^{s_3 t}$$

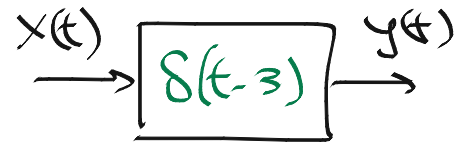
Obs: aparecen los mismos componentes a la salida.

$$x(t) = \sum_{k=-\infty}^{\infty} \underline{a_k} \cdot e^{\underline{s_k t}} \xrightarrow{T} y(t) = \sum_{k=-\infty}^{\infty} \underline{a_k \cdot H(s_k)} \cdot e^{\underline{s_k t}}$$

Ejemplo 3.1: $x(t) = e^{j2t}$

$$a) \quad y(t) = x(t-3) = e^{j2(t-3)} = \underbrace{e^{-6j}}_{\substack{\text{cte} \\ H(z_j)}} \underbrace{e^{j2t}}_{\text{mismo exp}}$$

$$s = j2 \quad a = 1$$



$$y(t) = x(t-3)$$

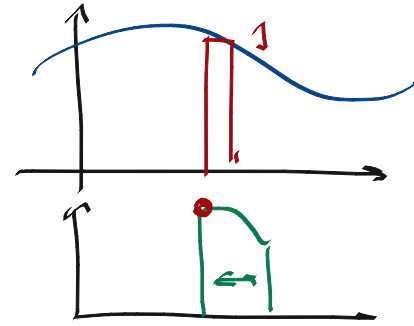
$$y(t) = h(t) * x(t)$$

$$b) \quad H(s) = \int_{-\infty}^{\infty} \underbrace{\delta(\tau-3)}_{\tau=3} \cdot e^{-s\tau} d\tau = e^{-s \cdot 3}$$

$$H(z_j) = e^{-3 \cdot j2} = e^{-6j}$$

$$y(t) = H(z_j) \cdot e^{j2t} = e^{-6j} \cdot e^{j2t}$$

$$= e^{-3s}$$



otra forma:

$$H(s) = \int_{-\infty}^{\infty} \delta(\tau-3) \cdot e^{-s\tau} d\tau = e^{-3s} \underbrace{\int_{-\infty}^{\infty} \delta(\tau-3) d\tau}_{1} = e^{-3s}$$

obs:

$$\int \delta(\tau) \cdot f(\tau) \cdot d\tau = f(0)$$

Ejemplo 3.1

c) $x(t) = \cos(4t) + \text{sen}(7t)$

i) $y(t) = \cos(4t - 12) + \text{sen}(7t - 21)$

ii) $x(t) = \frac{e^{j4t} + e^{-j4t}}{2} + \frac{e^{j7t} - e^{-j7t}}{2j}$

$$x(t) = \frac{1}{2} e^{j4t} + \frac{1}{2} e^{-j4t} + \frac{1}{2j} e^{j7t} - \frac{1}{2j} e^{-j7t}$$

$$y(t) = \frac{1}{2} H(4j) e^{j4t} + \frac{1}{2} H(-4j) e^{-j4t} + \frac{1}{2j} H(j7) e^{j7t} - \frac{1}{2j} H(-j7) e^{-j7t}$$

$$y(t) = \frac{1}{2} e^{-12j} e^{j4t} + \frac{1}{2} e^{12j} e^{-j4t} + \frac{1}{2j} e^{-21j} e^{j7t} - \frac{1}{2j} e^{21j} e^{-j7t}$$

$$y(t) = \frac{1}{2} \left(e^{j4t-12j} + e^{-j4t+12j} \right) + \frac{1}{2j} \left(e^{j7t-21j} - e^{-j7t+21j} \right)$$

$$y(t) = \cos(4t - 12) + \text{sen}(7t - 21)$$

SERIES DE FOURIER

Comb. lineal

$$x(t) = \sum_k a_k \cdot e^{s_k t} \xrightarrow{T} y(t) = \sum_k a_k \cdot H(s_k) \cdot e^{s_k t}$$

Definición

$x(t)$ periódica T , $x(t+T) = x(t)$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \cdot \underbrace{e^{j \frac{2\pi k}{T} t}}_{\text{ingredientes}} \quad \begin{array}{l} \text{ec. síntesis} \\ \text{(reconstruyo)} \end{array}$$

* elijo exponenciales armónicos

→ imaginarios puros

→ las "frecuencias" son múltiplos de una frec. base

→ $s_k = j \cdot \frac{2\pi k}{T}$, $k \in \mathbb{Z}$

* notación:

→ a_k : coeficientes (cantidad de cada ingrediente)

→ T : período (fundamental = T_0)

→ f_0 : frecuencia fundamental $f_0 = \frac{1}{T}$

→ ω_0 : frec. (fundamental) angular $\omega_0 = 2\pi \cdot f_0 = \frac{2\pi}{T}$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \cdot e^{j \underbrace{k \omega_0}_{\text{frec. angular}} t} \quad \omega_k = k \cdot \omega_0$$

Determinación de los coef. a_k

$$a_k = \langle x(t), e^{jk\omega_0 t} \rangle \leftarrow \text{proyección}$$

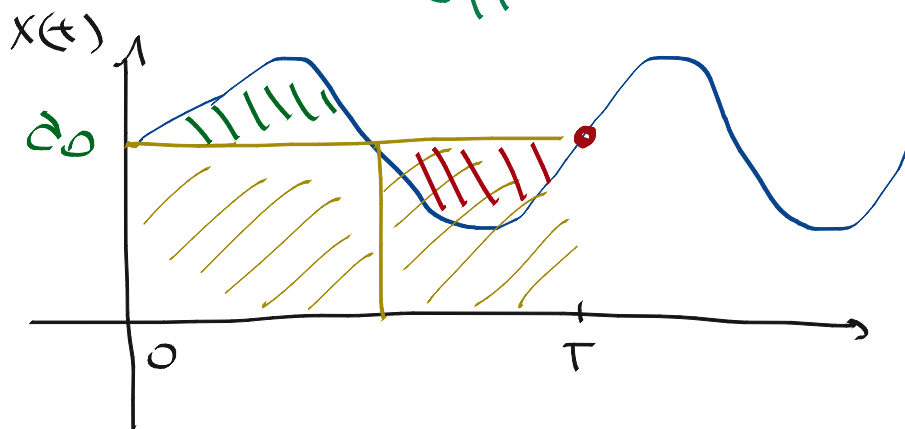
obs: * $\vec{n} = a_1 \vec{e}_1 + a_2 \vec{e}_2 \Rightarrow a_1 = \langle \vec{n}, \vec{e}_1 \rangle$

* $\langle f(t), g(t) \rangle = \int_{\langle T \rangle} f(t) g^*(t) dt$
 $\langle T \rangle \rightarrow$ cualquier período T

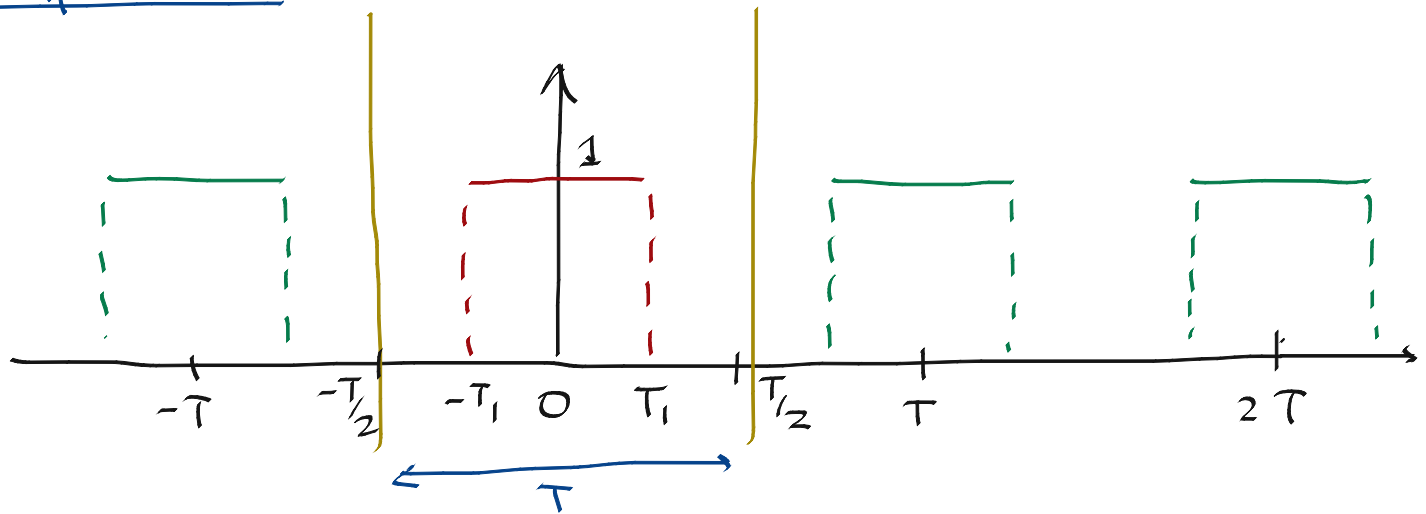
$$a_k = \frac{1}{T} \int_0^T x(t) \cdot e^{-jk\omega_0 t} dt$$

ec. análisis
(proyección)

obs: $k=0$ $a_0 = \frac{1}{T} \int_0^T x(t) dt$ $\leftarrow$ valor medio
(continua.)
offset



Ejemplo 3.5 "onda cuadrada"



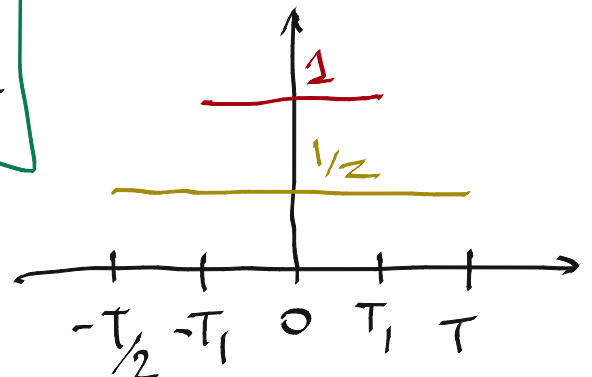
$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & T_1 < |t| < T/2 \end{cases}$$

→ cualquier período de largo T

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) \cdot e^{-jk\omega_0 t} dt = \frac{1}{T} \int_{-T_1}^{T_1} 1 \cdot e^{-jk\omega_0 t} dt \quad \begin{matrix} k \neq 0 \\ \downarrow \end{matrix} = \frac{1}{T} \cdot \frac{e^{-jk\omega_0 t}}{-jk\omega_0} \Big|_{-T_1}^{T_1}$$

$$a_k = \frac{1}{T} \cdot \left(\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2 \cdot jk\omega_0} \right) \cdot 2 = 2 \cdot \frac{\text{sen}(k\omega_0 T_1)}{k\omega_0 T}$$

$$k=0 \quad a_0 = \frac{1}{T} \int_{-T_1}^{T_1} 1 \cdot dt = \frac{2T_1}{T}$$



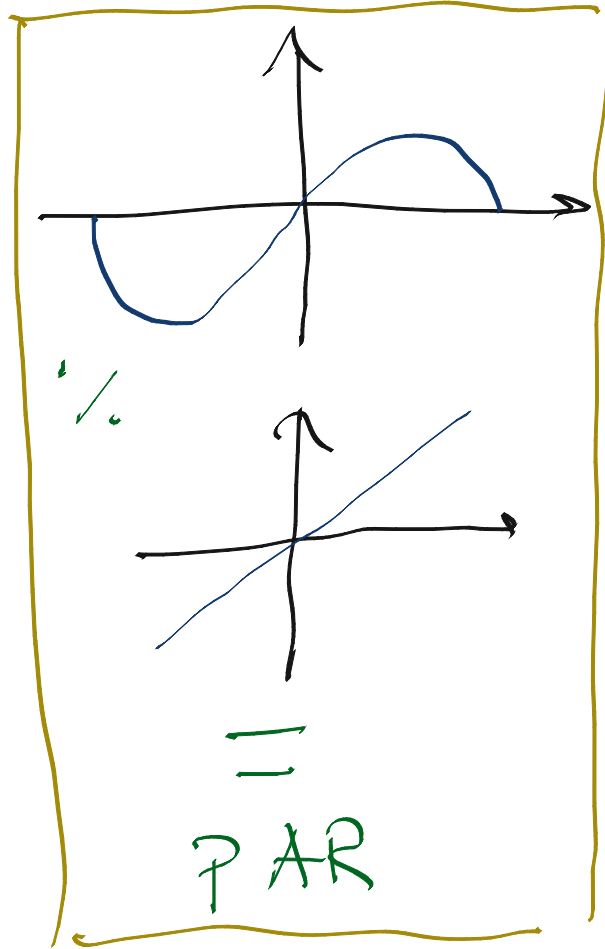
a) $T=1s$, $T_1=\frac{T}{4}$, $\omega_0=\frac{2\pi}{T}$, $\omega_0 T_1=\frac{2\pi}{T} \cdot \frac{T}{4} = \frac{\pi}{2}$

$$a_k = \frac{\frac{2}{T} \cdot \frac{\text{sen}\left(\frac{\pi}{2} \cdot k\right)}{k \cdot \frac{2\pi}{T}}}{\pi k} \rightarrow a_k = \frac{\text{sen}\left(\frac{\pi}{2} k\right)}{\pi k}$$

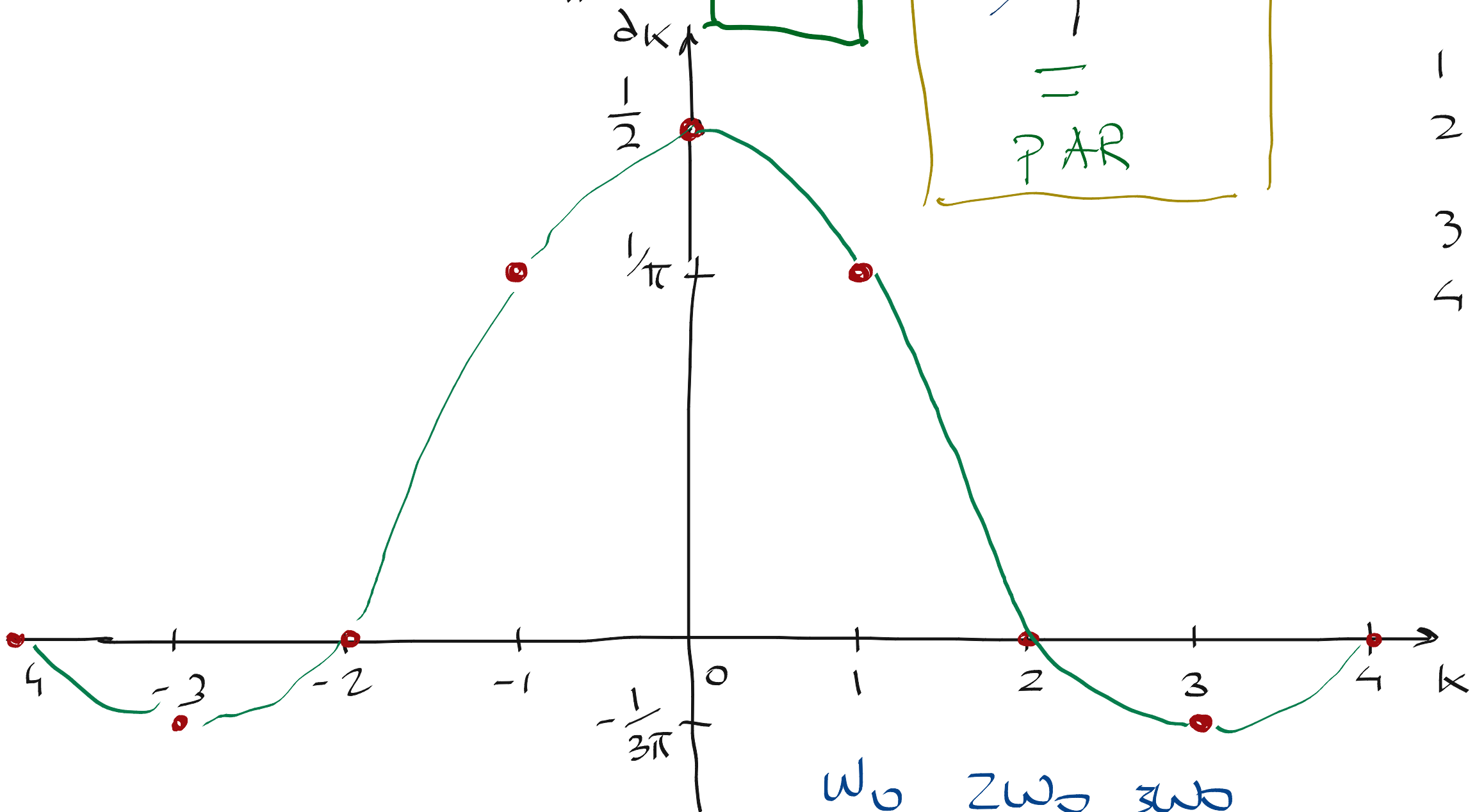
$$a_0 = \frac{2T_1}{T} = \frac{2 \cdot \frac{T}{4}}{T} = \frac{1}{2}$$

$\forall k \text{ par} \Rightarrow a_k = 0$

$$a_1 = \frac{1}{\pi} , a_3 = \frac{\text{sen}\left(\frac{3}{2}\pi\right)}{3\pi} = -\frac{1}{3\pi}$$



k	a _k
-4	0
-3	$-\frac{1}{3\pi}$
-2	0
-1	$\frac{1}{\pi}$
0	$\frac{1}{2}$
1	$-\frac{1}{\pi}$
2	0
3	$\frac{1}{3\pi}$
4	0



ω_0 $2\omega_0$ $3\omega_0$
 f_0 $2f_0$ $3f_0$
 1Hz 2Hz 3Hz

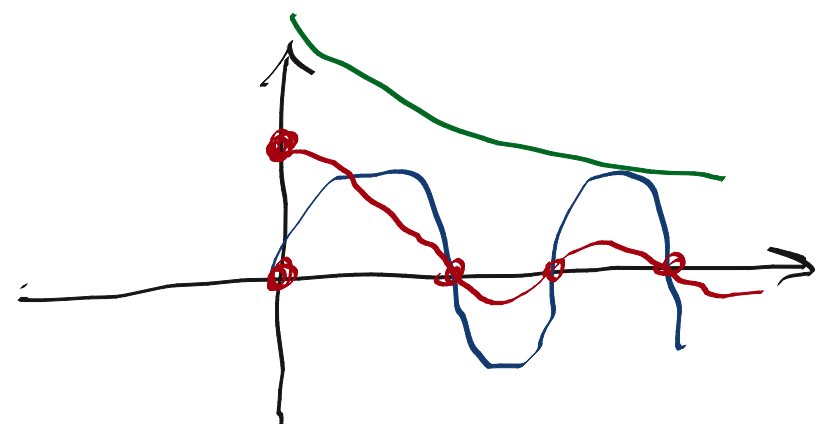
obs:

* función discreta: el verde es interpolado

* es par, discreta

* forma conocida:

$$\frac{\text{sen } x}{x}$$



Obs:

- * se anulan los coefs. pares $T=1s, f_0=1Hz$
- * las frecuencias son armónicas $\omega_k = k \cdot \omega_0$
- * el eje de frecuencias no tiene unidades
- * concentra energía en frecuencias bajas
- * es una señal "lenta" (termino relativo)
- * tiene continuidad ($\frac{1}{2}$)
- * Ancho de banda: "soporte en frecuencia"
 - concepto aproximado: ancho del primer lóbulo

→ $k=2 \Rightarrow \boxed{BW = 2Hz}$

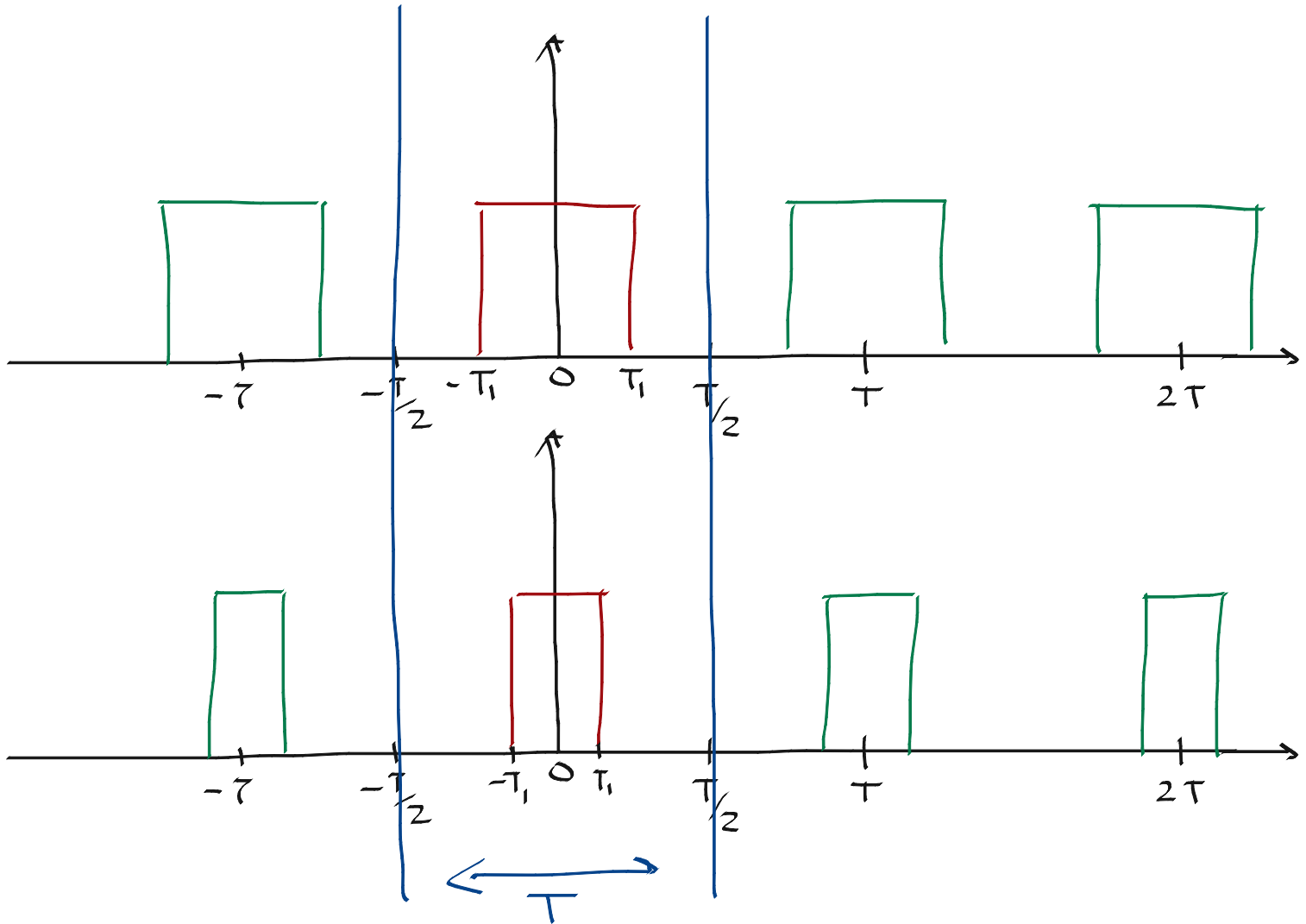
- * tanto el ancho de banda, como el eje de frecuencia son relativos a f_0 .

CLASE 11

Series de fourier

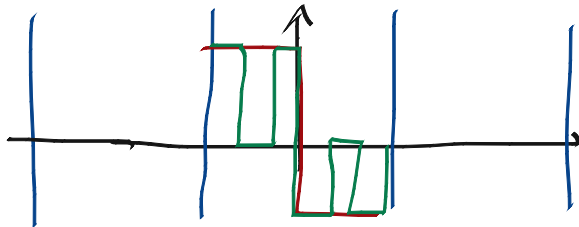
Convergencia

b) $T=1s$, $T_1=\frac{T}{8}$, $f_0=\frac{1}{T}=1Hz$



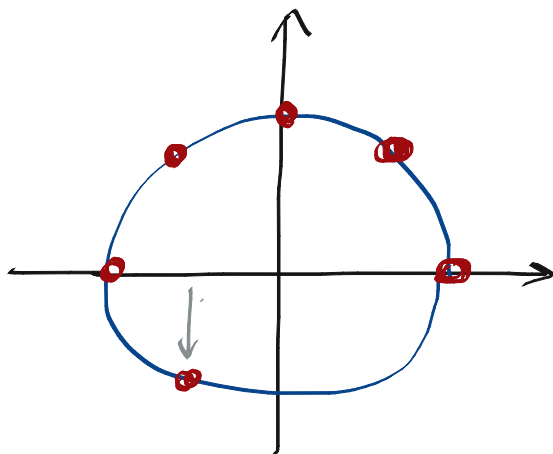
obs: "dos velocidades"

- * velocidad de repetición (periodicidad)
- * espectro:
 - los ingredientes de frecuencia.
 - la velocidad de cambio dentro de un período

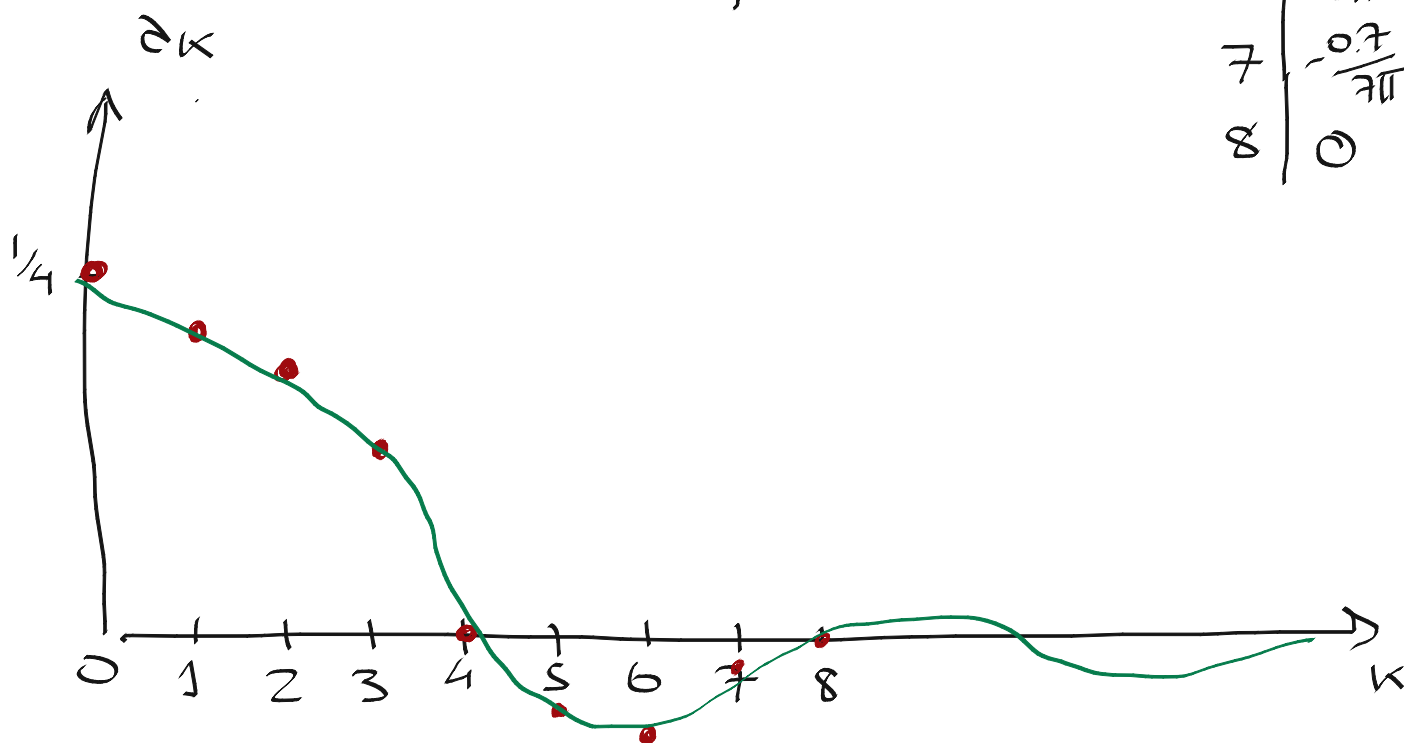


$$a_k = \frac{1}{\pi k} \cdot \sin\left(2\pi \left(\frac{T_1}{T}\right) k\right) = \boxed{\frac{\sin\left(\frac{\pi}{4} \cdot k\right)}{\pi k}}$$

$$a_0 = \frac{2T_1}{T} = \boxed{\frac{1}{4}}$$



k	a_k
0	$\frac{1}{4}$
1	$\frac{0.7}{\pi}$
2	$\frac{1}{2\pi}$
3	$\frac{0.7}{3\pi}$
4	0
5	$-\frac{0.7}{5\pi}$
6	$-\frac{1}{6\pi}$
7	$-\frac{0.7}{7\pi}$
8	0



obs:

* es una señal "lenta" (concentra energía en baja frec.)

* BW, $k=4 \Rightarrow \boxed{BW = 4 \cdot f_0 = 4 \text{ Hz}}$

$$BW = 4 \cdot 2\pi = 8\pi$$

Definición: $\text{senc}(x) = \begin{cases} \frac{\text{sen}(x)}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$

Obs: * $\lim_{x \rightarrow 0^{\pm}} \frac{\text{sen}(x)}{x} \approx \frac{x}{x} = 1$

* Continua.

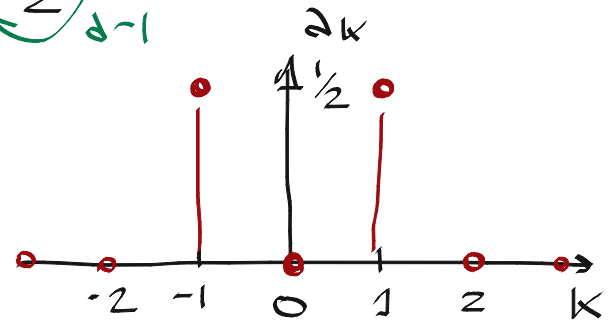
Obs: a) $a_k = \frac{1}{2} \cdot \frac{\text{sen}\left[\frac{\pi}{2}k\right]}{\frac{\pi}{2}k} = \boxed{\frac{1}{2} \cdot \text{senc}\left[\frac{\pi}{2}k\right]}$

b) $a_k = \frac{1}{4} \cdot \frac{\text{sen}\left[\frac{\pi}{4}k\right]}{\frac{\pi}{4}k} = \boxed{\frac{1}{4} \cdot \text{senc}\left[\frac{\pi}{4}k\right]}$

Ejemplo: "coseno"

$$x(t) = \cos(\omega_0 t) = \left(\frac{1}{2}\right)_{a_1} e^{j\omega_0 t} + \left(\frac{1}{2}\right)_{a_{-1}} e^{-j\omega_0 t}$$

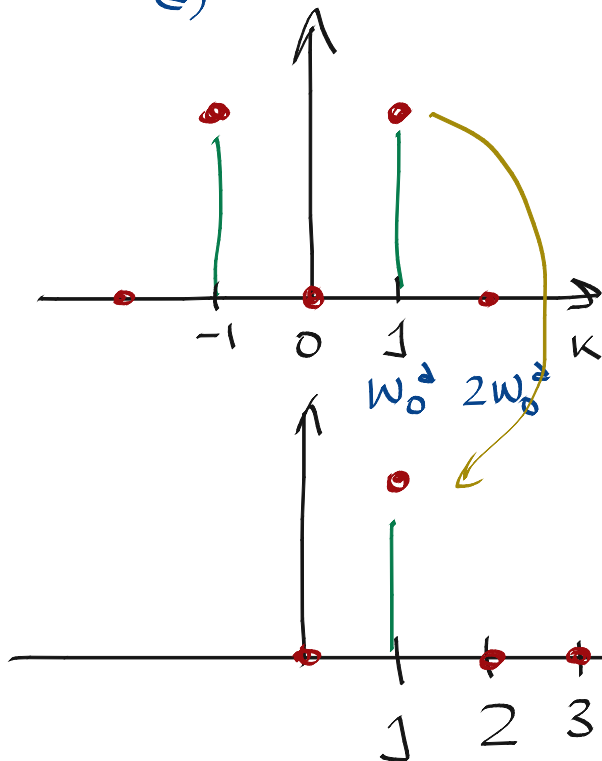
$$a_k = \begin{cases} \frac{1}{2} & k = \pm 1 \\ 0 & \text{e.o.c.} \end{cases}$$



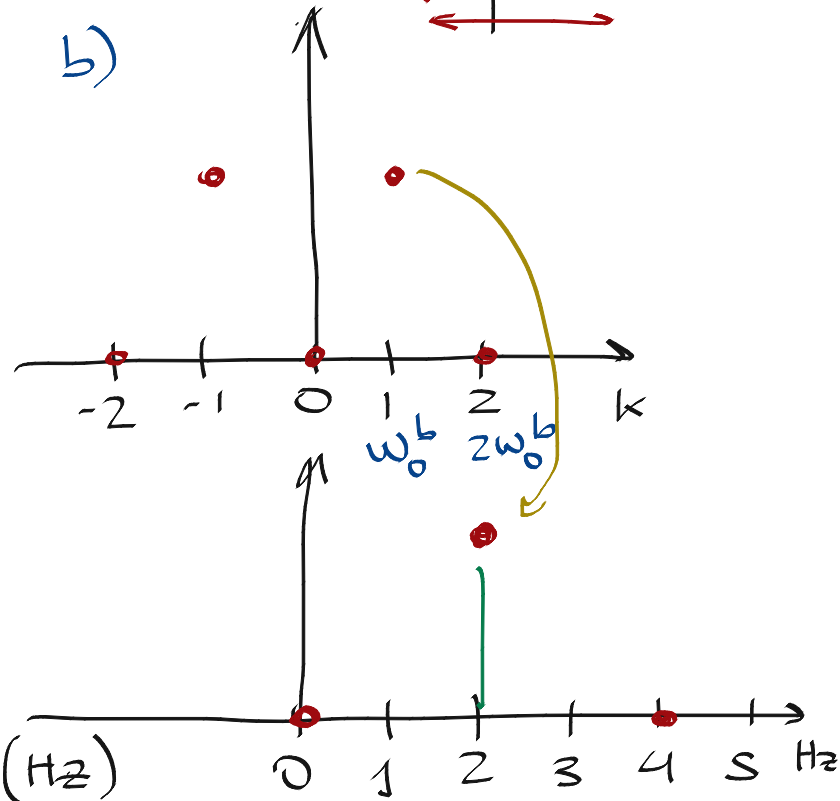
a) $x_1(t) = \cos(\overbrace{2\pi \cdot 1 \cdot t}^{\omega_0})$

b) $x_2(t) = \cos(\underbrace{2\pi \cdot 2 \cdot t}_{\omega_0})$

a)



b)



OLS: * tienen el "mismo" espectro

* $a_k \in \mathbb{R}$

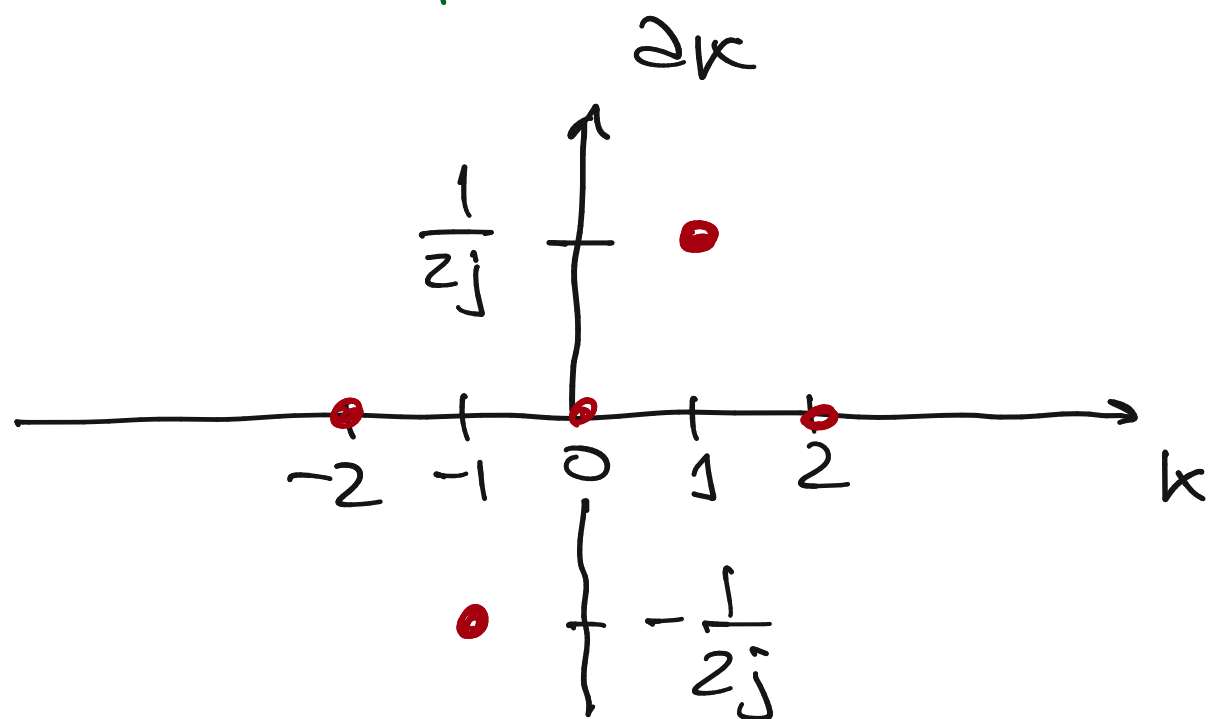
* a_k son relativos a ω_0 !!

* $a_0 = 0 \Rightarrow$ el coseno no tiene continua.

Ejemplo 3.3 : (p 187) "seno"

$$x(t) = \sin(\omega_0 t) = \underbrace{\frac{1}{2j}}_{a_1} \cdot e^{j\omega_0 t} - \underbrace{\frac{1}{2j}}_{a_{-1}} \cdot e^{-j\omega_0 t}$$

$$a_k = \begin{cases} \frac{1}{2j} & k=1 \\ -\frac{1}{2j} & k=-1 \\ 0 & \text{e.o.c.} \end{cases}$$



obs:

* $a_k \in \mathbb{C}$

* no puedo graficar así nomás.

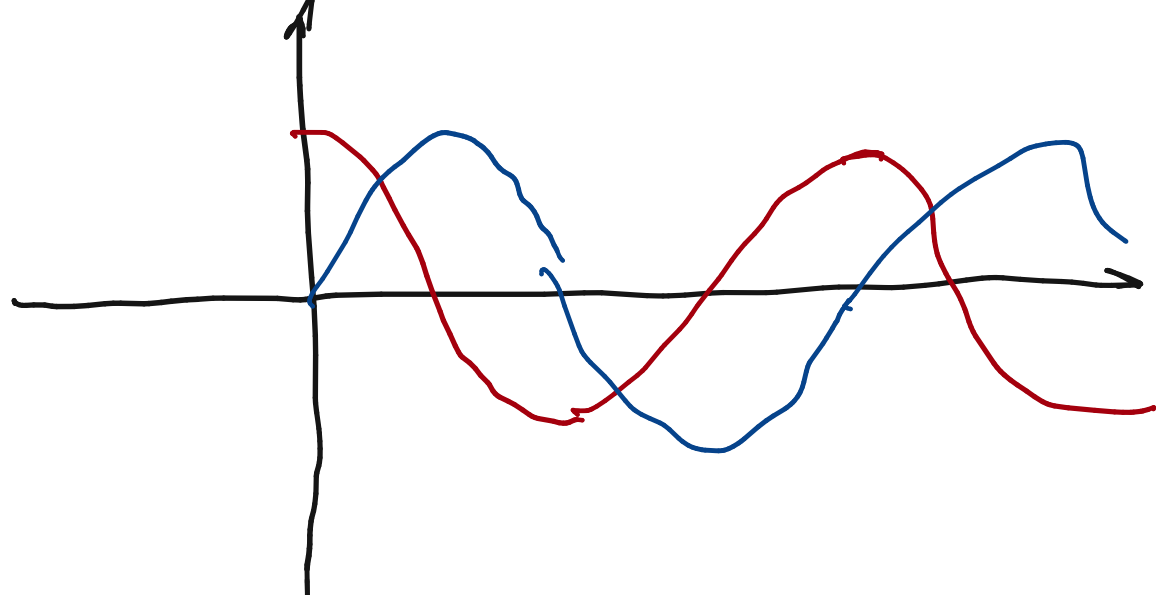
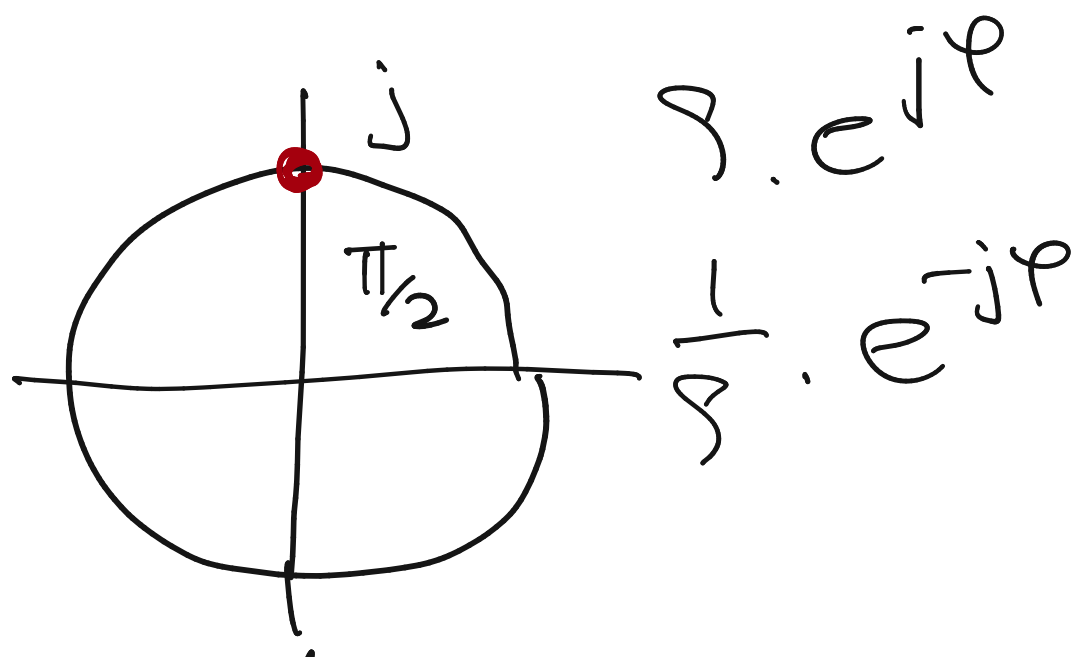
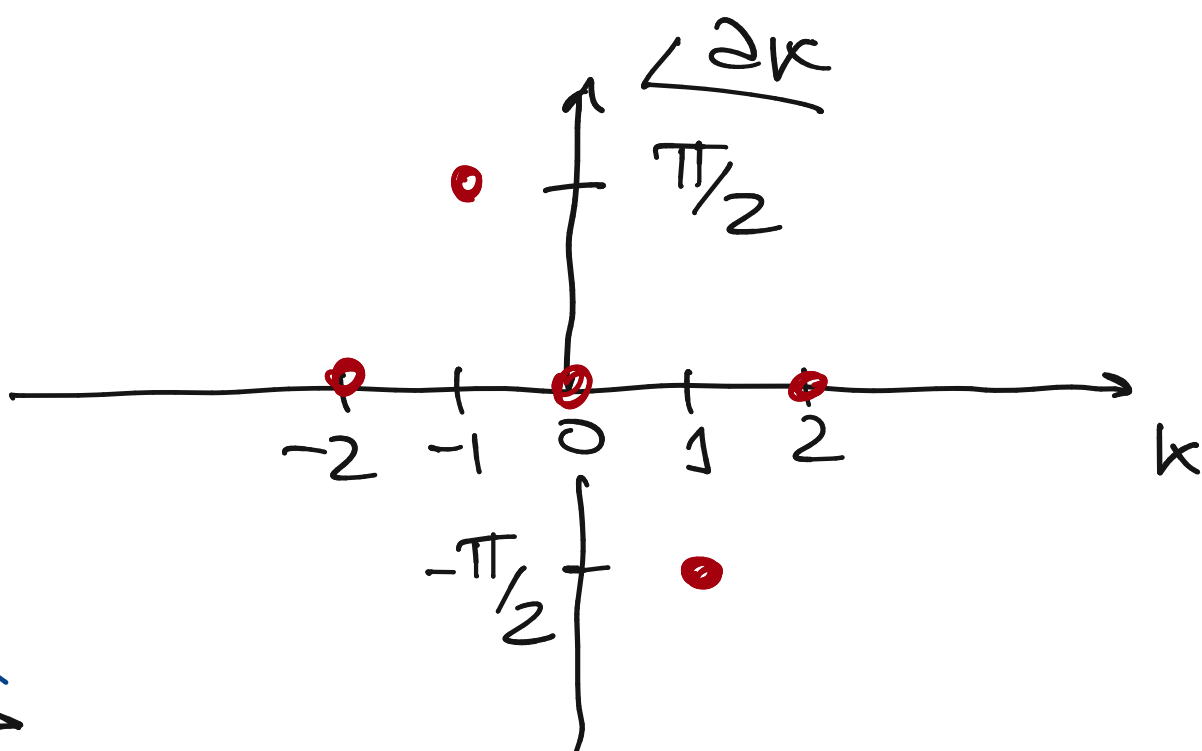
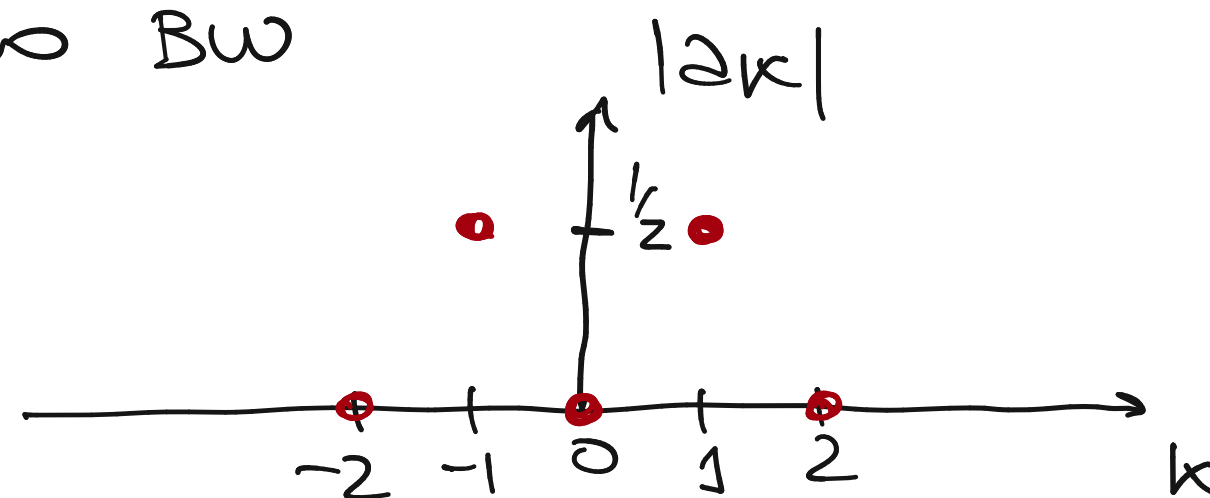
* tengo que mirar módulo y fase.

* tienen el mismo BW

* módulo: velocidad

* fase: posición

$$a_k: \mathbb{Z} \rightarrow \mathbb{R}^2$$



CONVERGENCIA

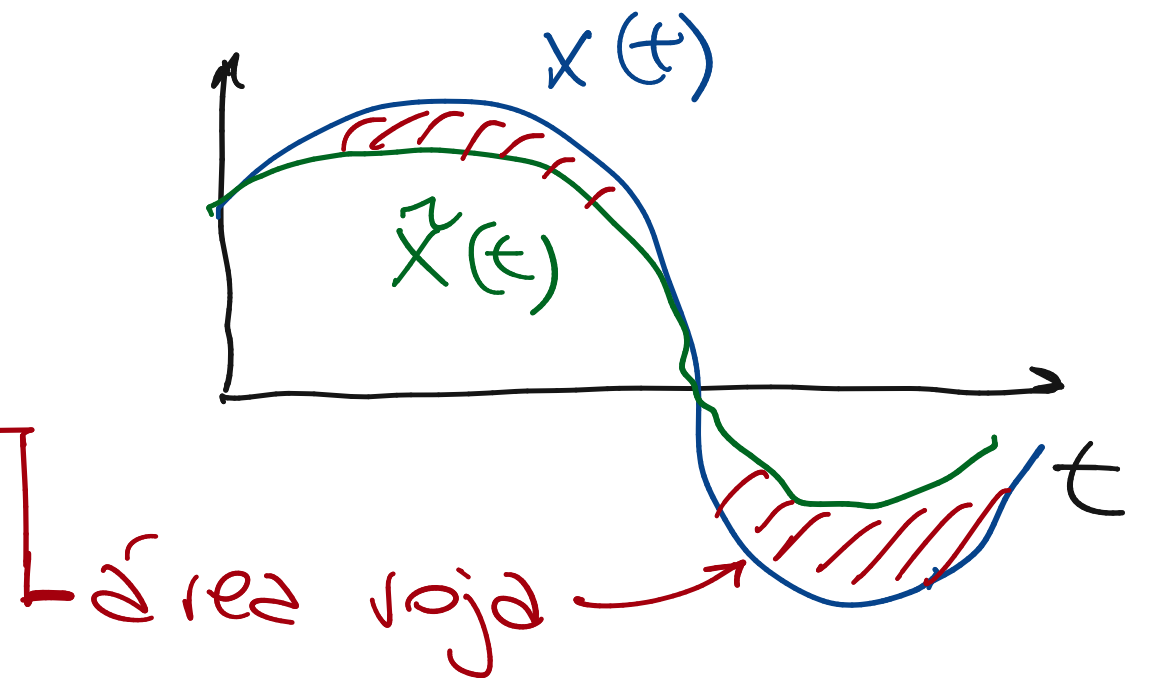
$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k \cdot e^{j k \omega_0 t}$$

$$\tilde{x}_N(t) = \sum_{k=-N}^N a_k \cdot e^{j k \omega_0 t}$$

$$x(t) = \lim_{N \rightarrow \infty} \tilde{x}_N(t), \text{ converge?} \quad \tilde{x}(t) = x(t)?$$

$$e_N(t) = x(t) - \tilde{x}_N(t)$$

$$E_N^2 = \int_T |e_N(t)|^2 dt$$



$$\exists \text{ representación}, \quad \lim_{N \rightarrow \infty} E_N = 0$$

CONVERGENCIA

Condición 1

$$\textcircled{H} \quad \int_T |x(t)|^2 \cdot dt < \infty$$

$\textcircled{T}$

1) $\exists a_k$.

2) $\hat{x}(t) \rightarrow x(t)$

Dem:

$$\begin{aligned} 1) \quad |a_k|^2 &= \left| \frac{1}{T} \cdot \int_T x(t) \cdot e^{-jk\omega_0 t} \cdot dt \right|^2 \\ &\leq \frac{1}{T^2} \cdot \int_T \underbrace{|x(t) \cdot e^{-jk\omega_0 t}|^2}_{=1} \cdot dt = \frac{1}{T^2} \cdot \underbrace{\int_T |x(t)|^2 dt}_{\rightarrow \infty} \end{aligned}$$

Obs:

* Garantía

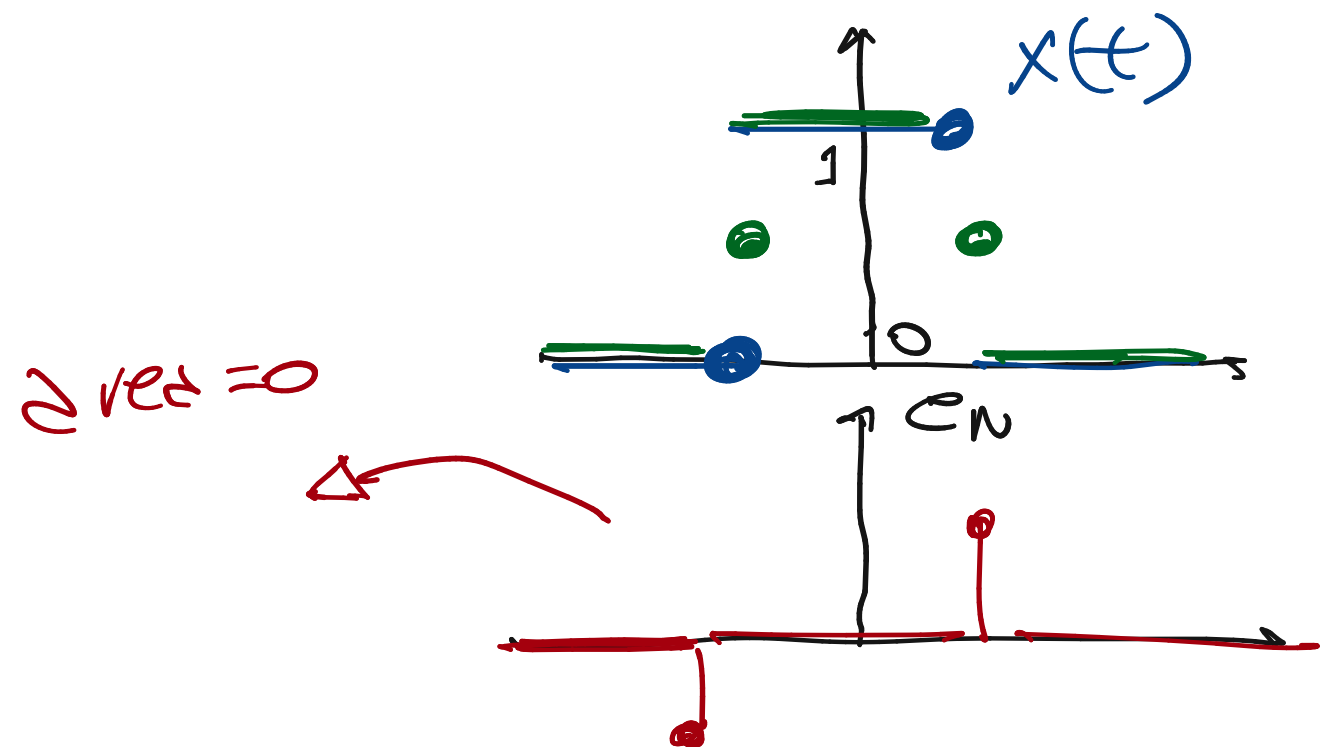
* Energía finita. $\rightarrow$ mayoría de las señales reales.

* Necesaria.

* Conservadora. pero fácil

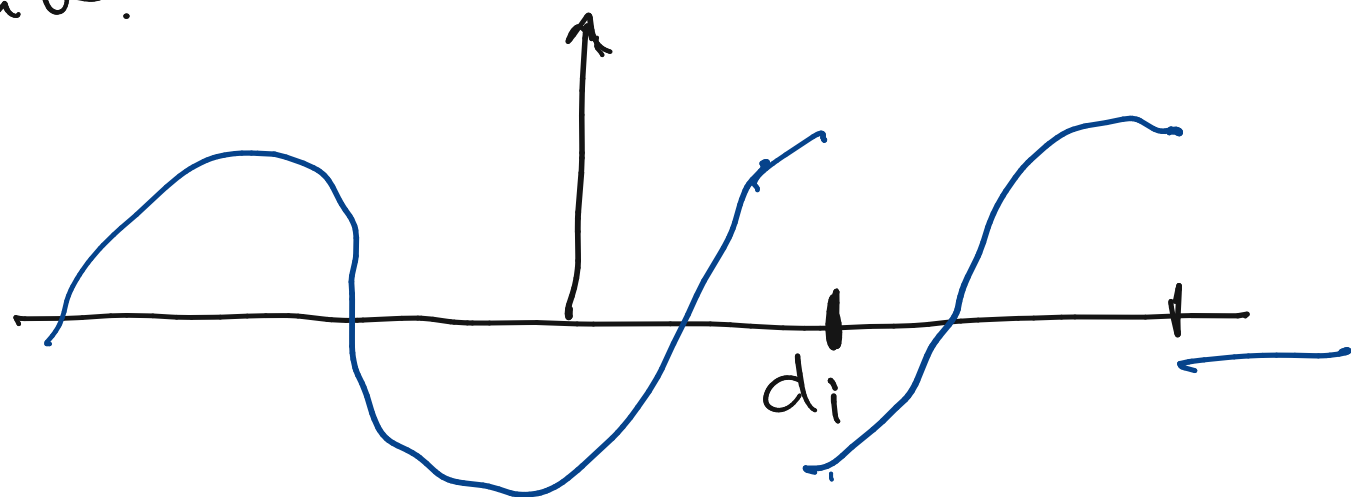
Condiciones de Dirichlet

- * $\tilde{x}(t)$ es igual a $x(t)$ en casi todo punto
- * salvo en las discontinuidades donde converge al punto medio

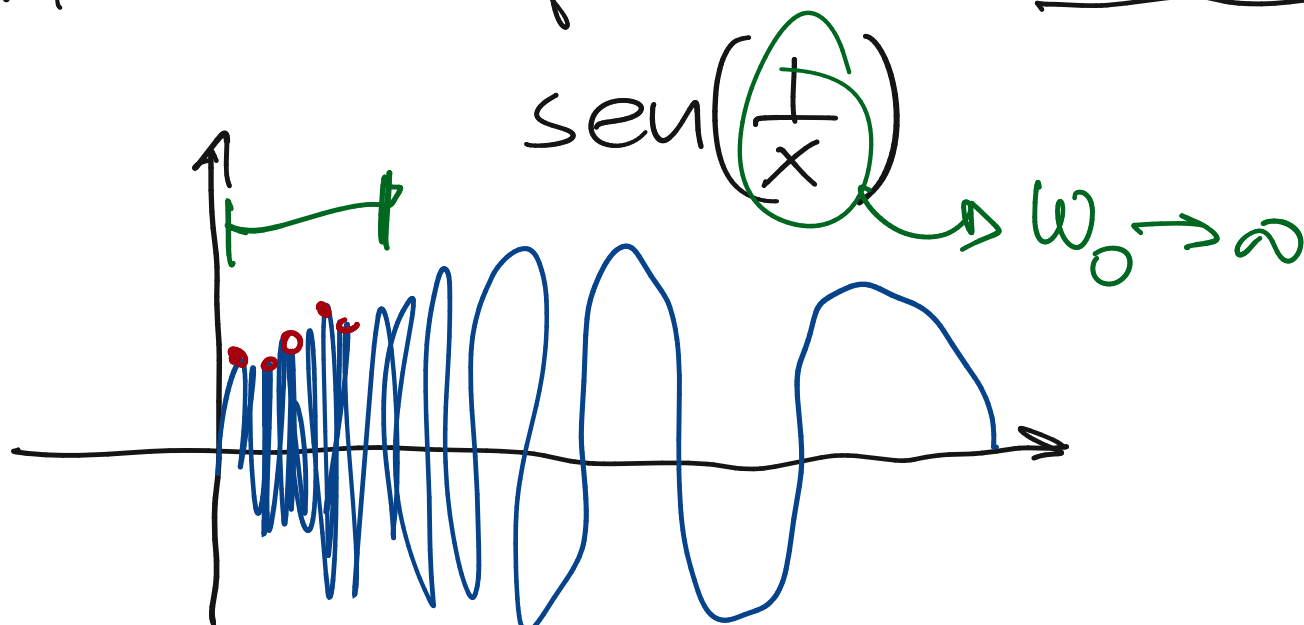
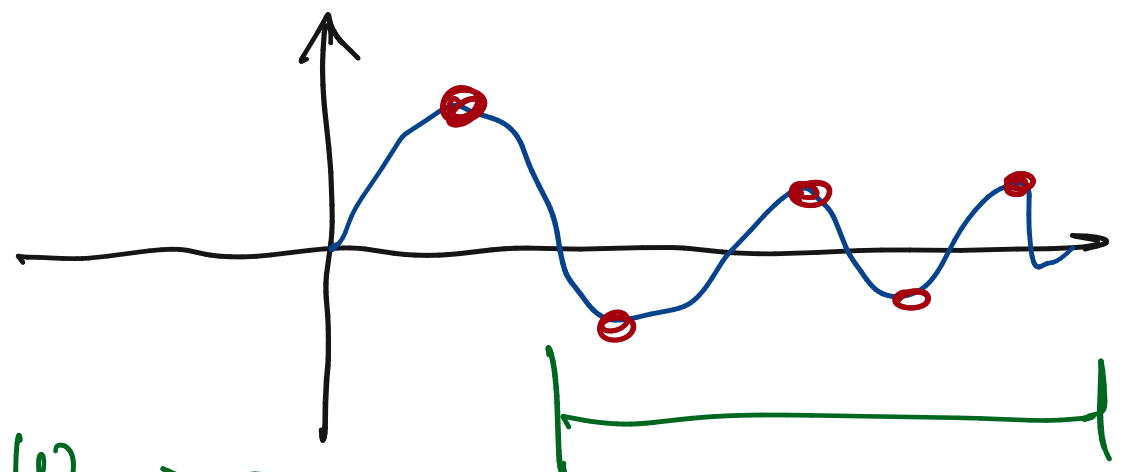


1) $\int_T |x(t)| < \infty$ (menos que energía finita)

2) Conjunto finito de discontinuidades finitas en un intervalo finito.



3) número finito de extremos relativos en un intervalo finito



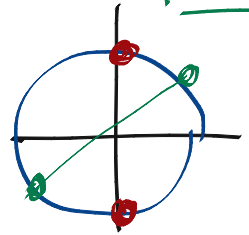
Ejemplo 3.5:

$$\tilde{X}(T_1) = \sum_{k=-\infty}^{\infty} \frac{1}{2} \cdot \text{sinc}\left(\frac{\pi}{2}k\right) \cdot e^{jk\omega_0 T_1} \quad \begin{matrix} T_1 = \frac{T}{4} \\ \downarrow \\ = \end{matrix}$$

$$\tilde{X}(T_1) = \sum_{k=-\infty}^{\infty} \frac{1}{2} \cdot \text{sinc}\left(\frac{\pi}{2}k\right) \cdot e^{jk \cdot \frac{2\pi}{T} \cdot \frac{T}{4}}$$

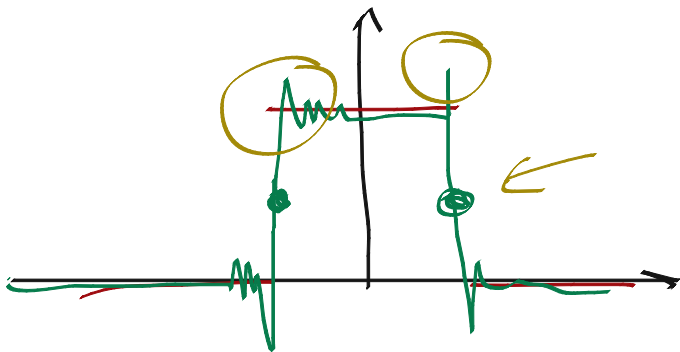
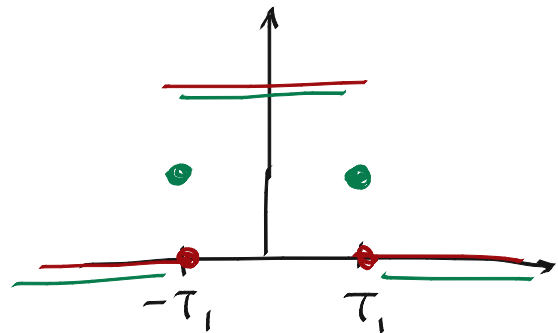
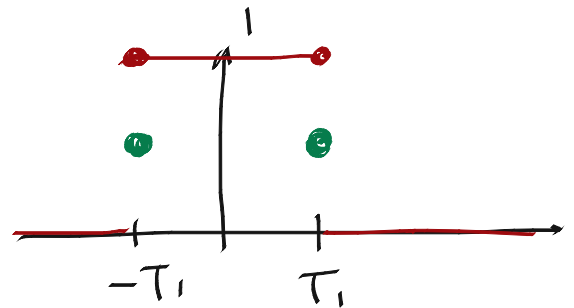
$$\tilde{X}(T_1) = \frac{1}{2} \left(1 + \underbrace{\frac{1}{\pi} \cdot e^{j\frac{\pi}{2}}}_{a_1} + \underbrace{\frac{1}{\pi} \cdot e^{-j\frac{\pi}{2}}}_{a_{-1}} + \dots \right) = \frac{1}{2}$$

se anulan



obs: *

- * fenómeno de Gibbs
- * oscilaciones
- * nunca se arregla.



PASOS PARA CONSTRUIR UNA HERRAMIENTA



- I dea
- D efinición
- E jemplos (abstractos, cuentas)
- C onvergencia (garantía)
- P ropiedades (ahorrar cuentas)
- E jemplos (aplicar propiedades)
- A plicaciones (sistemas y señales reales)

funciones
(abstracto)

Optimalidad de los coeficientes

$$\varepsilon(a_k) = \frac{1}{T} \int_T \left(x(t) - \sum_{k=-\infty}^{\infty} a_k \cdot \phi_k(t) \right)^2 dt$$

$$\frac{d \varepsilon(a_k)}{d a_k} = \frac{2}{T} \int_T \left[\left(x(t) - a_k \phi_k(t) \right) \right] \phi_k(t) dt = 0$$

$$\int_T x(t) \cdot \phi_k(t) dt = \int_T a_k \phi_k^2(t) dt = a_k \int_T \phi_k^2(t) dt$$

$$a_k = \frac{\int_T x(t) \cdot \phi_k(t) dt}{\int_T \phi_k^2(t) dt} = \frac{\langle x(t), \phi_k(t) \rangle}{\|\phi_k(t)\|^2} = \uparrow$$

$\|\phi_k(t)\| = 1$

$a_k = \langle x(t), \phi_k(t) \rangle$

+ fórmula
proyección

Optimalidad instantánea no
está dada

$$\frac{d^2 \varepsilon}{d a_k^2} = \frac{2}{T} \int_T \phi_k^2(t) dt \geq 0 \Rightarrow \exists \text{ máximo } \checkmark$$

CLASE 12

Series de Fourier

Propiedades y ejemplos

PROPIEDADES

Linealidad

(R)

$$\begin{array}{ccc} x(t) & \xrightarrow{\mathcal{F}} & a_k \\ y(t) & \xrightarrow{\mathcal{F}} & b_k \\ z(t) & \xrightarrow{\mathcal{F}} & c_k \end{array}$$

$$z(t) = A \cdot x(t) + B \cdot y(t)$$

(T)

$$z(t) \xrightarrow{\mathcal{F}} \underbrace{A \cdot a_k + B \cdot b_k}_{c_k}$$

Obs: $x(t), y(t)$ periódicas $\Rightarrow x(t) + y(t)$ periódica

Desplazamiento en el tiempo

$$x(t) \xrightarrow{\mathcal{F}} a_k$$

$$x(t-\tau) \xrightarrow{\mathcal{F}} b_k = ?$$

$$t' = t - \tau \begin{cases} t \in [0, T] \\ t' \in [-\tau, T-\tau] \end{cases}$$

$$b_k = \frac{1}{T} \int_{\langle T \rangle} x(t-\tau) \cdot e^{-jk\omega_0 t} \cdot dt = \frac{1}{T} \int_{\langle T \rangle} x(t') \cdot e^{-jk\omega_0(t'+\tau)} \cdot dt'$$

$$b_k = \frac{1}{T} \int_{\langle T \rangle} x(t') \cdot e^{-jk\omega_0 t'} \cdot \underbrace{e^{-jk\omega_0 \tau}}_{cte} \cdot dt'$$

$$= \frac{1}{T} \cdot e^{-jk\omega_0 \tau} \int_{\langle T \rangle} x(t') \cdot e^{-jk\omega_0 t'} \cdot dt' = \underbrace{\frac{1}{T} \int_{\langle T \rangle} x(t') \cdot e^{-jk\omega_0 t'} \cdot dt'}_{a_k} \cdot e^{-jk\omega_0 \tau}$$

$$= e^{-jk\omega_0 \tau} \cdot a_k$$

Obs: * $|b_k| = |a_k|$
* Cambia la fase

* Cambia posición y no la velocidad.

Inversión temporal

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \cdot e^{jk\omega_0 t}$$

$$x(t) \xleftrightarrow{\mathcal{F}} b_k = ?$$

$$x(-t) = \sum_{k=-\infty}^{\infty} a_k \cdot e^{-jk\omega_0 t} = \sum_{k=-\infty}^{\infty} b_k \cdot e^{jk\omega_0 t}$$

$$\dots + \underbrace{a_{-1}}_{\text{green}} e^{j\omega_0 t} + \underbrace{a_0}_{\text{blue}} + \underbrace{a_1}_{\text{red}} e^{-j\omega_0 t} + \dots = \dots + \underbrace{b_{-1}}_{\text{red}} e^{-j\omega_0 t} + \underbrace{b_0}_{\text{blue}} + \underbrace{b_1}_{\text{green}} e^{j\omega_0 t} + \dots$$

$$\boxed{b_k = a_{-k}}$$

Corolario:

$$* \quad x(t) \text{ par} \Rightarrow x(t) = x(-t)$$

$$\begin{array}{c} \updownarrow \quad \updownarrow \\ \boxed{a_k = a_{-k}} \Rightarrow \boxed{a_k \text{ es par}} \end{array}$$

$$* \quad x(t) \text{ es impar} \Rightarrow \boxed{a_k \text{ es impar}}$$

Conjugación

$$x^*(t) \leftrightarrow d_{-k}^*$$

Caracterios

1) $x(t)$ real $\Rightarrow x(t) = x^*(t)$

$$\boxed{d_k = d_{-k}^*} \Rightarrow A_k \cdot e^{j\varphi_k} = A_{-k} \cdot e^{-j\varphi_{-k}}$$

$$d_k = A_k \cdot e^{j\varphi_k}$$

$$d_k^* = A_k \cdot e^{-j\varphi_k}$$

$$\Rightarrow \boxed{\begin{aligned} A_k &= A_{-k} \Rightarrow \text{módulo es par} \\ \varphi_k &= -\varphi_{-k} \Rightarrow \text{fase es impar} \end{aligned}}$$

2) $x(t)$ real y par $\Rightarrow d_k$ reales y pares

3) $x(t)$ real e impar $\Rightarrow d_k$ imaginarios puros e impares
 $d_0 = 0$

Escalaamiento temporal

$$x(at) \xleftrightarrow{\mathcal{F}} b_k = ?$$

$$x(at) = \sum_{k=-\infty}^{\infty} \underline{a_k} \cdot e^{j\underline{k\omega_0} \underline{at}}_{\omega_1}$$

$$x(t) = \sum_{k=-\infty}^{\infty} \underline{a_k} \cdot e^{j\underline{k\omega_0} \underline{t}}_{\omega_1}$$

obs: * los coeficientes son los mismos: $b_k = a_k$
* cambia la frecuencia fundamental: $\omega_1 = a\omega_0$

Ejemplo

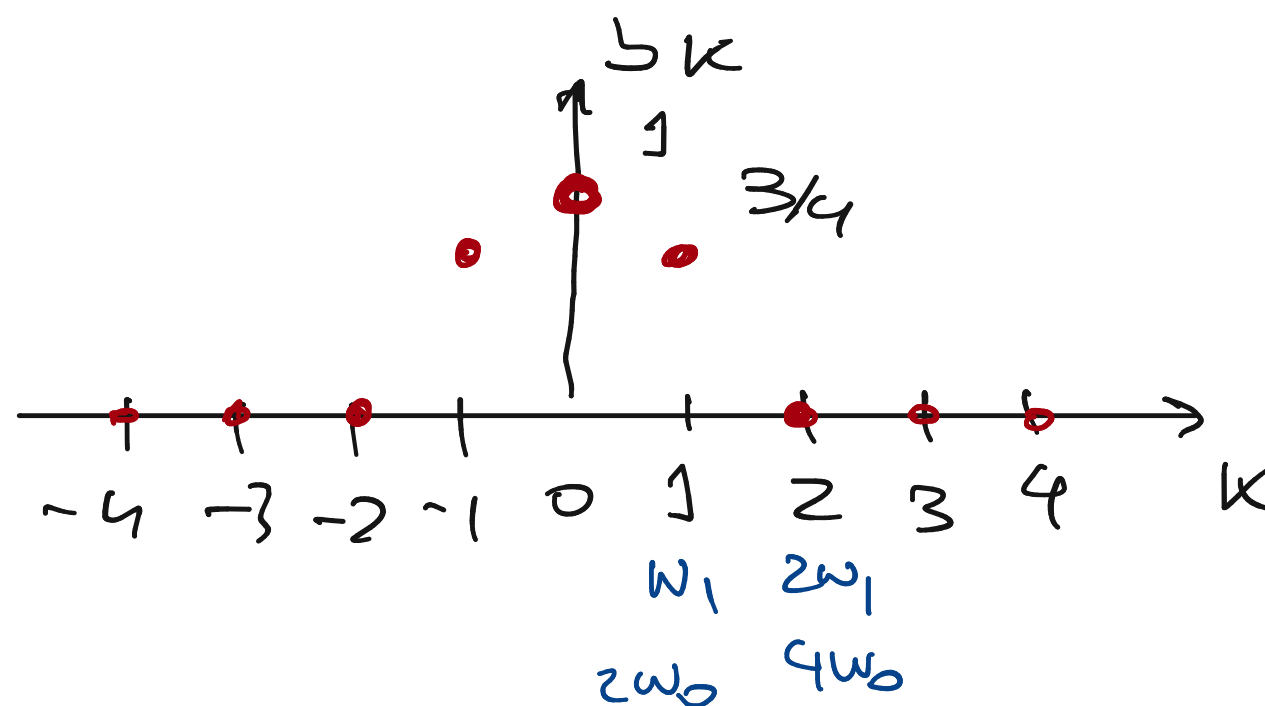
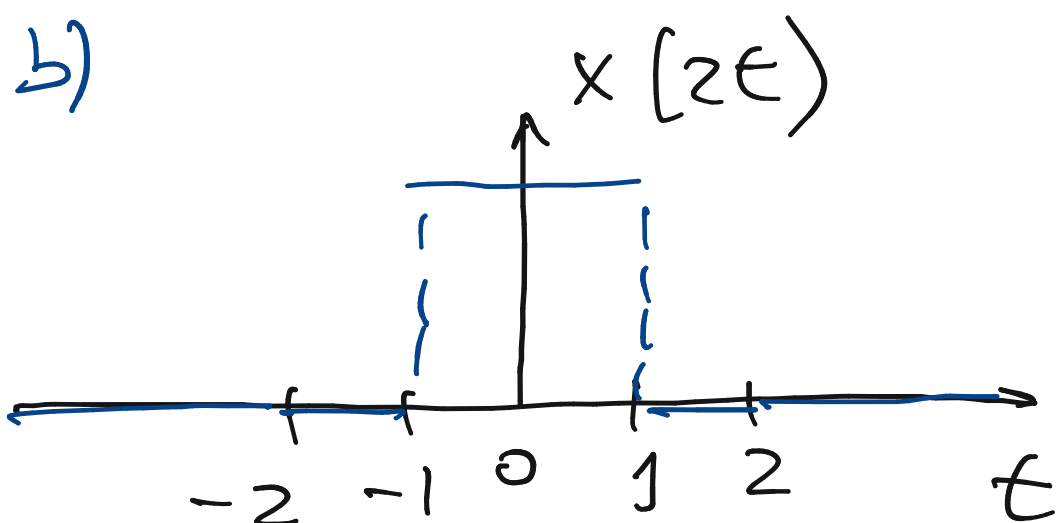
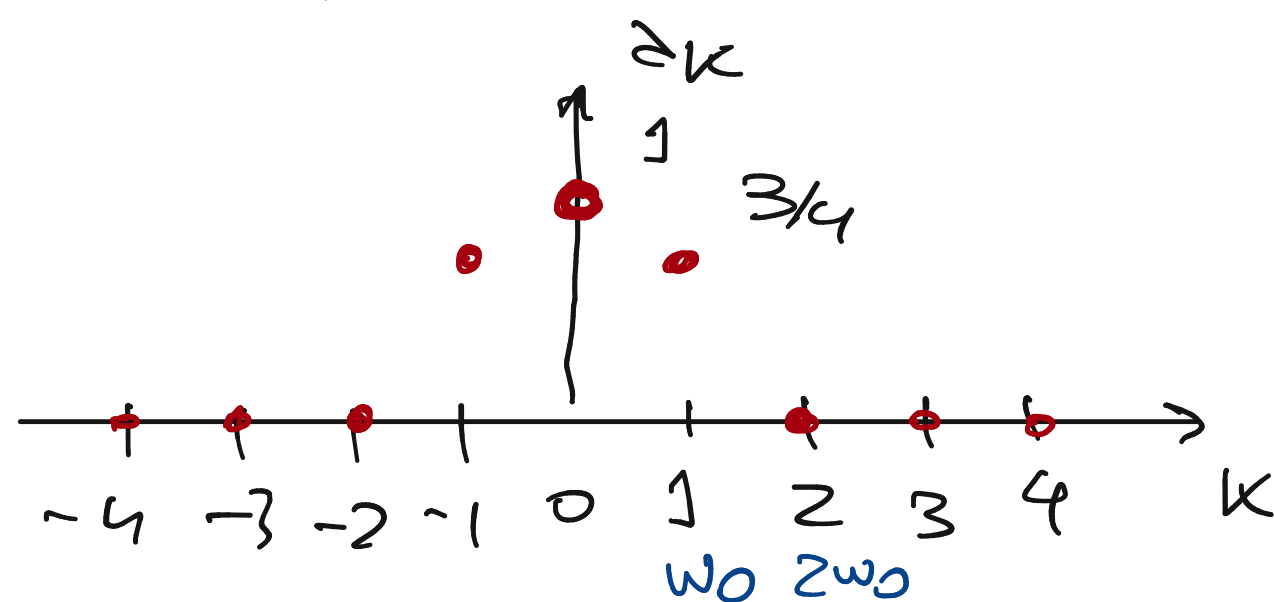
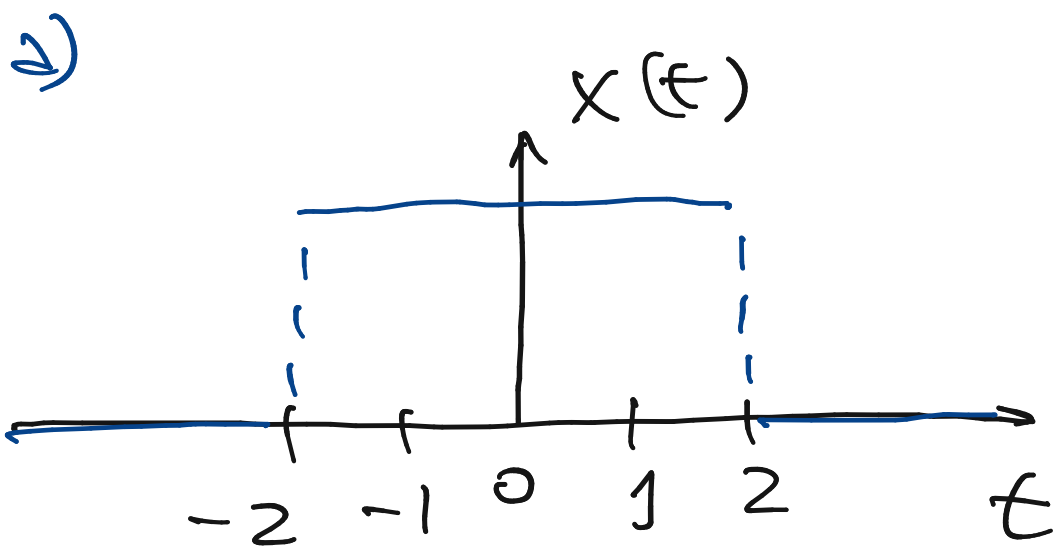
$$\Delta = 2$$

$$X(2t)$$

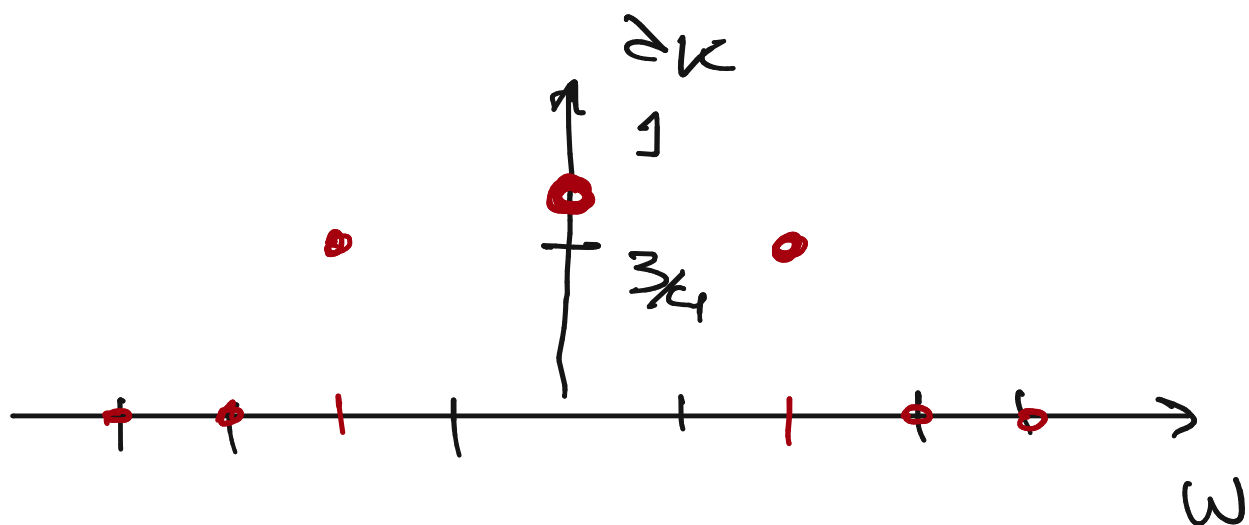
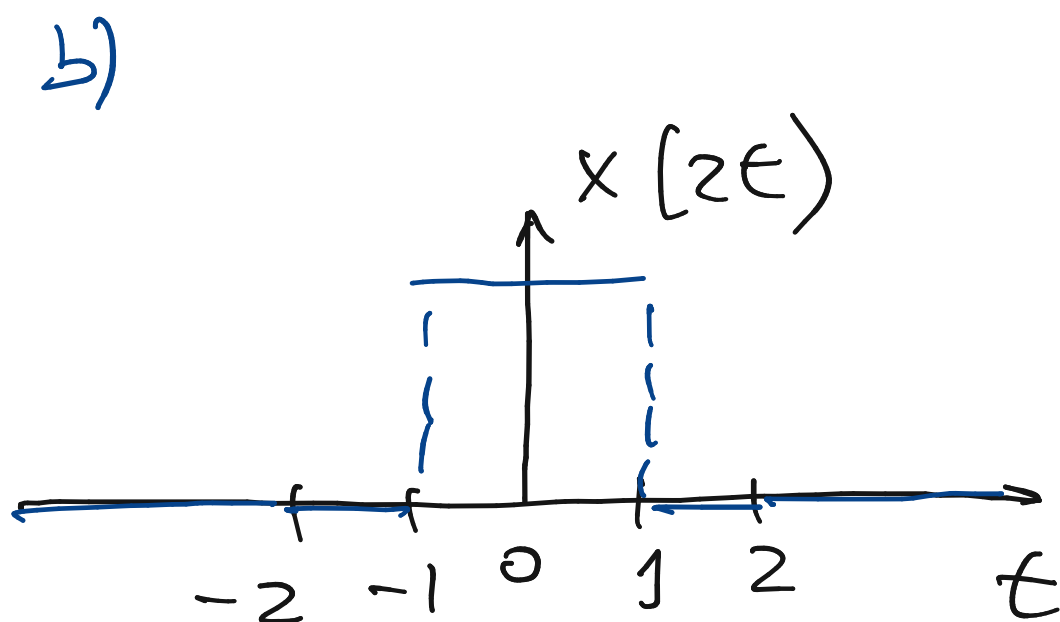
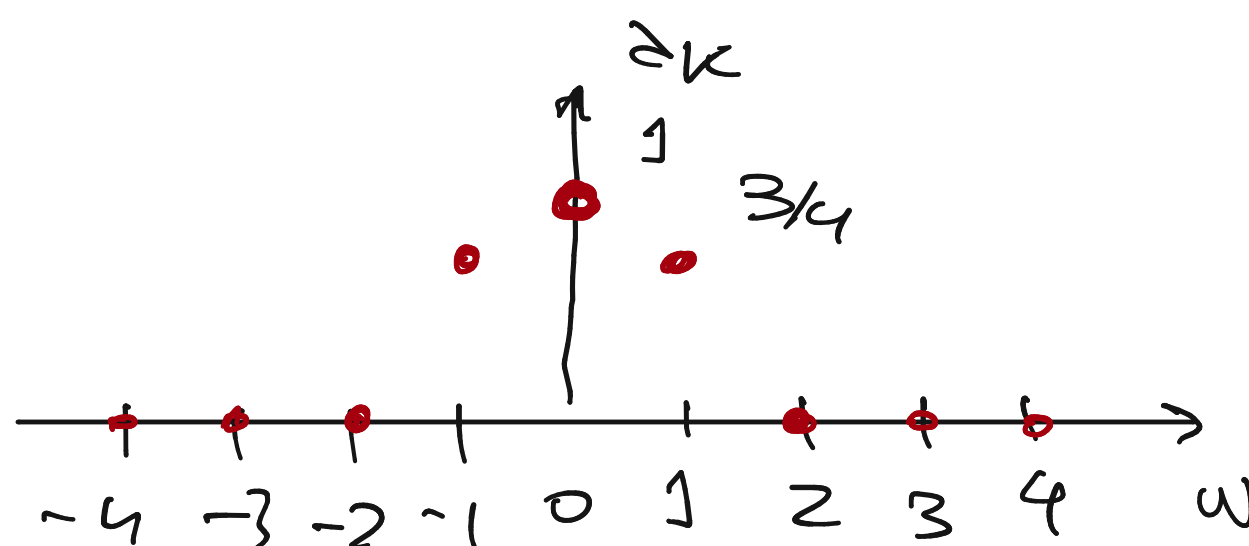
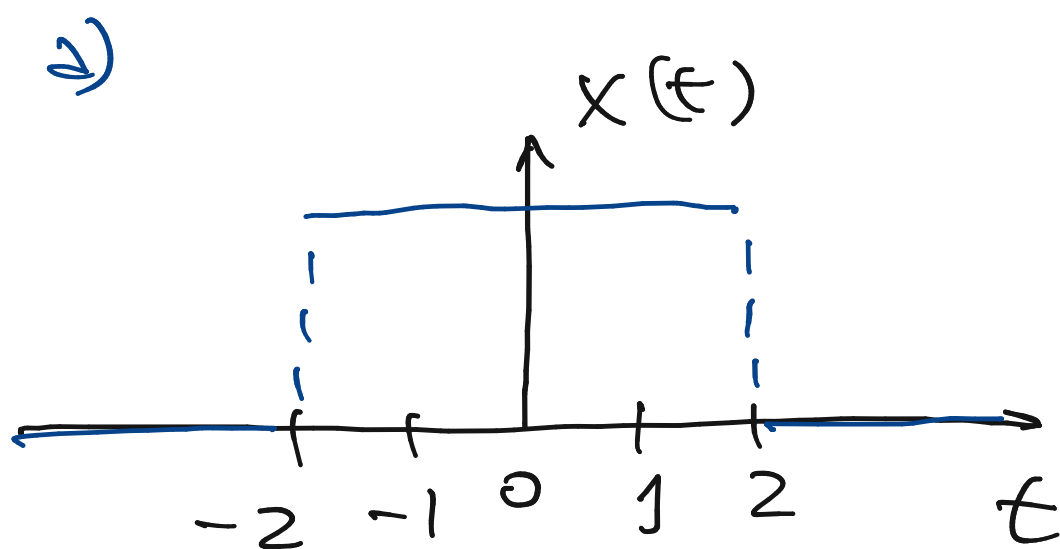
$$\omega_1 = 2 \cdot \omega_0 \Rightarrow \omega_1 = 2\omega_0$$

tiempo

frecuencia



[repetido pero con unidades]



Multiplicación (modulación)

$$x(t) \xrightarrow{\mathcal{F}} a_k, \quad y(t) \xrightarrow{\mathcal{F}} b_k \Rightarrow c(t) = x(t) \cdot y(t) \xrightarrow{\mathcal{F}} c_k = ?$$

$$c(t) = \sum_{k=-\infty}^{\infty} a_k \cdot e^{jk\omega_0 t} \cdot \sum_{n=-\infty}^{\infty} b_n \cdot e^{jn\omega_0 t}$$

$$c(t) = \sum_k \sum_n a_k \cdot b_n \cdot e^{jk\omega_0 t} \cdot e^{jn\omega_0 t} = \sum_n \sum_k a_k \cdot b_n \cdot e^{j(k+n)\omega_0 t} \Rightarrow$$

$$\begin{aligned} n &= k - m \\ \uparrow \\ m &= k + n \end{aligned}$$

$$c(t) = \sum_m \underbrace{\sum_k a_k \cdot b_{m-k}}_{c_m} \cdot e^{j\underline{m} \cdot \underline{\omega_0 t}}$$

$$c(t) = \sum_{m=-\infty}^{\infty} c_m \cdot e^{jm\omega_0 t}$$

$$c_m = \sum_{k=-\infty}^{\infty} a_k \cdot b_{m-k}$$

convolución discreta

$$c_m = a_m * b_m$$

Identidad de Parseval

$$\underbrace{\frac{1}{T} \int_{\langle T \rangle} |x(t)|^2 dt}_{\text{Energía calculada en el tiempo}} = \underbrace{\sum_{k=-\infty}^{\infty} |a_k|^2}_{\text{Energía calculada en frecuencia}}$$

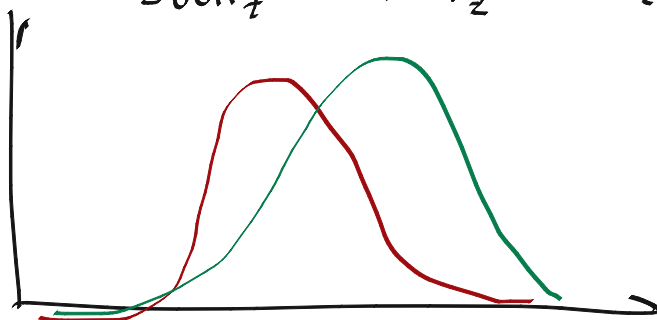
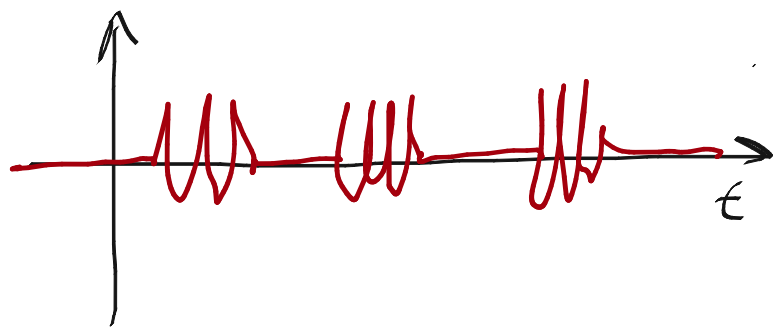
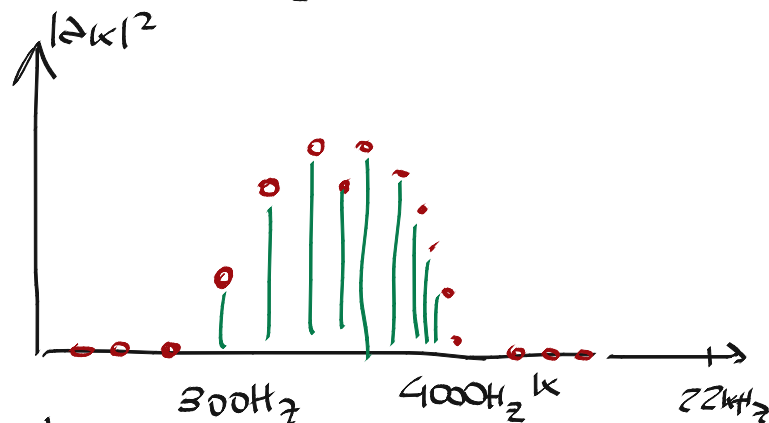
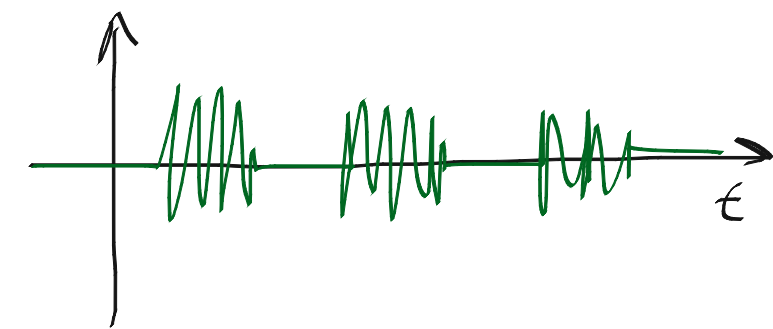
Energía calculada
en el tiempo.

Energía calculada
en frecuencia

obs:

* permite calcular el valor total de la energía en ambos dominios

* de otra visión del reparto de energía



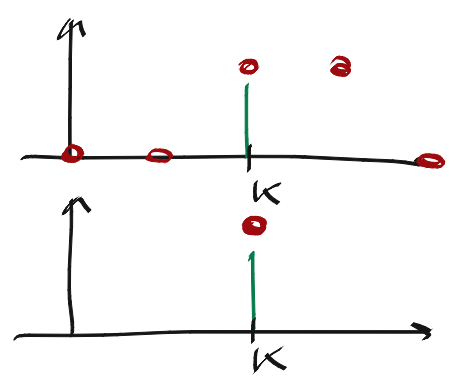
Ejemplo: "energía de un coeficiente"

$$X_k(t) = a_k \cdot e^{jk\omega_0 t}$$

$$E_k = \frac{1}{T} \int_{\langle T \rangle} |a_k \cdot \underbrace{e^{jk\omega_0 t}}_{\text{"}}|^2 dt$$

$$= \frac{1}{T} \int_{\langle T \rangle} |a_k|^2 dt = |a_k|^2 \cdot \underbrace{\frac{1}{T} \int_{\langle T \rangle} 1 \cdot dt}_{\text{"}} = |a_k|^2$$

$$E_k = |a_k|^2$$



Parseval

No se da!!

$$\frac{1}{T} \int_{\langle T \rangle} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

Dem:

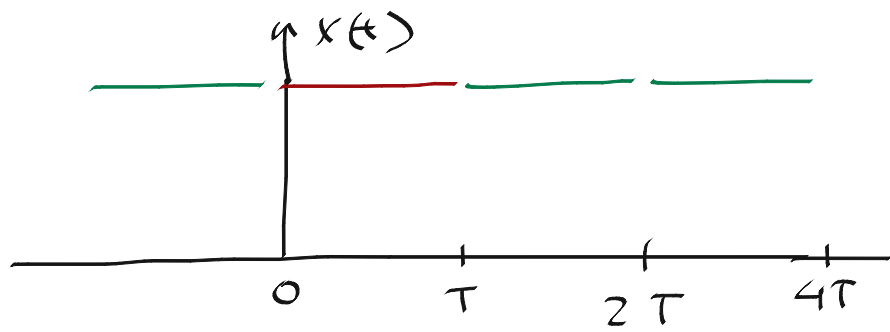
$$x(t) \cdot x^*(t) \xleftrightarrow{\mathcal{F}} \underbrace{a_k * a_{-k}^*}_{\substack{C_k \\ \text{valor medio}}} \leftarrow \begin{array}{l} \text{multiplicación} \\ + \\ \text{conjugación} \end{array}$$

$$C_0 = \frac{1}{T} \int_{\langle T \rangle} x(t) \cdot x^*(t) dt \stackrel{\substack{\downarrow \\ \text{valor medio}}}{=} \frac{1}{T} \int_{\langle T \rangle} |x(t)|^2 dt$$

$$C_k = \sum_{n=-\infty}^{\infty} a_k \cdot a_{k-n}^* \stackrel{\substack{n=0 \\ \downarrow}}{\Rightarrow} C_0 = \sum_{n=-\infty}^{\infty} a_k \cdot a_{k-n}^* = \sum_{k=-\infty}^{\infty} |a_k|^2$$

Ejemplo: "constante"

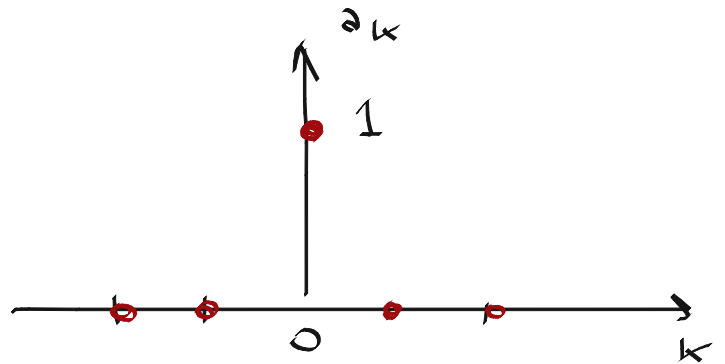
$$x(t) = 1, \quad t \in [0, T)$$



por inspección

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \dots + a_{-1} e^{-j\omega_0 t} + \underbrace{a_0}_{=1} + a_1 e^{j\omega_0 t} + \dots$$

$$a_k = \begin{cases} 1 & k=0 \\ 0 & k \neq 0 \end{cases}$$

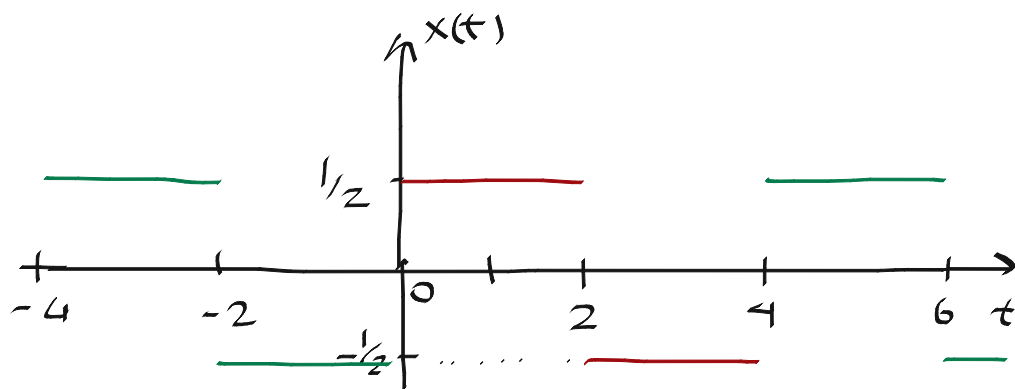


$a_k = \delta[k]$

Ejemplo 3.6:

$x(t)$ período T

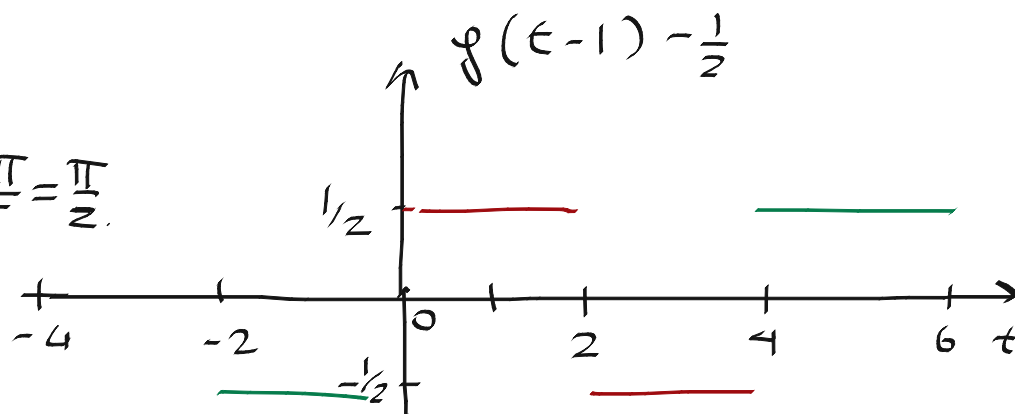
$$x(t) = \begin{cases} \frac{1}{2} & t \in [0, 2) \\ -\frac{1}{2} & t \in [2, 4) \end{cases}$$



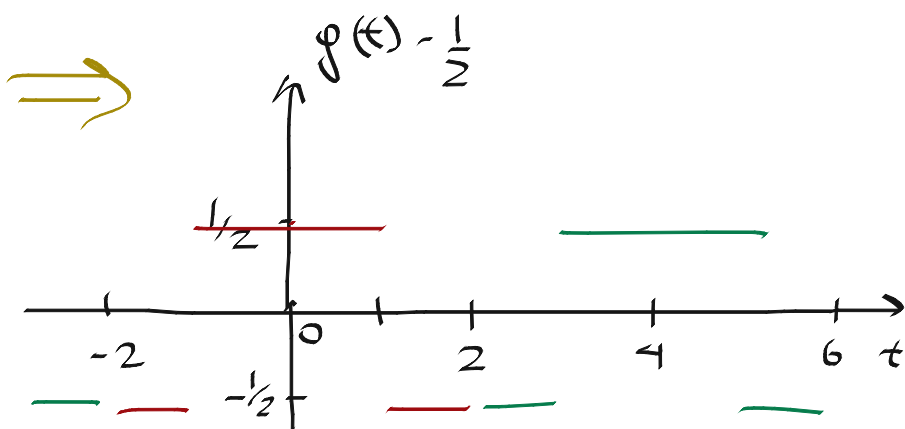
$$T = 4, T_1 = 1, \omega_0 = \frac{2\pi}{T} = \frac{\pi}{2}$$

$$x(t) = g(t-1) - \frac{1}{2}$$

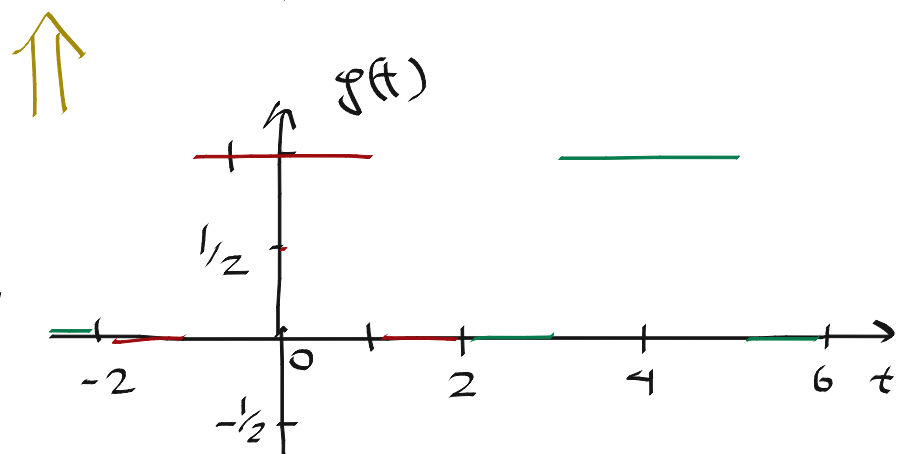
$$g_k = \frac{1}{2} \cdot \text{sinc}\left(\frac{\pi}{2}k\right)$$



$$\begin{aligned} g(t) &\xrightarrow{\mathcal{F}} g_k \\ g(t-1) &\xrightarrow{\mathcal{F}} e^{-jk\omega_0} g_k \end{aligned}$$



$$g(t-1) \xrightarrow{\mathcal{F}} e^{-jk\frac{\pi}{2}} g_k$$



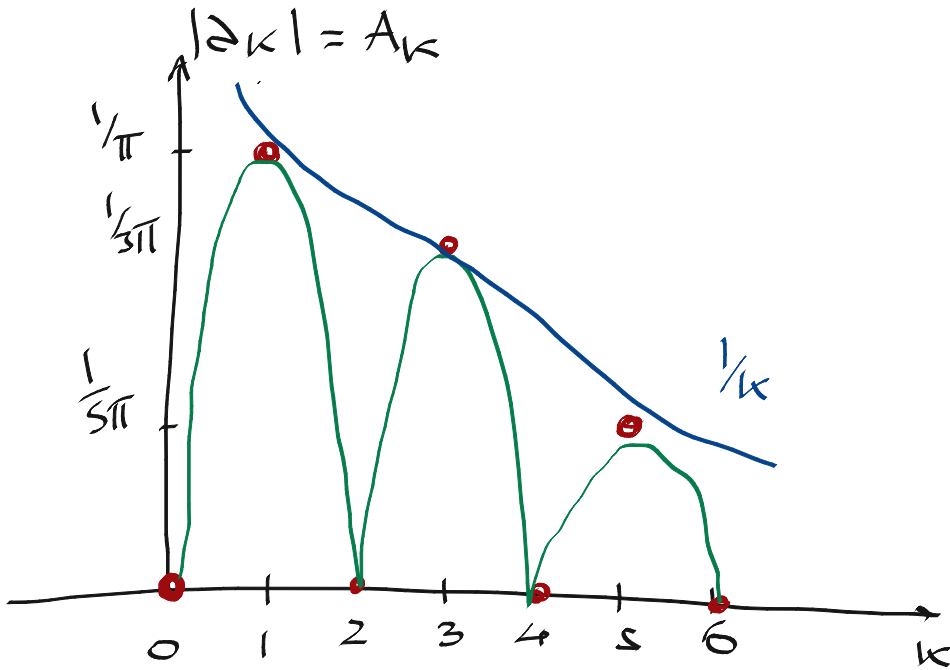
$$g(t-1) - \frac{1}{2} \xrightarrow{\mathcal{F}} e^{-jk\frac{\pi}{2}} g_k - \frac{1}{2} \delta[k]$$

$$x(t) \xrightarrow{\mathcal{F}} \begin{cases} \frac{1}{2} e^{-jk\frac{\pi}{2}} \text{sinc}\left(\frac{\pi}{2}k\right) & k \neq 0 \\ 0 & k = 0 \end{cases}$$

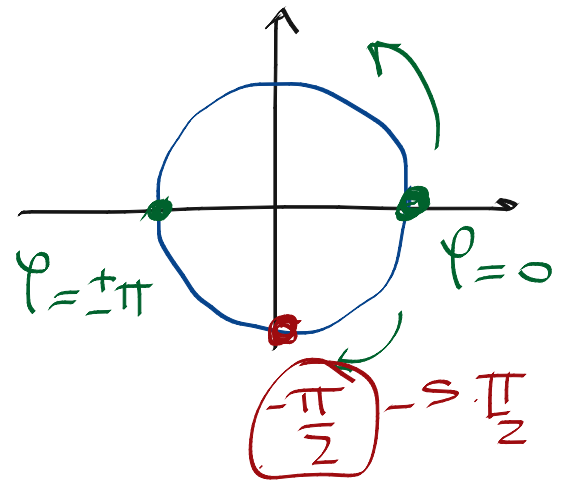
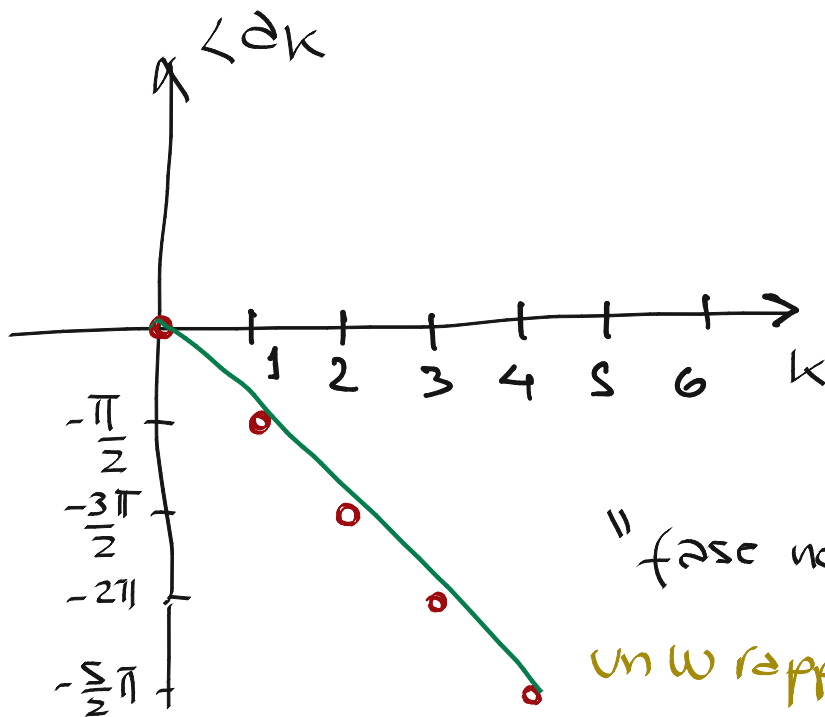
$k \neq 0$

$k = 0$

$$a_k = \begin{cases} \frac{1}{2} \cdot \underline{e^{-jk\frac{\pi}{2}}} \cdot \text{sinc}\left(\frac{\pi}{2}k\right) & k \neq 0 \\ 0 & k = 0 \end{cases}$$



k	A _k	"φ _k "	φ _k
0	0	0	0
1	1/π	-π/2	-π/2
2	0	-π	0
3	1/3π	-3/2π	-3/2π = -π/2
4	0	-2π	-π
5	1/5π	-5/2π	-3/2π
6	0	-3π	

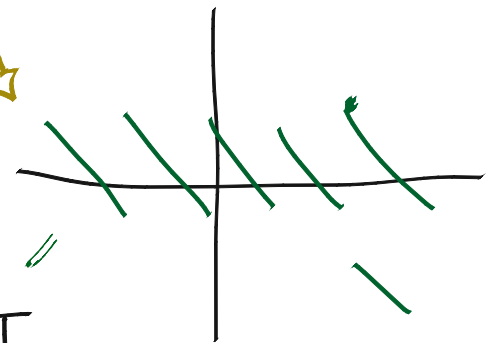


"fase negativa"

unw rapp

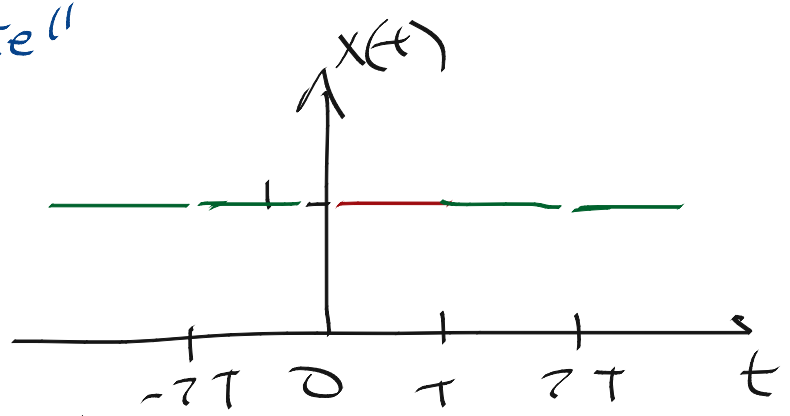
obs: * $\angle a_k = -\frac{\pi}{2} \cdot k$

* $\angle a_k = -\frac{\pi}{2} \cdot k + [(-1)^{k+1} + 1]\pi$



Ejemplo: "constante"

$$x(t) = 1, t \in [0, T)$$



$$a_k = \frac{1}{T} \int_{\langle T \rangle} x(t) \cdot e^{-jk\omega_0 t} dt$$

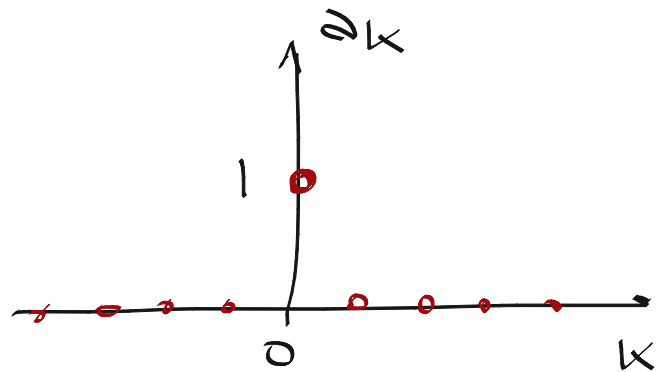
$$a_k = \frac{1}{T} \int_0^T 1 \cdot e^{-jk\omega_0 t} dt = \begin{cases} 1 & k=0 \\ \left. \frac{e^{-jk\omega_0 t}}{-jk\omega_0} \right|_0^T & k \neq 0 \end{cases}$$

$$= \frac{e^{-jk\omega_0 T} - e^{-jk\omega_0 \cdot 0}}{-jk\omega_0} = \frac{e^{-jk \cdot \frac{2\pi}{T} \cdot T} - 1}{-jk\omega_0}$$

$\omega_0 = \frac{2\pi}{T}$

$$a_k = \begin{cases} 1 & k=0 \\ 0 & k \neq 0 \end{cases}$$

$$a_k = \delta[k]$$



Derivada

$$\begin{array}{ccc} x(t) & \xrightarrow{\mathcal{D}} & a_k \\ x'(t) & \xrightarrow{\mathcal{D}} & j\omega_0 \cdot a_k \end{array}$$

Convolución (recíproca modulación)

$$x(t) * y(t) \xrightarrow{\mathcal{F}} a_k \cdot a_k$$

Clase 13

Series de Fourier

Tiempo discreto

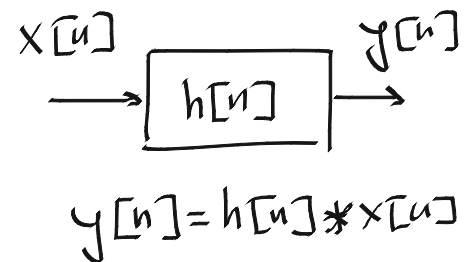
SERIES DE FOURIER EN TIEMPO DISCRETO

Idea: "exponenciales complejas"

- * bases sencillas: — C^∞ y periódicas
- * descomposición: — escribir señal como comb. lineal
- * superposición: — salida fácil para SLIT
— combinación lineal de salidas
- * universalidad: — aplicable a muchos sist.

Respuesta de un SLIT discreto a un exp. complejo

$$x[n] = z^n, \quad y[n] = \sum_{k=-\infty}^{\infty} h[k] \cdot x[n-k]$$



$$y[n] = \sum_{k=-\infty}^{\infty} h[k] \cdot z^{n-k} = z^n \cdot \underbrace{\sum_{k=-\infty}^{\infty} h[k] \cdot z^{-k}}_{H(z)} \quad \text{sólo depende de } z$$

$$H(z) = \sum_{k=-\infty}^{\infty} h[k] \cdot z^{-k}$$

$$y[n] = \underbrace{H(z)}_{\substack{\text{escalar} \\ \text{valor} \\ \text{propio}}} \cdot \underbrace{z^n}_{\substack{\text{vector} \\ \text{propio}}}$$

Definición

Función periódica:

$$x[n] = x[n+N], \quad N \text{ período}, \quad \omega_0 = \frac{2\pi}{N} \text{ frec. fundamental}$$

Base: $\phi_k[n] = e^{jk\omega_0 n}$

* armónicas: $\omega_k = k \cdot \omega_0$

* sólo hay N frecuencias diferentes

↳ ω_k es periódica

← dif. con el caso continuo

Síntesis

$$x[n] = \sum_{k \in \langle N \rangle} a_k \cdot \phi_k[n] \Rightarrow$$

$$x[n] = \sum_{k \in \langle N \rangle} a_k \cdot e^{jk\omega_0 n}$$

Análisis:

$$a_k = \frac{1}{N} \sum_{n \in \langle N \rangle} x[n] \cdot e^{-jk\omega_0 n}$$

Obs:

* las ecuaciones son simétricas

* como las sumas son finitas $\Rightarrow$ siempre existe!!

* no hace falta estudiar la convergencia

Cálculo N datos $\Rightarrow \omega_0 = \frac{2\pi}{N}$ también

$$\underline{X[0]} = \sum_{k=\langle N \rangle} a_k \cdot \underline{e^{j0}} = \underline{a_0} + \underline{a_1} + \dots + \underline{a_{N-1}}$$

$$\underline{X[1]} = \sum_{k=\langle N \rangle} a_k \cdot e^{j k \omega_0 \cdot 1} = \underline{a_0} + \underline{a_1} \cdot \underline{e^{j \omega_0}} + \dots + \underline{a_{N-1}} \cdot \underline{e^{j(N-1) \omega_0}}$$

⋮

$$\underline{X[N-1]} = \sum_{k=\langle N \rangle} a_k \cdot e^{j k \omega_0 (N-1)} = \underline{a_0} + \underline{a_1} \cdot \underline{e^{j(N-1) \omega_0}} + \dots + \underline{a_{N-1}} \cdot \underline{e^{j(N-1)(N-1) \omega_0}}$$

obs:

* síntesis: $\rightarrow X[n]$ incógnitas $\left\{ \Rightarrow \text{suma directa} \right.$
 $\leftarrow \rightarrow a_k$ datos

* análisis: $\rightarrow X[n]$ datos $\left\{ \Rightarrow \text{sistema lineal} \right.$
 $\leftarrow a_k$ incógnitas $N \times N$
cálculo eficiente

obs: "periodicidad"

$$d) \sum_{k=\langle N \rangle} e^{j k \omega_0 n} = \begin{cases} N & k = \pm m \cdot N \\ 0 & \text{e.o.c.} \end{cases}$$

$$\phi_k[n] = e^{j k \frac{2\pi}{N} n} \Rightarrow \phi_0[n] = e^{j \cdot 0} = 1$$
$$\phi_N[n] = e^{j N \cdot \frac{2\pi}{N} n} = 1 \quad \left. \vphantom{\phi_k[n]} \right\} \Rightarrow \boxed{\phi_0[n] = \phi_N[n]}$$

$$b) X[n] = \underbrace{a_0 \cdot \phi_0[n]}_{||} + \underbrace{a_1 \cdot \phi_1[n] + \dots + a_{N-1} \cdot \phi_{N-1}[n]}_{||}$$

$$X[n] = \underbrace{a_1 \cdot \phi_1[n] + a_2 \cdot \phi_2[n] + \dots + a_{N-1} \cdot \phi_{N-1}[n]} + \underbrace{a_N \cdot \phi_N[n]}$$

$$\Rightarrow \underbrace{a_0 \cdot \phi_0[n]}_{\text{son iguales por a)}} = \underbrace{a_N \cdot \phi_N[n]}$$

$$\Rightarrow \boxed{a_0 = a_N}$$

c)

$$* \text{ en general: } \boxed{a_n = a_{n+N}}$$

* coef. de Fourier en T.D. son periódicos

← dif. con caso continuo

Ejemplo 3.10 (p 214) $X[n] = \sin[\omega n]$, $\omega_0 = \frac{2\pi}{N}$, $N \in \mathbb{N}$

→ similar al ej. 3.3: $x(t) = \sin(\omega t)$

→ $x[n]$ no siempre es periódica!!

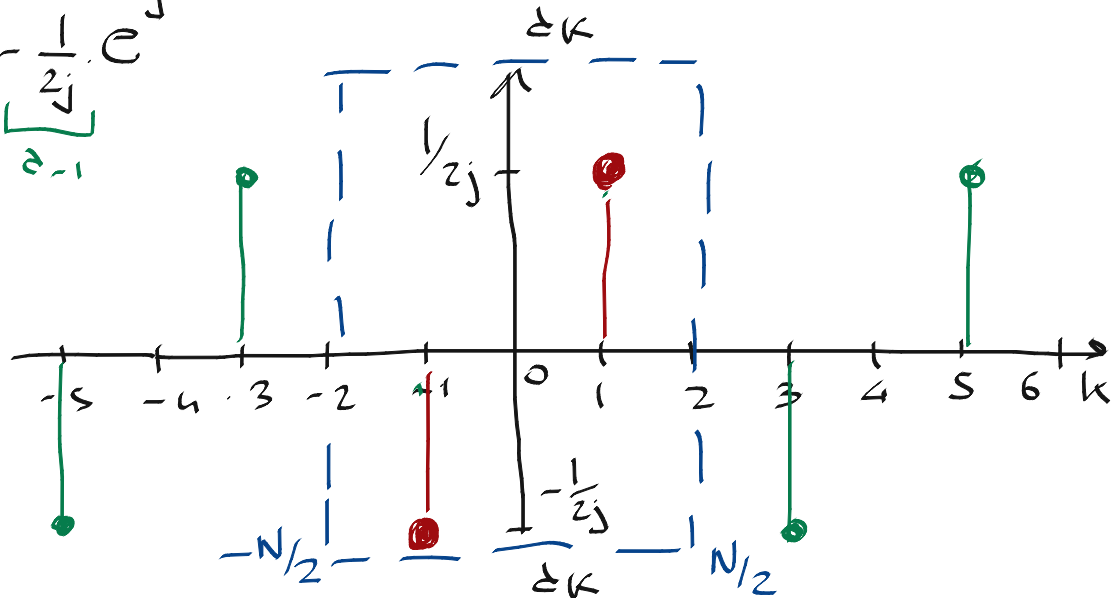
$$N=4$$

a) $\omega = \frac{2\pi}{N} = \omega_0 \rightarrow$ frecuencia ω y el período N están relacionados

$$x[n] = \underbrace{\frac{1}{2j}}_{\Delta_1} e^{j\omega_0 n} - \underbrace{\frac{1}{2j}}_{\Delta_{-1}} e^{-j\omega_0 n}$$

$$\Delta_k = \begin{cases} \frac{1}{2j} & k=1 \\ -\frac{1}{2j} & k=-1 \\ 0 & \text{e.o.c} \end{cases}$$

igual que T.C.

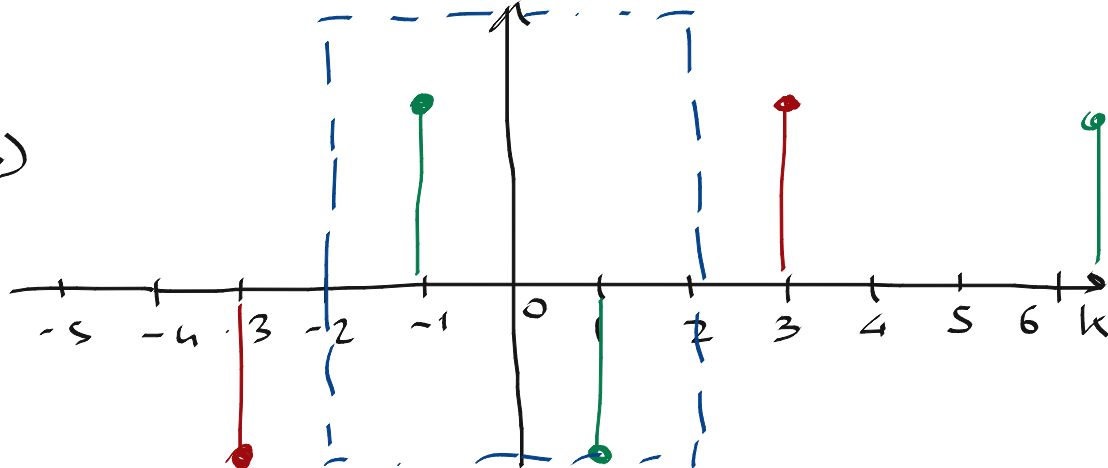


b) $\omega = M \cdot \frac{2\pi}{N}$ ($M=3$)

$$\omega = M \cdot \omega_0$$

↑ más rápido!

$$\Delta_k = \begin{cases} \frac{1}{2j} & k=M \\ -\frac{1}{2j} & k=-M \\ 0 & \text{e.o.c} \end{cases}$$



$$\tilde{X}[n] = -\frac{1}{2j} e^{j\omega_0 n} + \frac{1}{2j} e^{-j\omega_0 n} = \boxed{-\sin[\omega_0 n]}$$

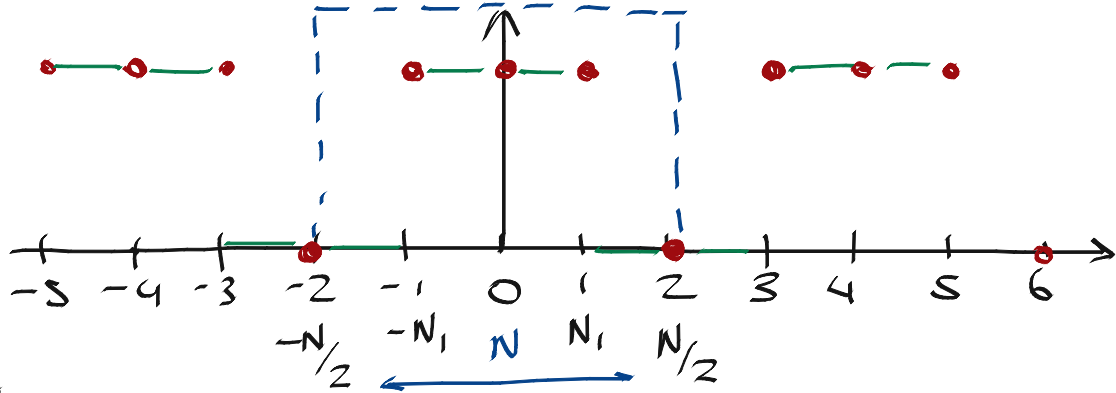
$$x[n] = \sin\left[\frac{3 \cdot 2\pi}{4} n\right] \neq \tilde{X}[n] = -\sin\left[\frac{2\pi}{4} n\right]$$

SOLAPAMIENTO
O ALIASING!!

Ejemplo 3.12

a) calculo la serie.

$$N=4, N_1=1$$



$$a_k = \frac{1}{N} \sum_{n=-\frac{N}{2}}^{\frac{N}{2}} x[n] \cdot e^{-jk\omega_0 n}$$

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} e^{-jk\omega_0 n} \xrightarrow{m=n+N_1} \frac{1}{N} \sum_{m=0}^{2N_1} e^{-jk\omega_0 (m-N_1)} = \frac{e^{jKN_1\omega_0}}{N} \sum_{m=0}^{2N_1} e^{-jk\omega_0 m}$$

$$a_k = \frac{e^{jKN_1\omega_0}}{N} \sum_{m=0}^{2N_1} \underbrace{(e^{-jk\omega_0})^m}_r = \frac{e^{jKN_1\omega_0}}{N} \sum_{m=0}^{2N_1} r^m \xrightarrow{\text{serie geom. } (r \neq 1)} = \frac{e^{jKN_1\omega_0}}{N} \cdot \frac{1-r}{1-r}$$

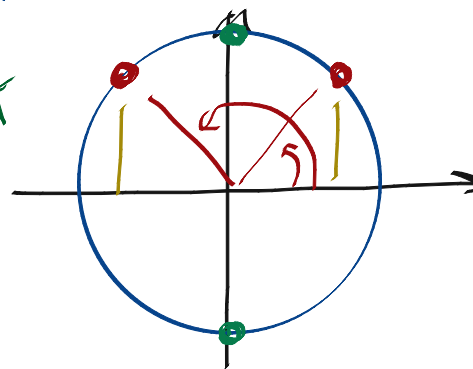
$$\xrightarrow{\text{desahago c.v.}} a_k = \frac{e^{jKN_1\omega_0}}{N} \cdot \frac{1 - e^{-j(2N_1+1)k\omega_0}}{1 - e^{-jk\omega_0}} \cdot \frac{e^{j\frac{1}{2}k\omega_0}}{e^{j\frac{1}{2}k\omega_0}} = \frac{1}{N} \cdot \frac{e^{j(N_1+\frac{1}{2})k\omega_0} - e^{-j(N_1+\frac{1}{2})k\omega_0}}{e^{j\frac{1}{2}k\omega_0} - e^{-j\frac{1}{2}k\omega_0}}$$

$$a_k = \frac{1}{N} \cdot \frac{e^{j(N_1+\frac{1}{2})k\omega_0} - e^{-j(N_1+\frac{1}{2})k\omega_0}}{e^{j\frac{1}{2}k\omega_0} - e^{-j\frac{1}{2}k\omega_0}} \cdot \frac{z_j}{z_j}$$

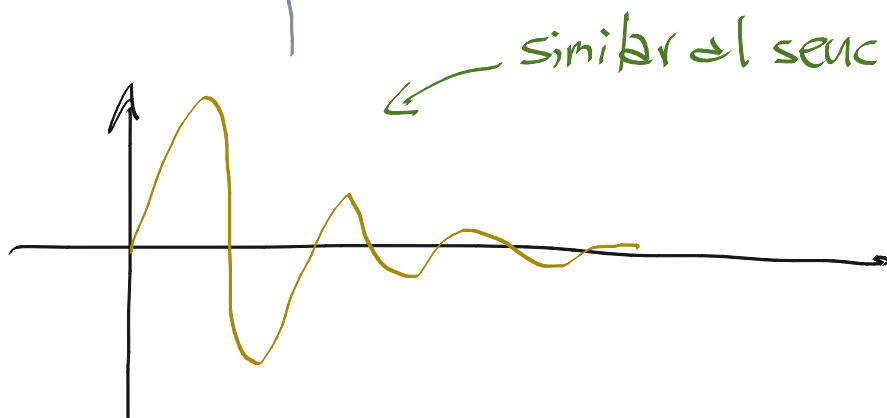
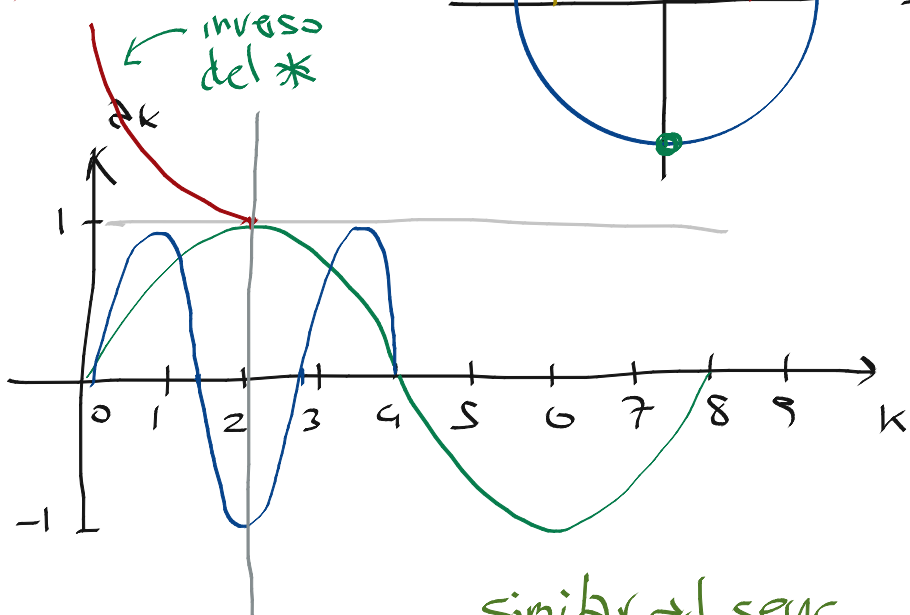
$$a_k = \frac{1}{N} \cdot \frac{\text{sen}[(N_1+\frac{1}{2})\omega_0 k]}{\text{sen}[\frac{\omega_0}{2} k]}$$

b) representación gráfica (módulo y fase)

$$a_k = \frac{1}{4} \cdot \frac{\sin\left[\frac{3}{2} \cdot \frac{2\pi}{4} k\right]}{\sin\left[\frac{2\pi}{4 \cdot 2} k\right]} = \frac{1}{4} \cdot \frac{\sin\left[\frac{3\pi}{4} k\right]}{\sin\left[\frac{\pi}{4} k\right]}$$



k	*	*	a _k
0	—	—	3/4
1	√2/2	√2/2	1/4
2	-1	-1	-1/4
3	√2/2	√2/2	1/4
4	—	—	3/4
5	—	—	3/4
6	—	—	3/4



$$a_0 = \frac{1}{N} \sum_{n=-N_1}^{N_1} x[n]$$

$$a_0 = \frac{1}{N} \sum_{n=-N_1}^{N_1} 1 = \frac{2N_1+1}{N} = \frac{3}{4}$$

$$\sin\left[\frac{\omega_0}{2} k\right] = \sin\left[\frac{2\pi}{N \cdot 2} k\right]$$

$$\sin\left[\frac{\pi}{N} k\right]$$

$$\frac{\pi}{2} = \frac{\pi}{N} k$$

$$k = \frac{N}{2}$$

obs:

* ver simulación 

* las sumas parciales en computadora se hacen en el práctico 7

Clase 14

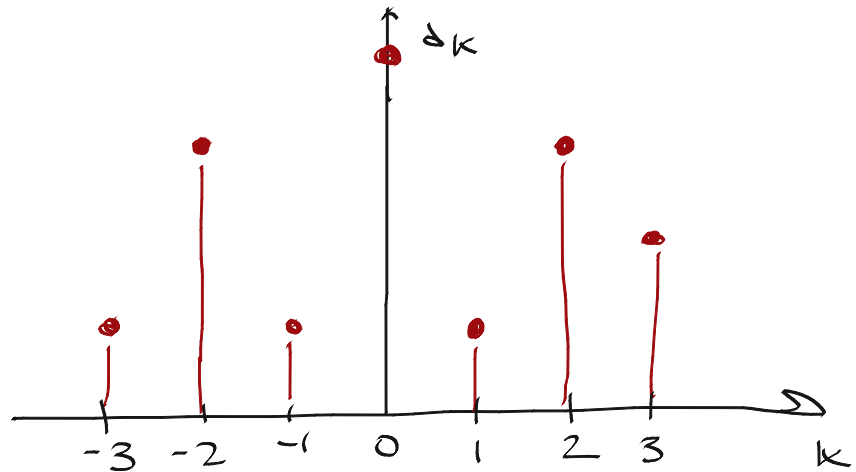
Series de Fourier

Análisis de sistemas LIT

Ejemplo 3.2 (p 187)

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \cdot e^{jk\omega_0 t}$$

$$a_0 = 1, a_{\pm 1} = \frac{1}{4}, a_{\pm 2} = \frac{1}{2}, a_{\pm 3} = \frac{1}{3}$$



$$x(t) = \underbrace{a_0 + \underbrace{a_1 e^{j\omega_0 t} + a_{-1} e^{-j\omega_0 t}}_{\text{conjugados}} + a_2 e^{j2\omega_0 t} + a_{-2} e^{-j2\omega_0 t} + a_3 e^{j3\omega_0 t} + a_{-3} e^{-j3\omega_0 t}}_{\text{conjugados}}$$

$$x(t) = 1 + \frac{1}{2} \cos(\omega_0 t) + \cos(2\omega_0 t) + \frac{2}{3} \cos(3\omega_0 t)$$

- obs: *
- * las series de Fourier pueden usar como base cosenos
 - * esto ocurre cuando los coef. son pares
 - * también puede hacerse con senos (imágenes IP6)

SERIES DE FOURIER

obs. * Fourier es una herramienta para analizar funciones. No tiene que ser señal o sistema

* Ahora vamos a ~~pasar~~ a aplicarlo

Tiempo continuo

Idea ✓

Definición ✓

Ejemplos ✓

Convergencia ✓

Propiedades. ✓

Ejemplos ✓

Aplicaciones

Tiempo discreto

Idea ✓

Definición ✓

Ejemplos ✓

Convergencia X

Propiedades. ✓

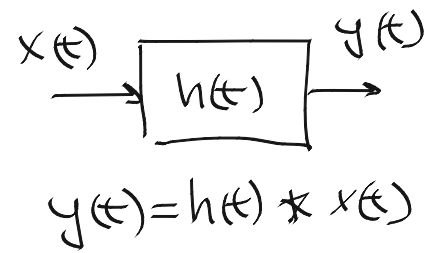
Ejemplos ✓

Aplicaciones

SERIES DE FOURIER Y SISTEMAS LIT

$$y(t) = \underline{H(s)} \cdot e^{st} \quad (*)_1$$

↳ función sistema



* caso periódico: $s = j\omega$

$$H(s) = \int_{-\infty}^{\infty} h(t) \cdot e^{-st} dt \Rightarrow \boxed{H(j\omega) = \int_{-\infty}^{\infty} h(t) \cdot e^{-j\omega t} dt}$$

resp. en frecuencia.

* $x(t) = \sum_{k=-\infty}^{\infty} a_k \cdot e^{j k \omega_0 t}$

↓ (*)₂

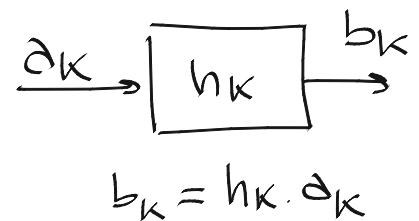
$$y(t) = \sum_{k=-\infty}^{\infty} \underbrace{a_k \cdot H(j k \omega_0)}_{\substack{\text{escalares} \\ (\sin t) \\ \text{coef } b_k}} \cdot \underbrace{e^{j k \omega_0 t}}_{\text{exp. armónicos}}$$

es una serie de Fourier !!

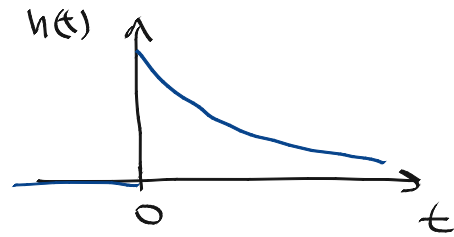
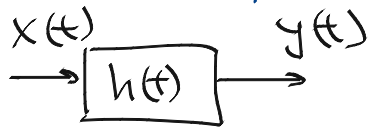
$$y(t) \xleftrightarrow{\mathcal{F}} b_k = H(j k \omega_0) \cdot a_k$$

$$H(j k \omega_0) = \int_{-\infty}^{\infty} h(t) \cdot e^{-j k \omega_0 t} dt \quad \boxed{= h_k}$$

$$y(t) \xleftrightarrow{\mathcal{F}} h_k \cdot a_k$$



Ejemplo 3.16 (p 228-229)



$$h(t) = u(t) \cdot e^{-t}$$

a) Calcular respuesta en frecuencia

$$h_k = H(jk\omega_0), \quad H(j\omega) = \int_{-\infty}^{\infty} u(t) \cdot e^{-t} \cdot e^{-j\omega t} dt = \int_0^{\infty} e^{-(j\omega+1)t} dt$$

$$H(j\omega) = - \frac{e^{-(j\omega+1)t}}{j\omega+1} \Big|_0^{\infty} = \boxed{\frac{1}{1+j\omega}}$$

obs:

$$\begin{aligned} * H(-j\omega) &= \frac{1}{1-j\omega} = \frac{1}{1+j\omega} = \overline{\frac{1}{1+j\omega}} = \overline{H(j\omega)} \\ &\rightarrow \text{módulo par} \\ &\rightarrow \text{fase impar} \end{aligned}$$

b) Calcular la salida (en frecuencia)

$$\text{sea } x(t) = 1 + \frac{1}{2} \cos(\omega_0 t) + \cos(2\omega_0 t) + \frac{2}{3} \cos(3\omega_0 t) \quad (\text{ej 3.2})$$

obs: * $x(t)$ real y par $\Rightarrow y(t)$ real

* $h(t)$ real

* admite una serie de cosenos

* datos: $\rightarrow T=1$, $f_0=1\text{Hz}$

$$\rightarrow \omega_0 = 2\pi \cdot 1 = 2\pi$$

$$\text{coef. } h_k: h_k = H(jk\omega_0) = \frac{1}{1 + j \frac{2\pi k}{\omega_0}}$$

$$\text{coef. salida: } b_k = H(jk\omega_0) \cdot a_k$$

$$b_0 = a_0 \cdot H(j^0) = 1$$

$$b_1 = a_1 \cdot H(j\omega_0) = \frac{1}{4} \cdot \frac{1}{1 + j2\pi}$$

$$b_{-1} = a_{-1} \cdot H(-j\omega_0) = \frac{1}{4} \cdot \frac{1}{1 - j2\pi}$$

$$b_2 = \frac{1}{2} \cdot \frac{1}{1 + j4\pi}, \quad b_{-2} = \frac{1}{2} \cdot \frac{1}{1 - j4\pi}$$

$$b_3 = \frac{1}{3} \cdot \frac{1}{1 + j6\pi}, \quad b_{-3} = \frac{1}{3} \cdot \frac{1}{1 - j6\pi}$$

c) Reconstruyo $y(t)$

$$y(t) = 1 + \underbrace{\frac{1}{1+j2\pi}}_{b_1} e^{j2\pi t} + \underbrace{\frac{1}{1-j2\pi}}_{b_{-1}} e^{-j2\pi t} + \dots$$

$$b_1 = B_1 e^{j\theta_1}, \quad B_1 = |b_1| \quad \theta_1 = \angle b_1 \Rightarrow b_{-1} = B_1 e^{-j\theta_1}$$

$$y(t) = 1 + B_1 e^{j\theta_1} e^{j2\pi t} + B_1 e^{-j\theta_1} e^{-j2\pi t} + \dots$$

$$y(t) = 1 + 2B_1 \cos(2\pi t + \theta_1) + 2B_2 \cos(4\pi t + \theta_2) + 2B_3 \cos(6\pi t + \theta_3)$$

k	B_k	θ_k
0	1	0
1	$\frac{1}{4} \cdot \frac{1}{\sqrt{1+4\pi^2}}$	$-\arctan(2\pi)$
2	$\frac{1}{2} \cdot \frac{1}{\sqrt{1+16\pi^2}}$	$-\arctan(4\pi)$
3	$\frac{1}{3} \cdot \frac{1}{\sqrt{1+36\pi^2}}$	$-\arctan(6\pi)$

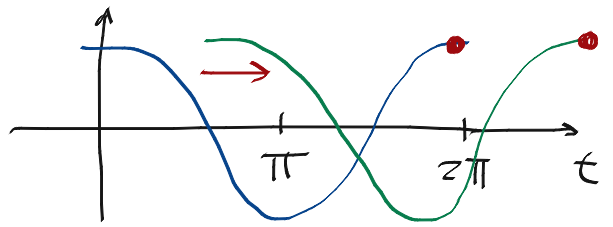
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k	B_k	θ_k
0	1	0
1	0.16	-0.45π
2	0.08	-0.47π
3	0.05	-0.48π

Obs:

- * atenua más las frecuencias altas $\Rightarrow$ filtro pasabajos
- * fase negativa $\Rightarrow$ retardo !! y confirmamos causalidad

d) Análisis del retardo



* buscar un punto singular:

↳ elijo el primer máximo.

* $\cos(\omega_k t)$, $t_{\text{máx}} = ? \Rightarrow \omega_k \cdot t_{\text{máx}} = 2\pi$

$$\boxed{t_{\text{máx}} = \frac{2\pi}{\omega_k}} \rightarrow \begin{cases} t_1 = \frac{2\pi}{2\pi} = 1 \\ t_2 = \frac{2\pi}{4\pi} = \frac{1}{2} \\ t_3 = \frac{2\pi}{6\pi} = \frac{1}{3} \end{cases}$$

* retardo: $\cos(t) \xrightarrow{\text{retardo}} \cos\left(t - \frac{0.5}{2\pi}\right) \Rightarrow t_{\text{máx}} - 0.5 = 2\pi$

$$t_{\text{máx}} = 2\pi + \underline{\underline{0.5}}$$

* $\cos(\omega_k t) \xrightarrow{\text{retardo}} \cos(\omega_k (t - \underline{\underline{0.5}})) \Rightarrow \omega_k (t_{\text{máx}} - \underline{\underline{0.5}}) = 2\pi$

directo.

$$t_{\text{máx}} - 0.5 = \frac{2\pi}{\omega_k} \Rightarrow t_{\text{máx}} = \frac{2\pi}{\omega_k} + \underline{\underline{0.5}}$$

* $\cos(\omega_k t) \xrightarrow{\text{fase}} \cos(\omega_k t - \underline{\underline{\theta_k}}) \Rightarrow \omega_k t_{\text{máx}} - \underline{\underline{\theta_k}} = 2\pi$

no pasa directo

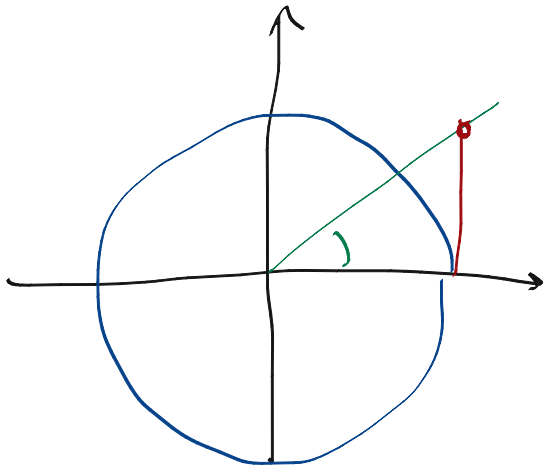
$$t_{\text{máx}} = \frac{2\pi}{\omega_k} + \underline{\underline{\frac{\theta_k}{\omega_k}}}$$

e) Análisis general

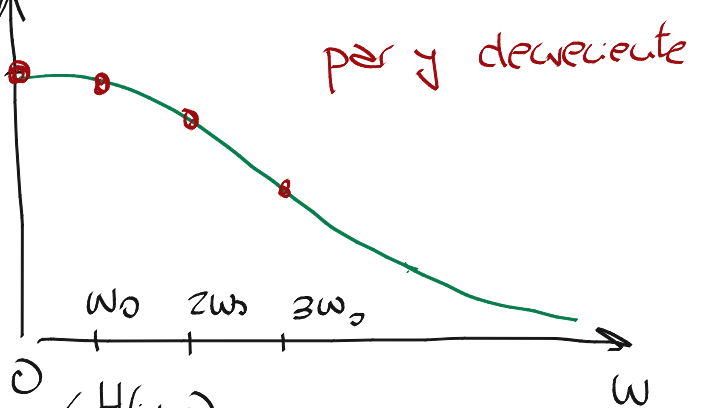
$$H(j\omega) = \frac{1}{1+j\omega}$$

$$|H(j\omega)| = \frac{1}{\sqrt{1+\omega^2}}$$

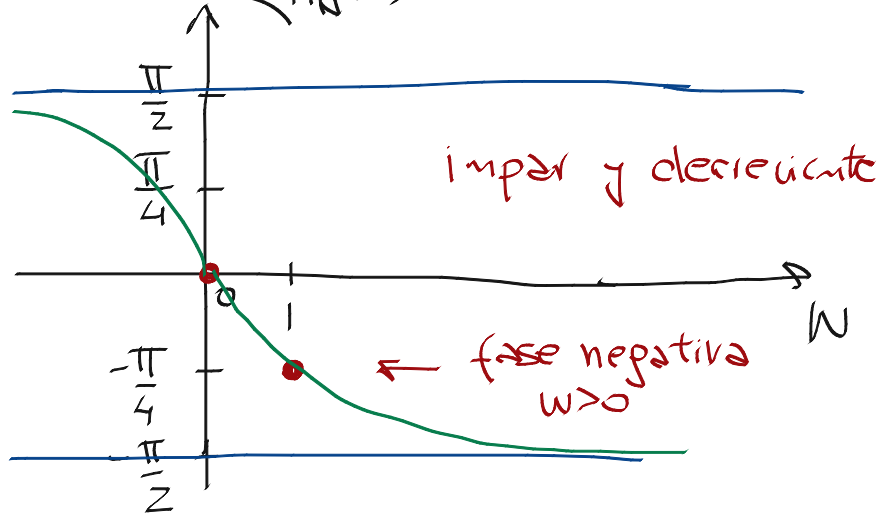
$$\angle H(j\omega) = -\arctan(\omega)$$



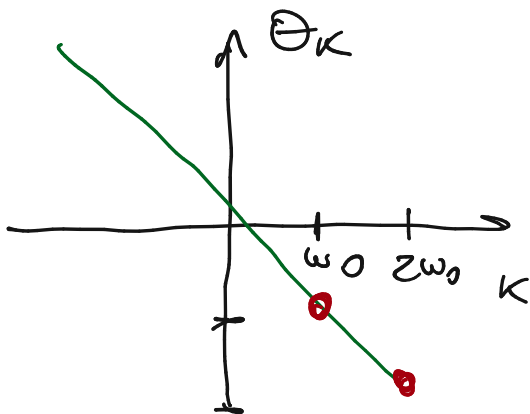
$$|H(j\omega)|$$



$$\angle H(j\omega)$$



f) coeficientes h_k .



$$|h_k| = \frac{1}{\sqrt{1+4\pi^2 k^2}}$$

$$\Theta_k = -\arctan(2\pi k)$$

Clase 15

Series de Fourier

Análisis de sistemas LTI y
Filtrado

Ejemplo 3.17 (p 230)

$$h[n] = u[n] \cdot a^n, \quad a < 1$$

$$x[n] = \cos[\omega_0 n] = \cos\left[\frac{2\pi}{N} \cdot n\right]$$

$$x[n] = \frac{1}{2} \cdot e^{j\omega_0 n} + \frac{1}{2} e^{-j\omega_0 n}$$

a) Resp. en frecuencia.
 $b_k = h_k \cdot \delta_k$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] \cdot e^{-j\omega n}$$

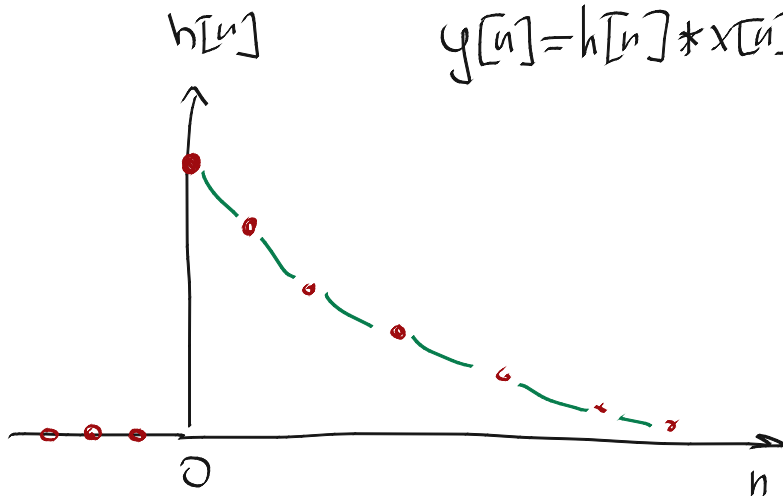
$$H(e^{j\omega}) = \sum_{n=0}^{\infty} a^n \cdot e^{-j\omega n} = \sum_{n=0}^{\infty} \underbrace{(a e^{-j\omega})^n}_r = \lim_{N \rightarrow \infty} \frac{1 - r^{N+1}}{1 - r} = \lim_{N \rightarrow \infty} \frac{1 - (a e^{-j\omega})^{N+1}}{1 - a e^{-j\omega}}$$

$$H(e^{j\omega}) = \frac{1}{1 - a e^{-j\omega}}$$

$$H(e^{j\omega}) = \frac{1}{1 - a \cos \omega - j a \sin \omega}$$



$$y[n] = h[n] * x[n]$$



$$z = r \cdot e^{j\omega} \Rightarrow z = e^{j\omega} \quad (s = j\omega)$$



b) Cálculo de salida: $y[n] = B_1 \cos[\omega_0 n + \Theta_1]$, $N=4 \Rightarrow \omega_0 = \frac{\pi}{2}$

$$h_1 = H(e^{j\omega}) \Big|_{\omega=\omega_0} = H(e^{j\omega_0}) = \frac{1}{1 - a \cdot e^{-j\frac{\pi}{2}}}$$

$$|h_1| = \frac{1}{\sqrt{\left(1 - a \cos\left(\frac{\pi}{2}\right)\right)^2 + a^2 \sin^2\left(\frac{\pi}{2}\right)}} = \boxed{\frac{1}{\sqrt{1 + a^2}}}$$

$$\phi(e^{j\omega}) = -a \tan\left(\frac{-a \sin(\omega)}{1 - a \cos \omega}\right) = \boxed{a \tan\left(\frac{a \sin(\omega)}{1 - a \cos \omega}\right)}$$

$$\Theta_1 = \phi(e^{j\frac{\pi}{2}}) = a \tan\left(\frac{a \sin(\frac{\pi}{2})}{1 - a \cos \frac{\pi}{2}}\right) = \boxed{a \tan(a)}$$

$$y[n] = \frac{1}{\sqrt{1+a^2}} \cdot \cos\left[\frac{\pi}{2} \cdot n + a \tan(a)\right]$$

$$a + 1 = \frac{1}{2} \quad h_1 = |h_1| \cdot e^{j\Theta_1}$$

$$b_1 = \frac{1}{2} \cdot |h_1| \cdot e^{j\Theta_1}$$

$$b_{-1} = \frac{1}{2} \cdot |h_1| \cdot e^{-j\Theta_1}$$

$$y[n] = \underline{\frac{1}{2} |h_1|} \cdot \underline{e^{j\Theta_1}} \cdot \underline{e^{j\frac{\pi}{2}n}} + \underline{\frac{1}{2} |h_1|} \cdot \underline{e^{-j\frac{\pi}{2}n}} \cdot \underline{e^{-j\Theta_1}}$$

$$y[n] = |h_1| \cdot \cos\left[\frac{\pi}{2}n + \Theta_1\right]$$

c) Analisis general

$$|H(e^{j\omega})| = \frac{1}{\sqrt{(1 - a \cos \omega)^2 + a^2 \sin^2 \omega}}$$

$$= \frac{1}{\sqrt{1 + \underbrace{a^2 \cos^2 \omega - 2a \cos \omega + a^2 \sin^2 \omega}_{a^2}}}$$

$$|H(e^{j\omega})| = \frac{1}{\sqrt{1 + a^2 - 2a \cos \omega}}$$

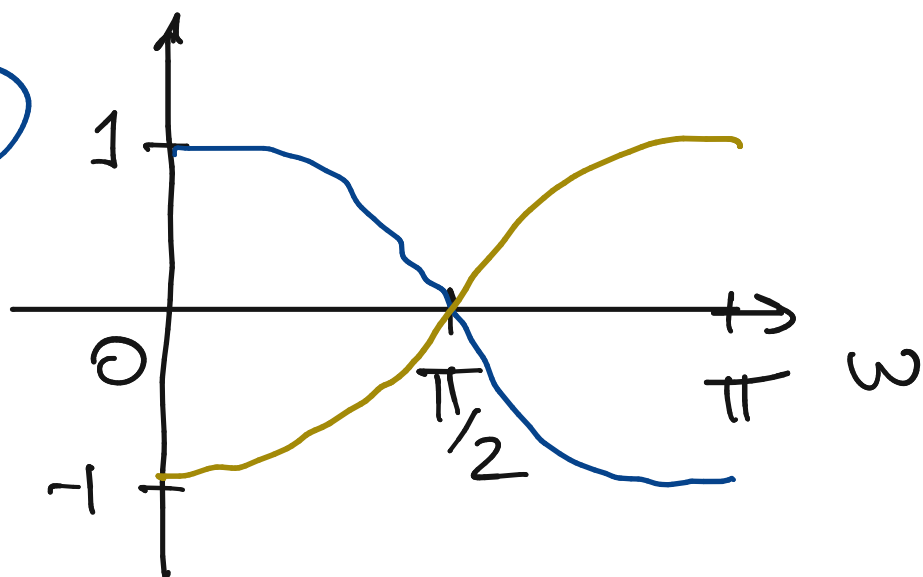
i) Analisis qualitativo

$$1 + a^2 - 2a \cos \omega$$

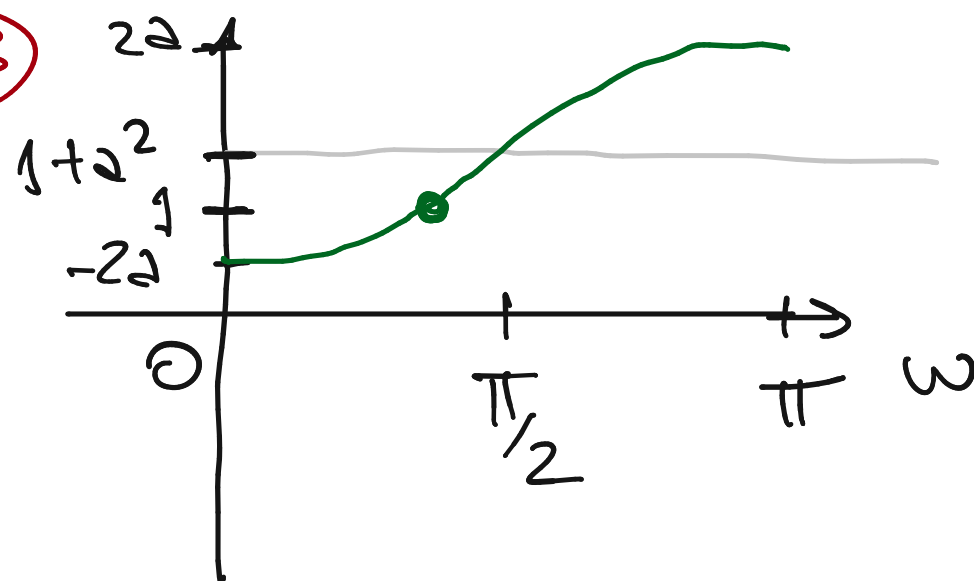
$$a = 0.2$$

$$\omega \in [0, \pi)$$

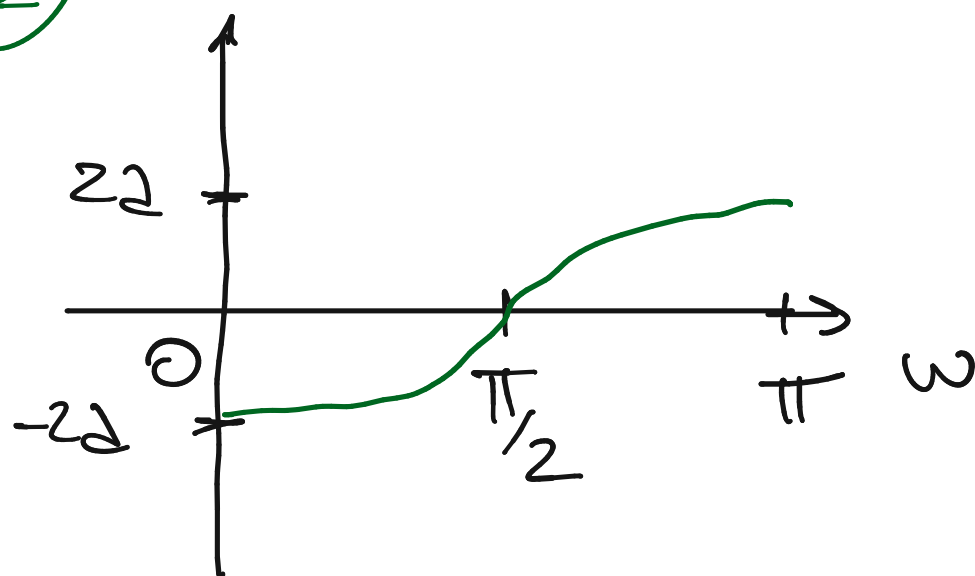
1



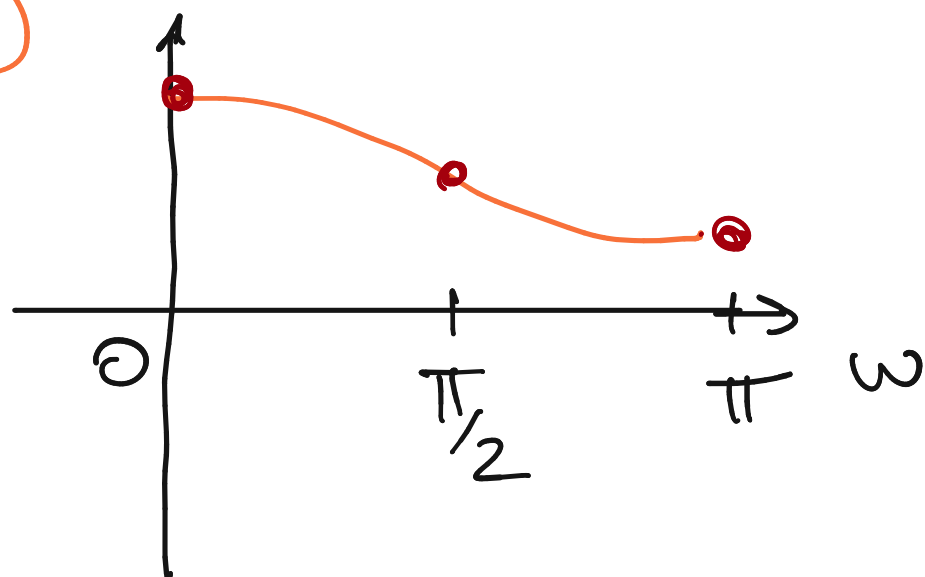
3



2



4



ii) pto. características

$$|H(e^{j0})| = \frac{1}{\sqrt{1-2a+a^2}}$$

$$|H(e^{j0})| = \frac{1}{\sqrt{(a-1)^2}}$$

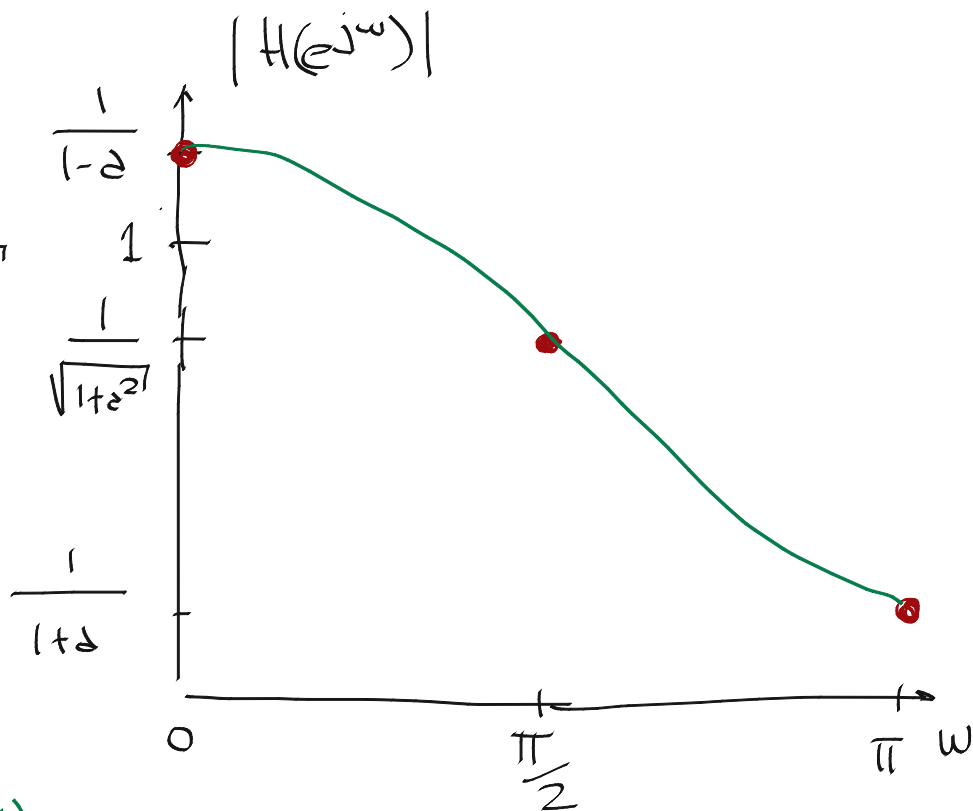
$$|H(e^{j0})| = \frac{1}{|a-1|}$$

$$|H(e^{j0})| = \frac{1}{1-a} \quad (a < 1) > 1$$

$$|H(e^{j\frac{\pi}{2}})| = \frac{1}{\sqrt{1+a^2}} < 1$$

$$|H(e^{j\pi})| = \frac{1}{\sqrt{1+2a+a^2}} = \frac{1}{\sqrt{(1+a)^2}} = \frac{1}{1+a} < 1$$

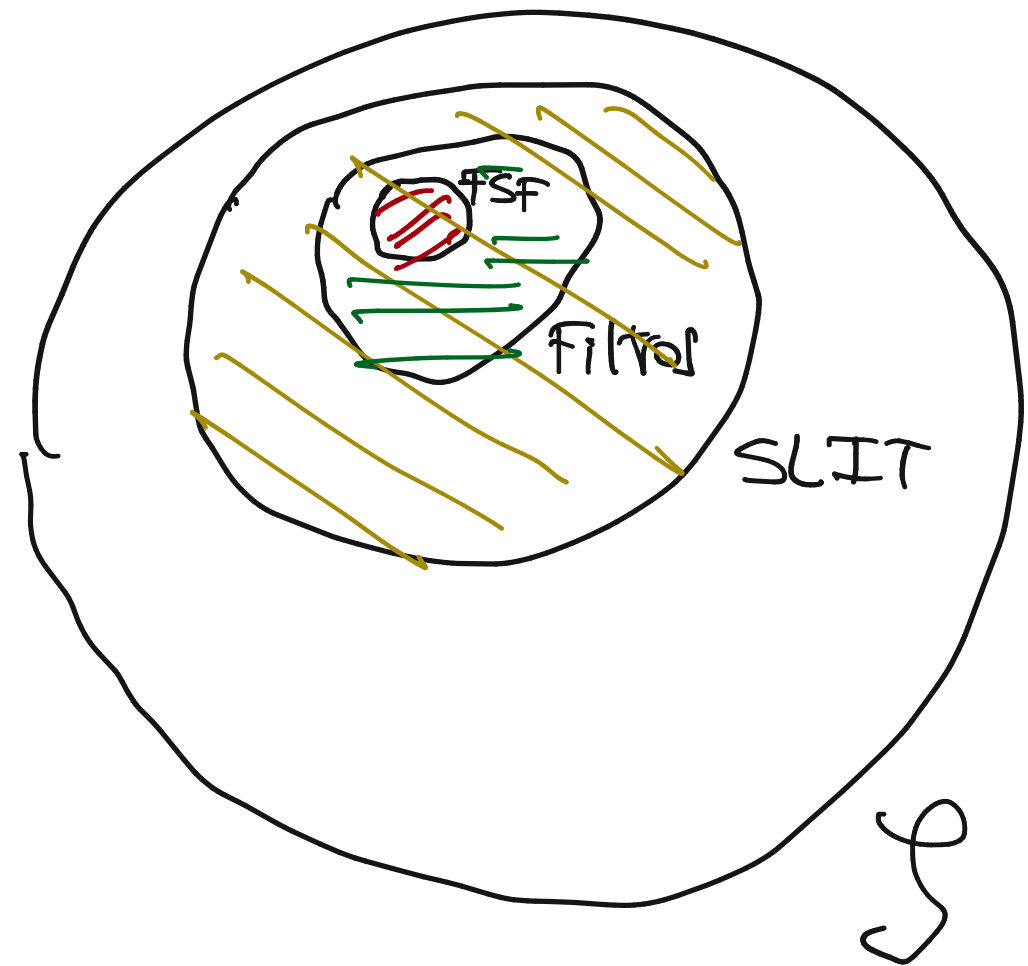
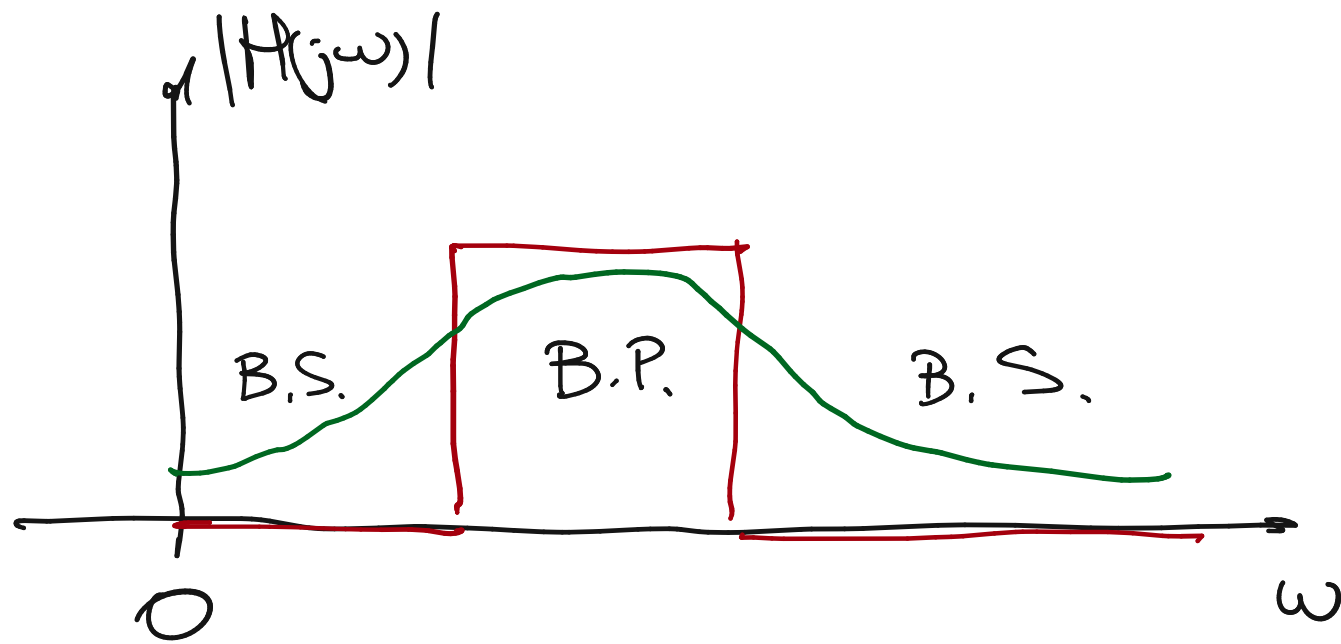
$$< \frac{1}{\sqrt{1+a^2}}$$



FILTRADO

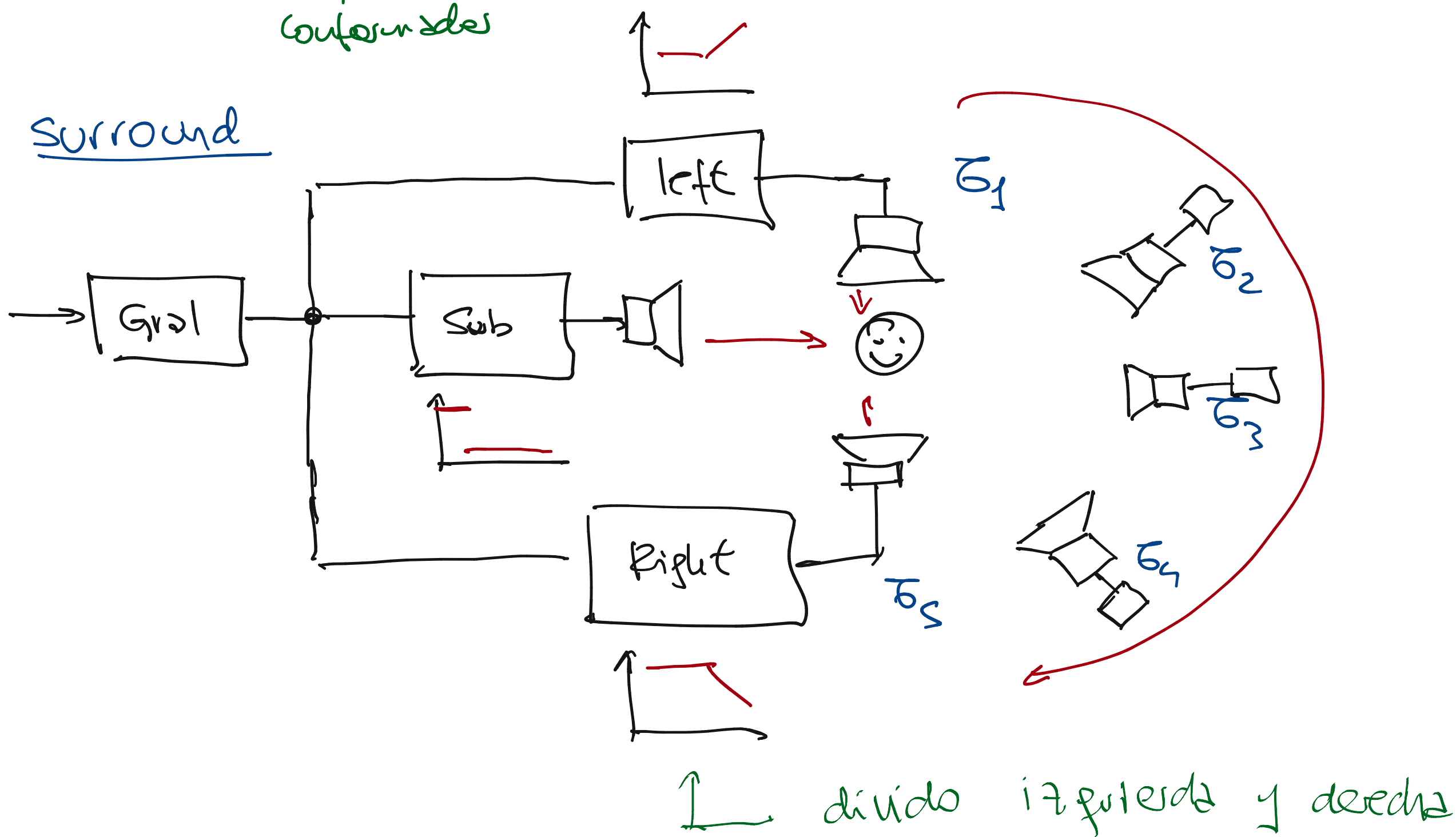
Filtros selectivos en frecuencia (FSF)

- 1) selectivos (pasabajos, pasaltos, etc)
- 2) Conformadores (modifican la frecuencia)



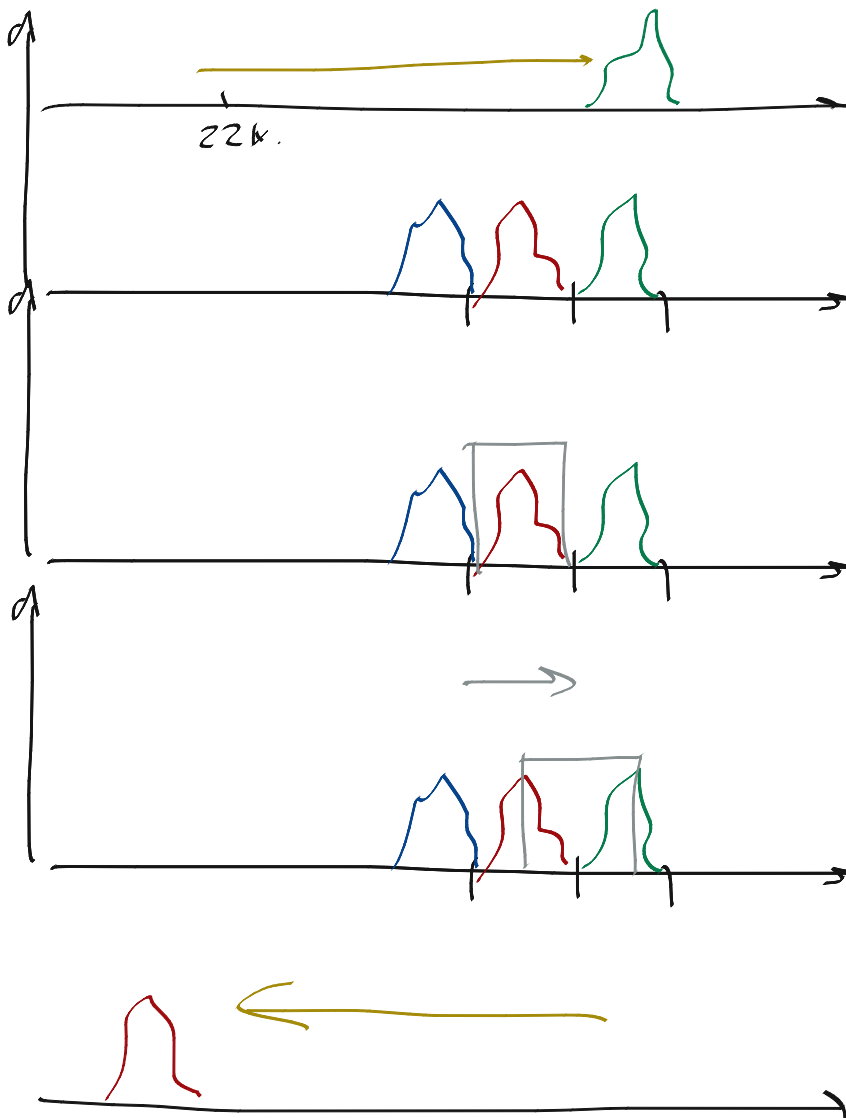
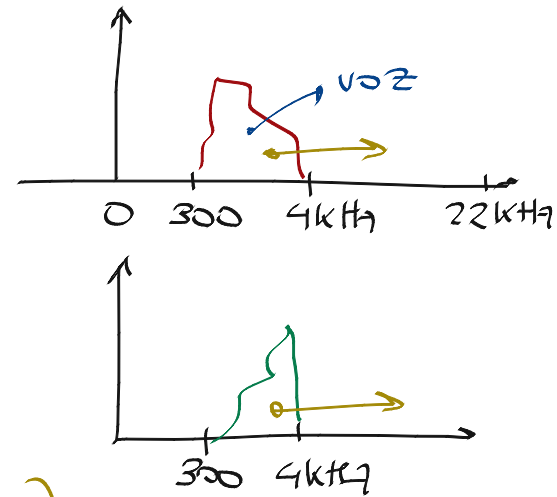
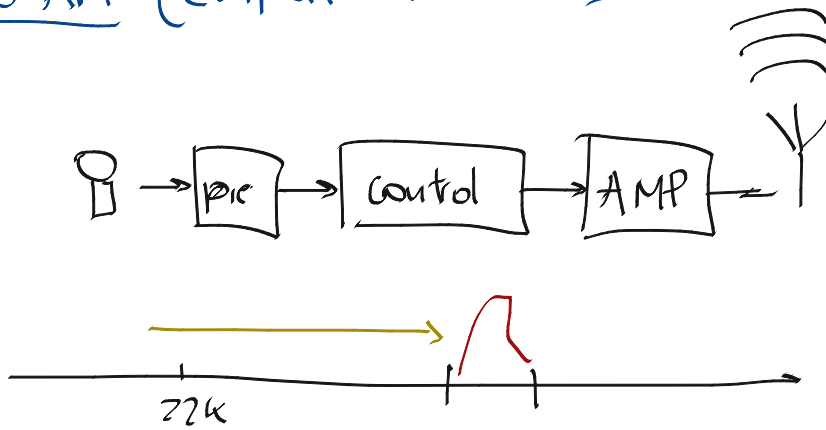
Sistema básico

1) (User) → ² Micro _{transducer} → ³ (val → e.p.) _{transducer} → (output)



Telecomunicaciones

Radio AM (amplitud modulada)



Módulo
Mezcla

Síntesis
(pasasenda)

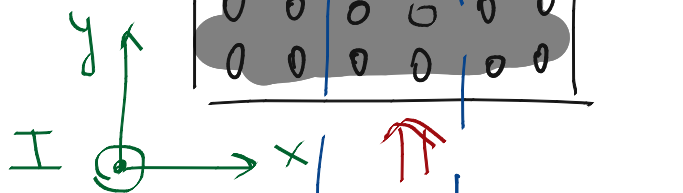
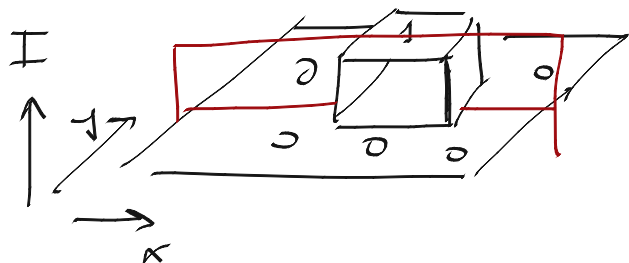
de-módulo

Imágenes

detección de bordes

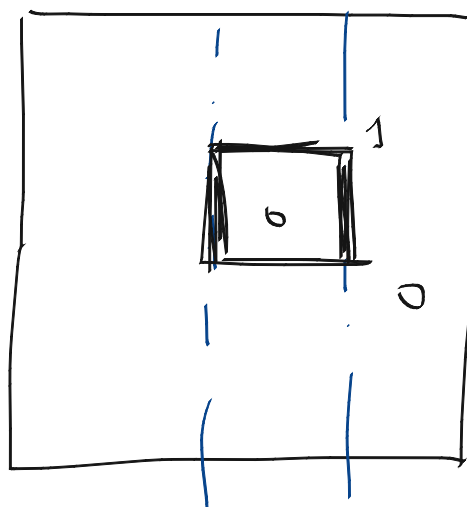
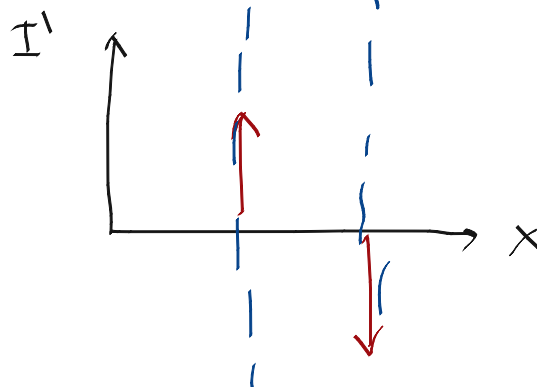
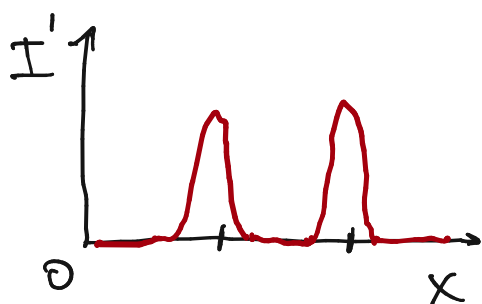
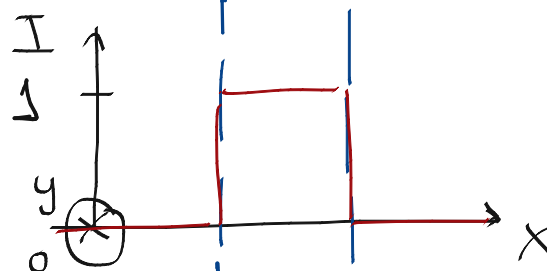
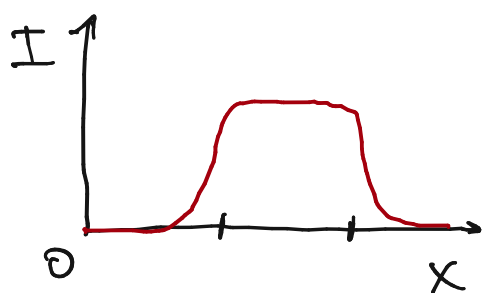
$$I(x, y) : \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$I(x, y) \in [0, 1] \subseteq [0, 255]$$



Imagen

0	0	0	0	0	0
0	0	1	1	0	0
0	0	1	1	0	0
0	0	0	0	0	0
0	0	0	0	0	0



Clase 16

Series de Fourier

Filtrado

ANÁLISIS DE SISTEMAS COMUNES

Derivador (T.C.)

$$y(t) = x'(t)$$

$$a) \quad x(t) \xrightarrow{\mathcal{F}} a_k$$

$$y(t) \xrightarrow{\mathcal{F}} b_k$$

$$x'(t) \xrightarrow{\mathcal{F}} j \cdot k \omega_0 \cdot a_k$$

ω_k

$$h_k = j \omega_k \rightarrow h_k = H(j\omega) \Big|_{\omega = \omega_k}$$

$$H(j\omega) = j\omega$$

$$1) \quad x(t) \rightarrow \left[\frac{d}{dt} \right] \rightarrow y(t)$$

$$y(t) = \int \dot{x}(t) dt$$

$$2) \quad x(t) \rightarrow \left[h(t) ? \right] \rightarrow y(t) = x'(t)$$

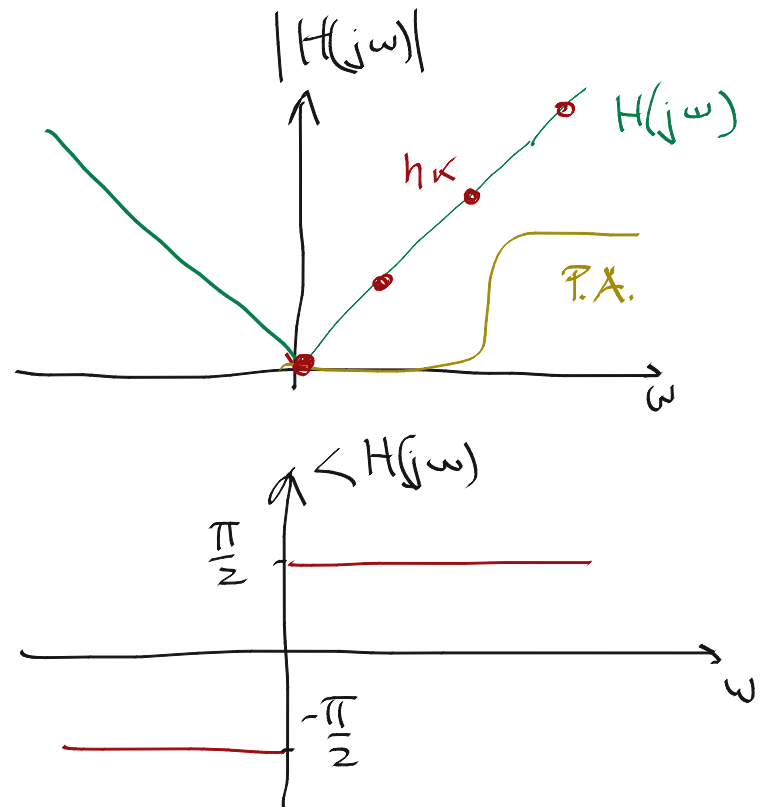
$$y(t) = h(t) * x(t)$$

$$3) \quad a_k \rightarrow \left[h_k ? \right] \rightarrow b_k$$

$$b_k = h_k \cdot a_k$$

$$b) \quad |H(j\omega)| = |\omega|$$

$$\angle H(j\omega) = \begin{cases} \frac{\pi}{2} & \omega > 0 \\ -\frac{\pi}{2} & \omega < 0 \end{cases}$$



obs:

* conformador: pasabajos (HPF)

* pasaseñal: energía infinita

$$E = \int |y(t)|^2 dt = \sum |b_k|^2 = \sum \underbrace{|h_k|^2}_{\infty} |a_k|^2 = \infty$$

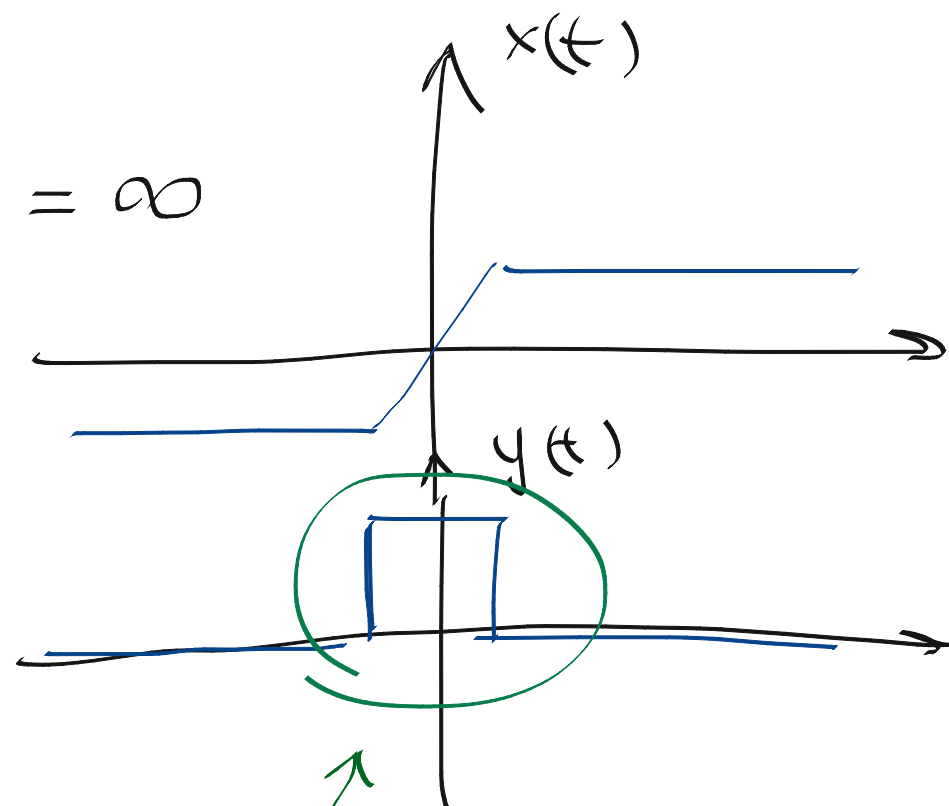
* inestable: no es realizable
en la práctica

* fase positiva ($\omega > 0$):

→ parece que adelanta (no causal)

→ la gráfica muestra lo contrario

→ en realidad p.i.o → no adelanta ni retrasa



Receta

Energía

Retardo

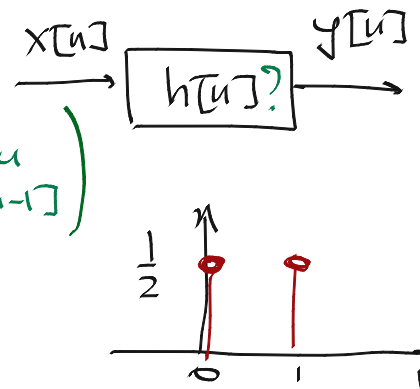
Interpretación

Media móvil (T.D.)

$L=2$

a) $y[n] = \underbrace{\frac{1}{2}}_{h_0} x[n] + \underbrace{\frac{1}{2}}_{h_1} x[n-1] \leftarrow \text{FIR} \begin{pmatrix} \text{no tiene} \\ \text{retroalimentación} \\ \text{no hay } y[n-1] \end{pmatrix}$

$$h[n] = \frac{1}{2} \delta[n] + \frac{1}{2} \delta[n-1]$$



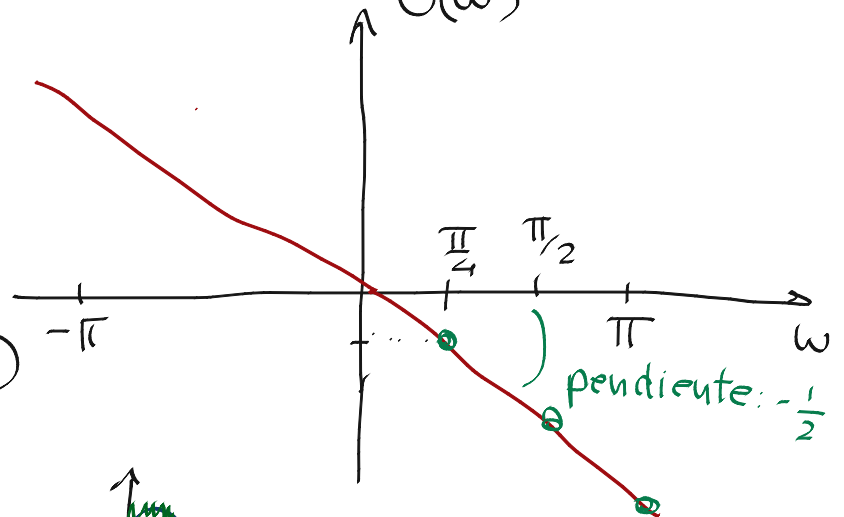
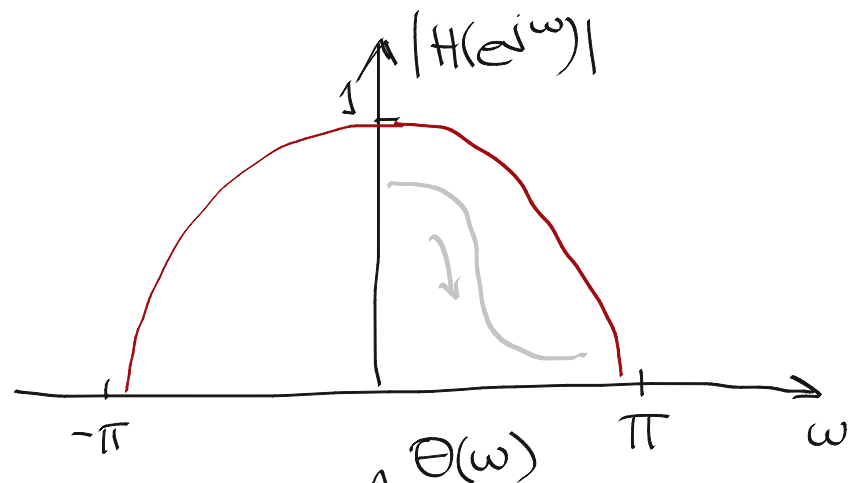
$$H(e^{j\omega}) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot e^{-j \cdot 1 \cdot \omega} \cdot 1 = \frac{1}{2} (1 + e^{-j\omega}) \cdot \frac{e^{j\frac{3\omega}{2}}}{e^{j\frac{3\omega}{2}}} = \frac{1}{2} \cdot \frac{e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}}}{e^{j\frac{3\omega}{2}}}$$

$$H(e^{j\omega}) = e^{-j\frac{3\omega}{2}} \cos\left(\frac{\omega}{2}\right)$$

$\varphi \cdot e^{j\varphi}$

b) $|H(j\omega)| = \left| \cos\left(\frac{\omega}{2}\right) \right|$

$$\Theta(\omega) = \angle H(j\omega) = -\frac{3\omega}{2}$$

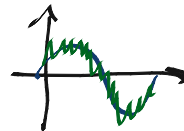


Obs:

* Conformador: pasabajos (LPF)

* fase negativa: retarda

* práctico: quita ruido (alta freq.)



* retardo: $\tau_r = \frac{\Theta(\omega)}{\omega} = -\frac{\frac{\pi}{4}}{\frac{\pi}{2}} = -\frac{1}{2} \text{ samples!}$

pendiente

$$\tau = -\frac{\Theta(\omega)}{\omega} = \frac{1}{2}$$

retardo de grupo

$$\tau(\omega) = \frac{d\theta(\omega)}{d\omega} \rightarrow \text{es función de } \omega \text{ (cuando la pend. no es recta)}$$

Ejemplo: $L=3$

$$a) y[n] = \frac{x[n+1] + x[n] + x[n-1]}{3}$$

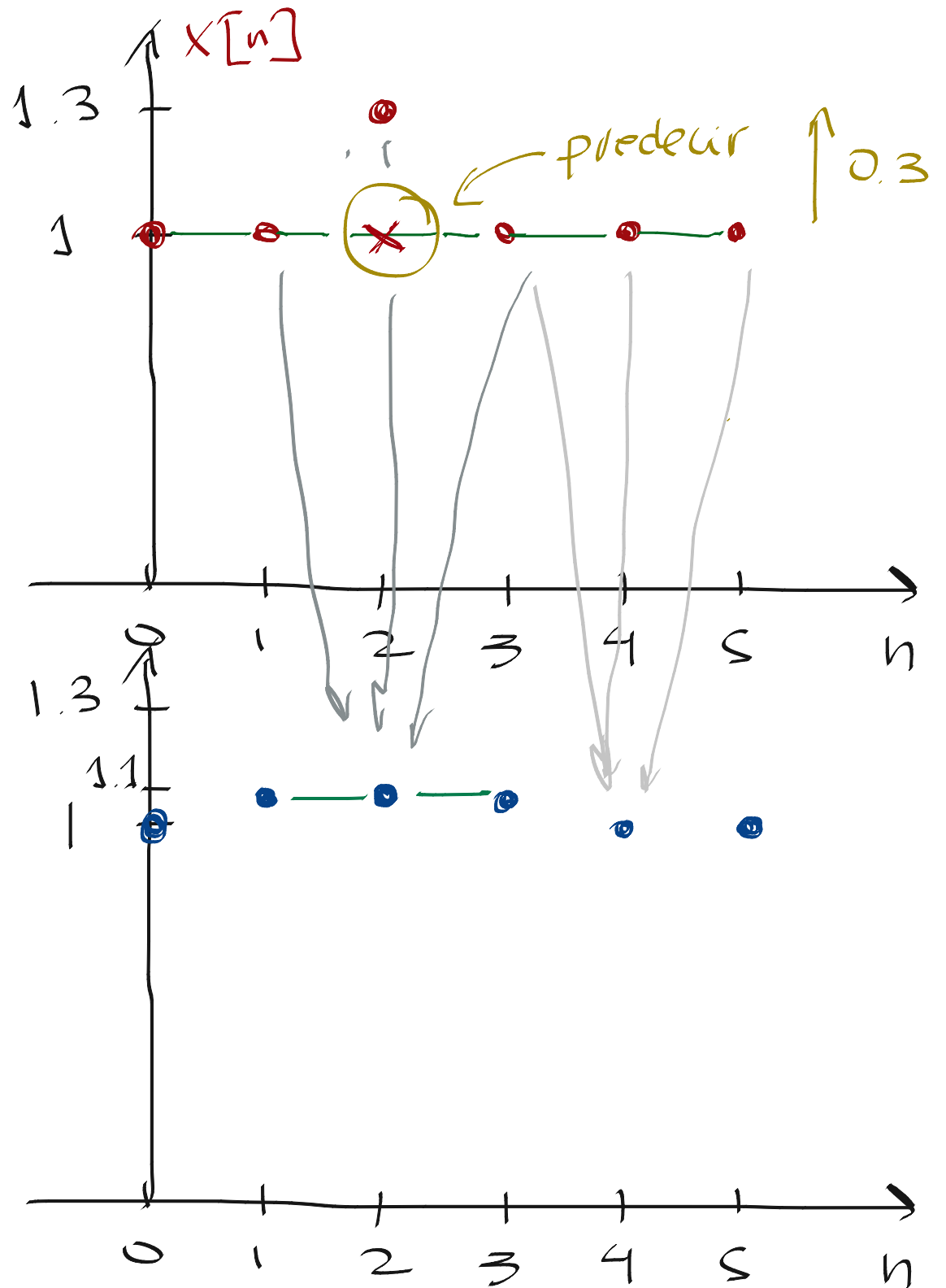
$$y[2] = \frac{1 + 1.3 + 1}{3} = \frac{3.3}{3} = 1.1$$

$$b) y[n] = \frac{x[n] + x[n-1]}{2}$$

* retardo: $\frac{1}{2}$

* retardo media muestra
(o medio instante de tiempo)

30% ruido $\rightarrow$



c) "¿cómo hago una media móvil que no retarde?"

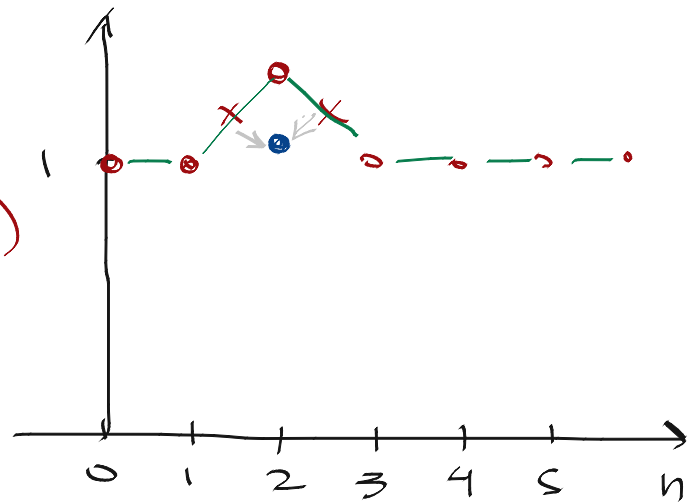
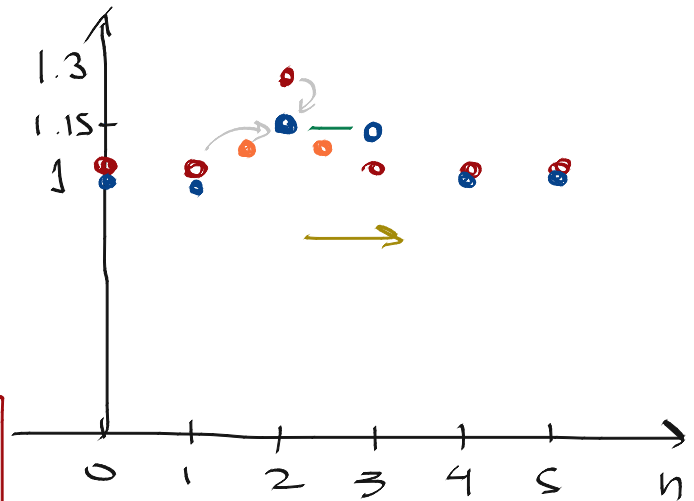
$$H(e^{j\omega}) = e^{-j\omega \frac{L}{2}} \cos\left(\frac{\omega}{2}\right)$$

$$H_2(e^{j\omega}) = \cos\left(\frac{\omega}{2}\right) = \frac{1}{2}(e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}})$$

$$h[n] = \frac{1}{2} \delta[n + \frac{1}{2}] + \frac{1}{2} \delta[n - \frac{1}{2}]$$

$$y[n] = \frac{1}{2} x[n + \frac{1}{2}] + \frac{1}{2} x[n - \frac{1}{2}]$$

ec. transformación de una
media móvil centrada ($L=2$)



Derivador discreto (T.D., hacia atrás)

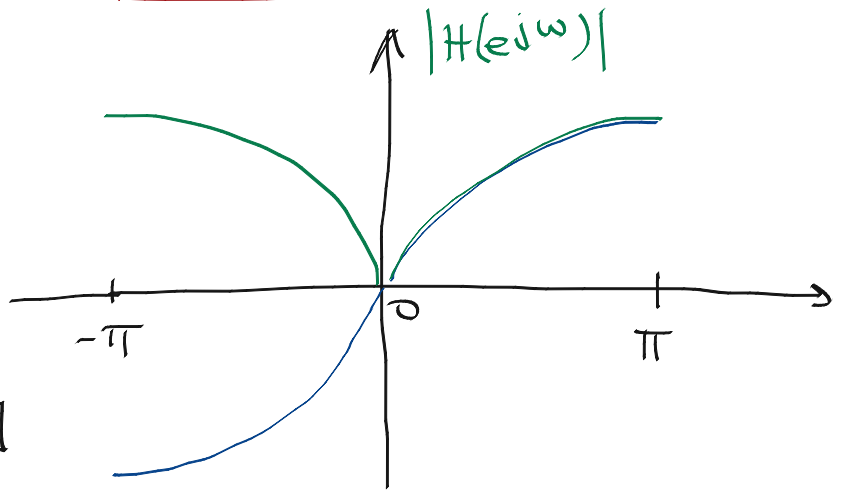
$$y[n] = \frac{x[n] - x[n-1]}{z} \Rightarrow h[n] = \frac{1}{2} \delta[n] - \frac{1}{2} \delta[n-1]$$

$$H(e^{j\omega}) = \frac{1 - e^{-j\omega}}{2} \cdot \frac{e^{j\frac{\omega}{2}}}{e^{j\frac{\omega}{2}}} = \frac{1}{2} \cdot \frac{e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}}{e^{j\frac{\omega}{2}}} = j \frac{\text{sen}\left(\frac{\omega}{2}\right)}{e^{j\frac{\omega}{2}}}$$

$$H(e^{j\omega}) = j e^{-j\frac{\omega}{2}} \text{sen}\left(\frac{\omega}{2}\right) \Rightarrow H(e^{j\omega}) = e^{j\left(\frac{\pi}{2} - \frac{\omega}{2}\right)} \text{sen}\left(\frac{\omega}{2}\right) \quad \text{9.º eix}$$

$$b) |H(e^{j\omega})| = |\text{sen}\left(\frac{\omega}{2}\right)|$$

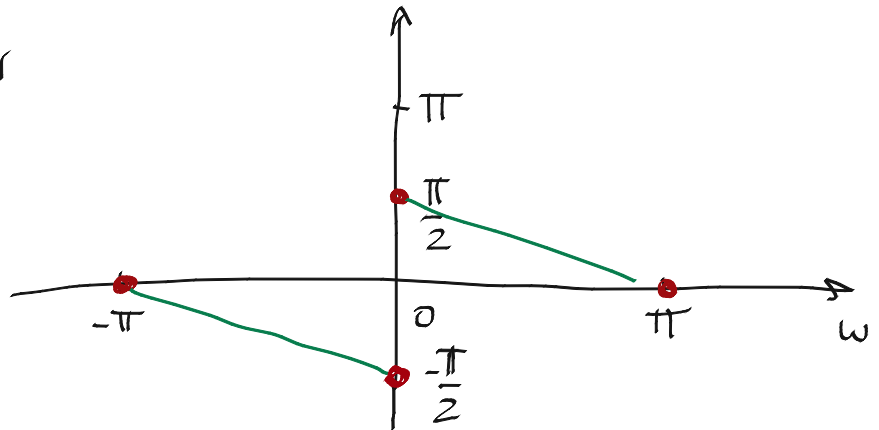
$$\Theta(\omega) = \begin{cases} \frac{\pi}{2} - \frac{\omega}{2} & \omega \in [0, \pi] \\ -\frac{\omega}{2} & \omega \in (-\pi, 0] \end{cases}$$



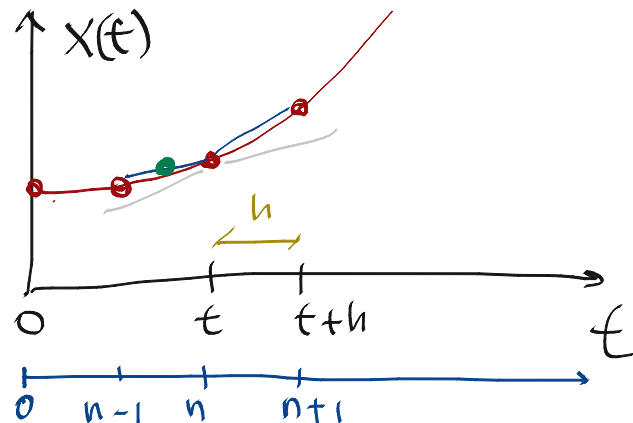
obs:

* módulo: filtro conformador
→ pasafaltos (HPF)

* fase ~~positiva~~: pend.
negativa → retarda



$$\tau(\omega) = -\frac{d\Theta(\omega)}{d\omega} = \boxed{\frac{1}{2}} \quad \text{retardo constante}$$

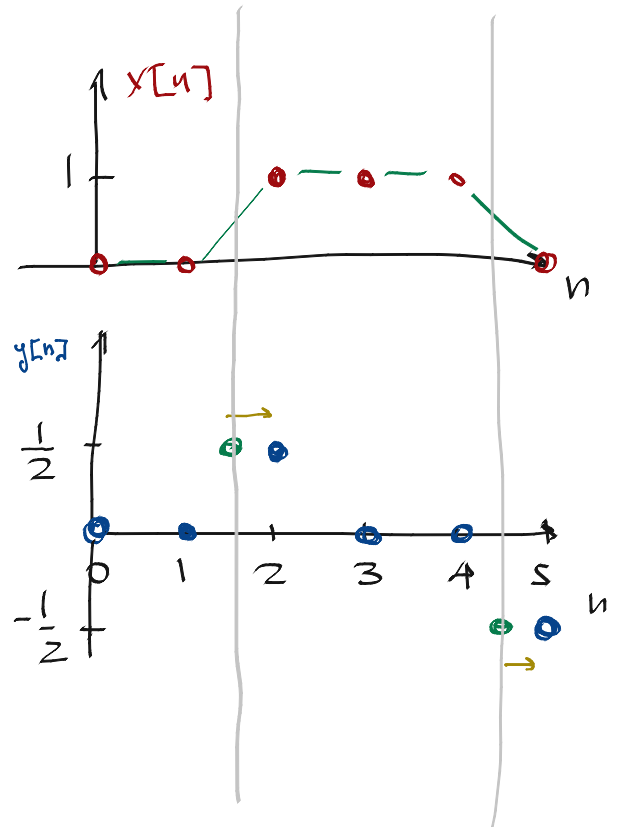


Aplicaciones reales

ejemplo: "cálculo de máximos relativos (picos)"

* en aplicaciones reales hay que descontar el retardo

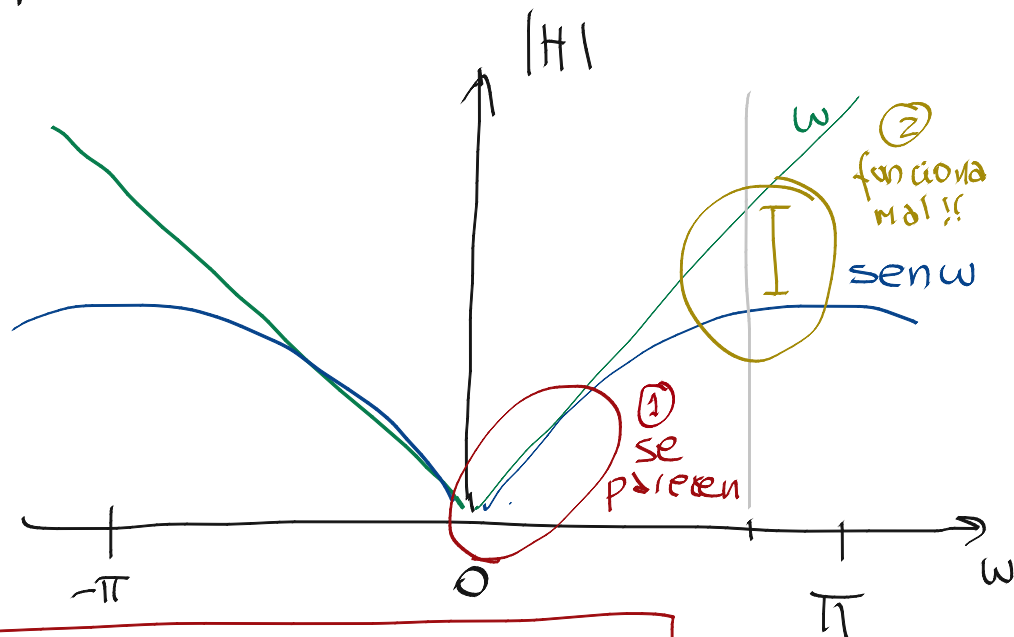
* extremo en $\frac{df(t)}{dt} = 0$



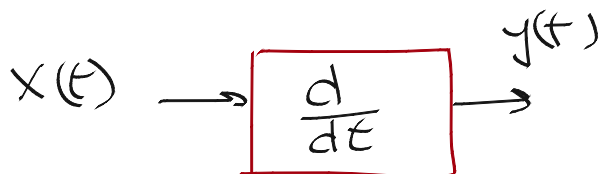
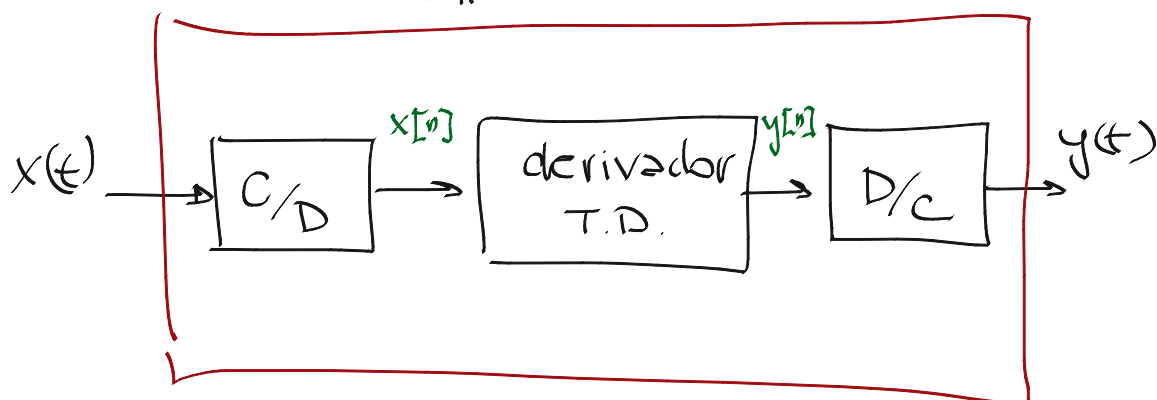
ejemplo: "comparación de derivadores"

a)

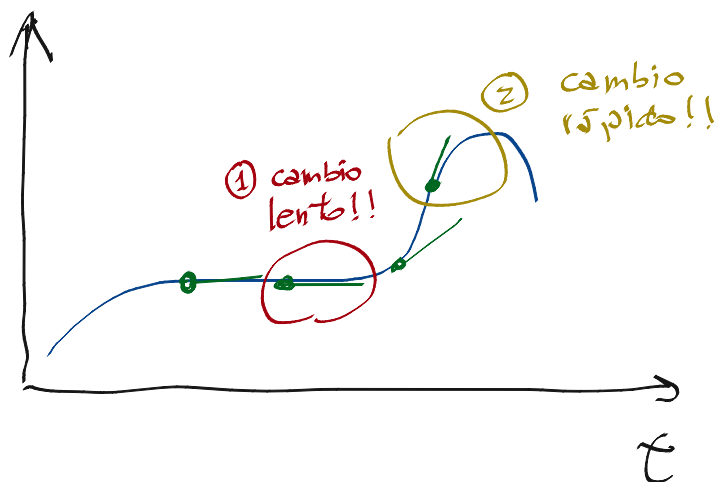
$\text{sen}(x) \sim x$
 $x \rightarrow 0$



b)



c)



Clase 17

Series de Fourier

Filtrado

FILTROS SELECTIVOS EN FRECUENCIA

Def: seleccionan con exactitud algunos rangos (bandas) de frecuencia.

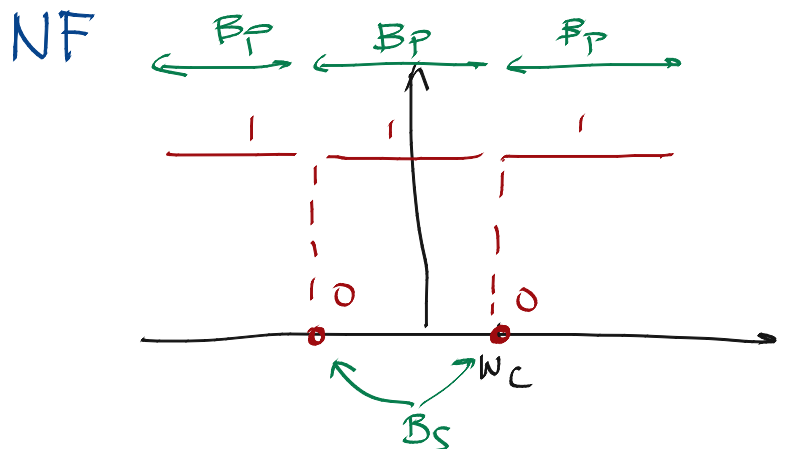
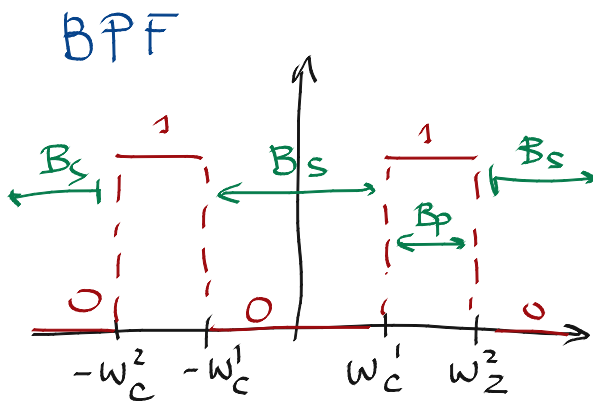
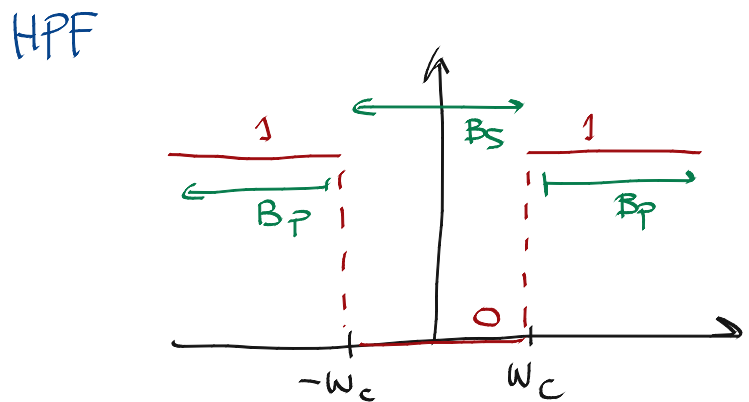
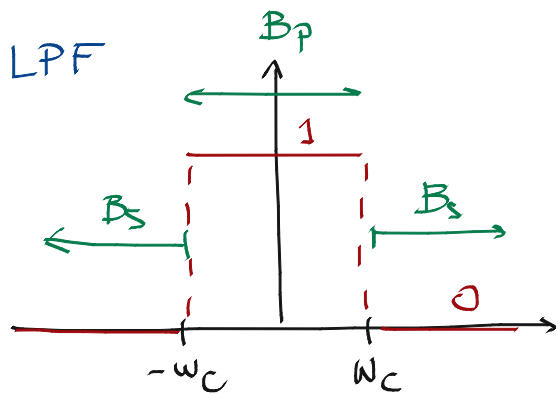
Tipos:

- * pasabajos (LPF)
- * pasaltos (HPF)
- * pasabanda (BPF)
- * notch (NF)

Características:

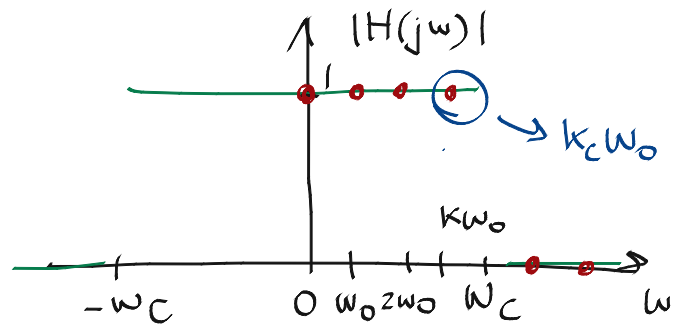
- * filtros ideales
- * frecuencia(s) de corte
- * ganancias: banda pasante y supresora

Filtros ideales (T.c.)



Ejemplo: "pasadizos ideal"

$$H(j\omega) = \begin{cases} 1 & \omega \in (-\omega_c, \omega_c) \\ 0 & \text{e.o.c.} \end{cases}$$

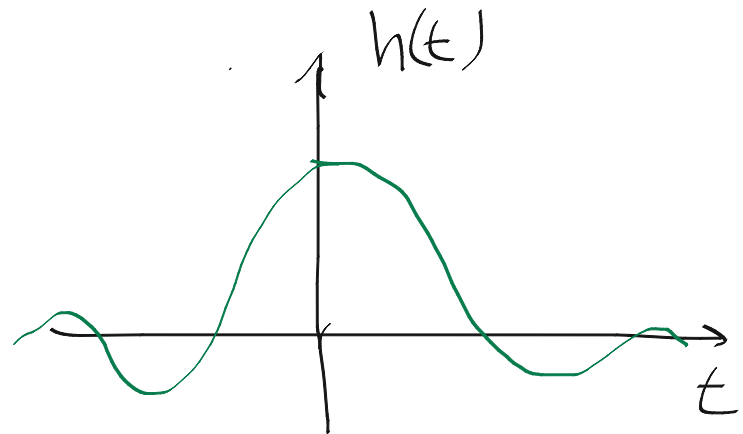


$$h(t) = \sum_{k=-\infty}^{\infty} h_k \cdot e^{jkw_0 t} = \sum_{k=-k_c}^{k_c} 1 \cdot e^{jkw_0 t} \stackrel{n=k+k_c}{=} \sum_{n=0}^{2k_c} e^{j(\underline{n-k_c})w_0 t}$$

$$h(t) = e^{-jk_c \omega t} \sum_{n=0}^{2k_c} e^{jn \omega t} = e^{-jk_c \omega t} \sum_{n=0}^{2k_c} (e^{j \omega t})^n = e^{-jk_c \omega t} \frac{1 - e^{j(k_c+1) \omega t}}{1 - e^{j \omega t}}$$

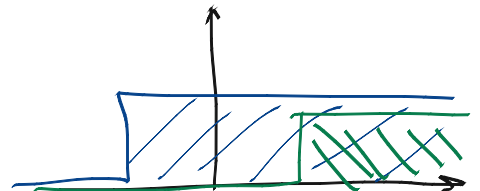
$$h(t) = \frac{e^{-jk_c \omega_0 t} - e^{j(k_c+1)\omega_0 t}}{1 - e^{j\omega_0 t}} \cdot \frac{e^{-j\frac{\omega_0}{2}t}}{e^{-j\frac{\omega_0}{2}t}} = \frac{e^{j(k_c+\frac{1}{2})\omega_0 t} - e^{-j(k_c+\frac{1}{2})\omega_0 t}}{e^{j\frac{\omega_0}{2}t} - e^{-j\frac{\omega_0}{2}t}} \cdot \frac{z_j}{z_j}$$

$$h(t) = \frac{\sin\left((k_c + \frac{1}{2})\omega_0 t\right)}{\sin\left(\frac{\omega_0}{2} t\right)}$$



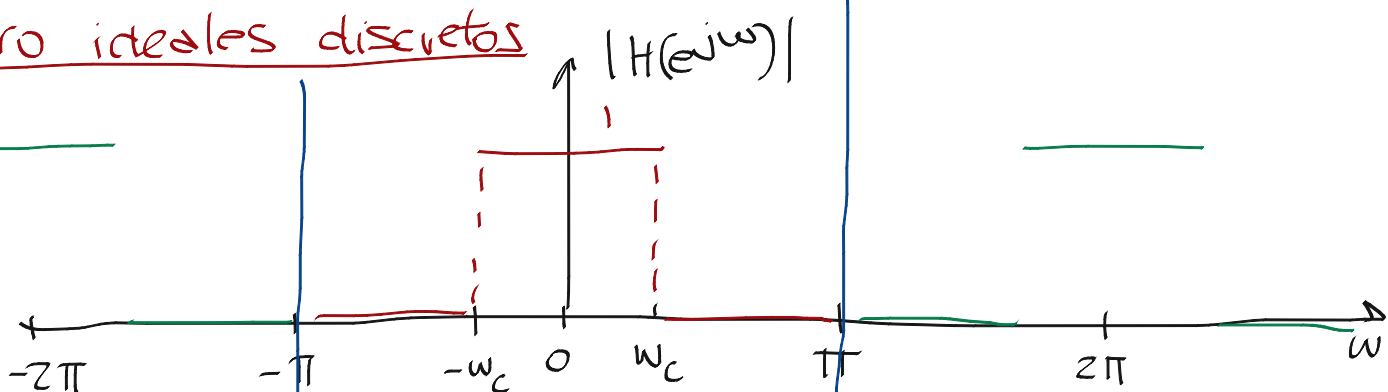
obs:

- * de un "senoc": es "dual" con la transformada
- * la respuesta al impulso tiene soporte infinito
- * no puede ser causal
- * no implementable en la práctica

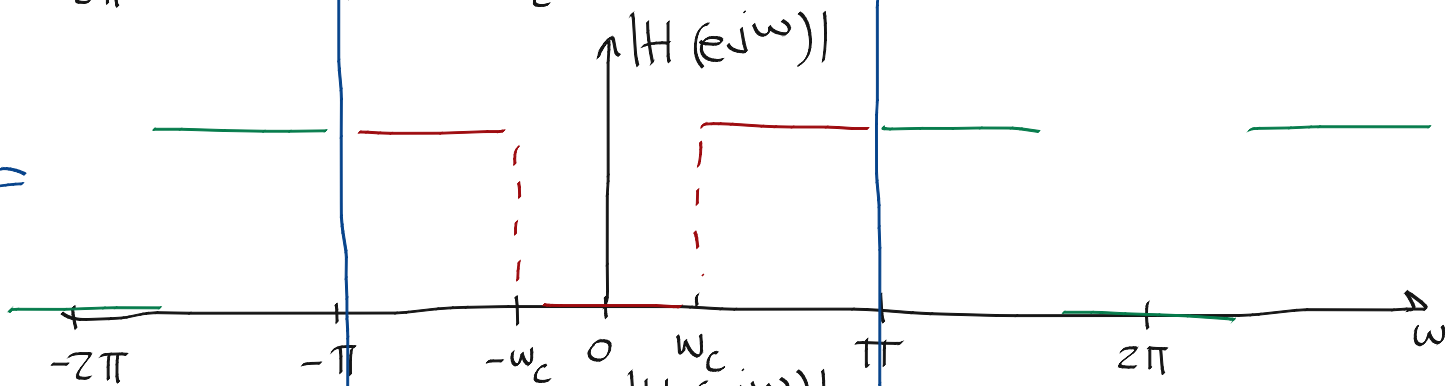


Filtros ideales discretos

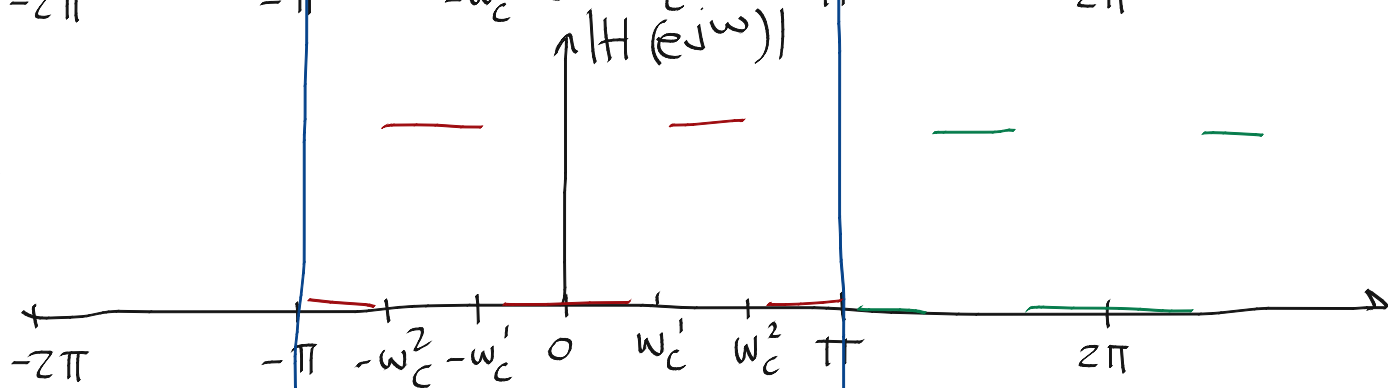
LPF



HPF



BPF



FILTROS REALES

* BW: $\rightarrow$ potencia vale 50%

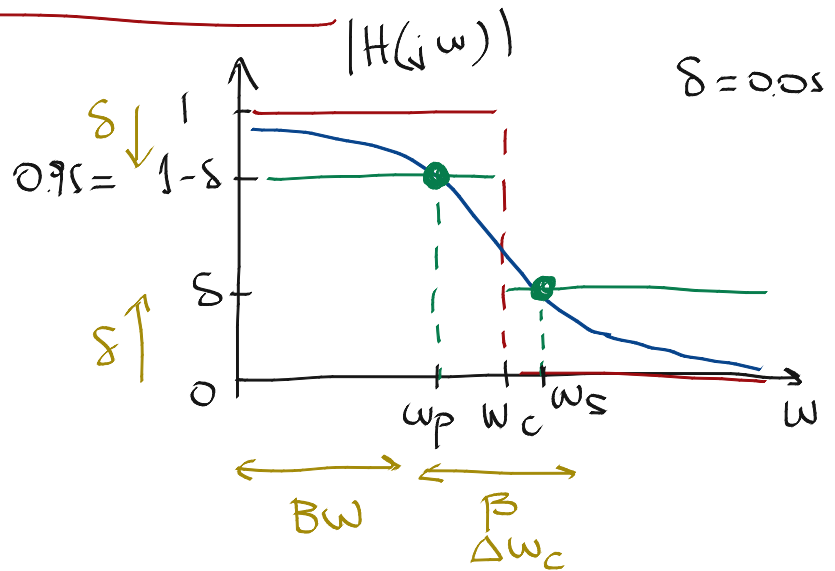
$\rightarrow$ se mide en dB

* dB: $10 \cdot \log_{10}(x)$

* $B(\omega) = 10 \cdot \log_{10}(P(\omega))$

$$B(\omega) = 10 \cdot \log_{10}(|H(j\omega)|^2)$$

$$B(\omega) = 20 \cdot \log_{10}(|H(j\omega)|)$$



* $\omega_c / |H(j\omega_c)|^2 = 0.5 \Rightarrow B(\omega_c) = 10 \cdot \log(0.5) = -3.01 \text{ dB} \approx 3 \text{ dB}$

* $|H(j\omega_c)| = \frac{1}{\sqrt{2}} \approx 0.7 \approx 70\%$

Ejemplo (3.16)

$$H(j\omega) = \frac{1}{1+j\omega}, \quad |H(j\omega)|^2 = \frac{1}{1+\omega^2}$$

$$B(\omega) = -10 \cdot \log(1+\omega^2)$$

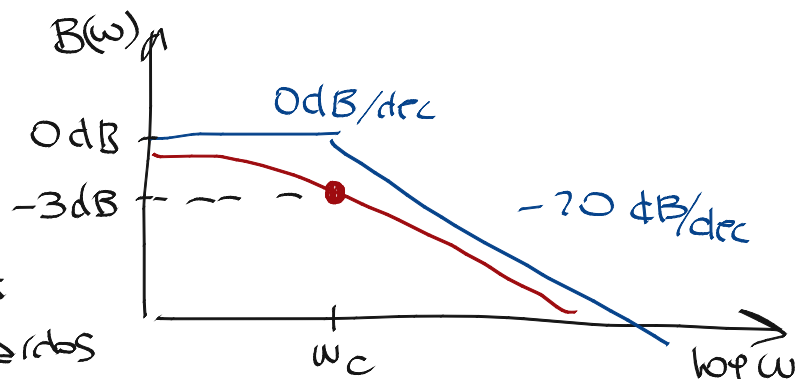
* Análisis asintótico. (Bode)

$\rightarrow \omega \approx 0 \Rightarrow B(\omega) \approx -10 \cdot \log(1) = 0 \text{ dB}$

$\rightarrow \omega \rightarrow \infty \Rightarrow B(\omega) \approx -10 \cdot \log \omega^2 = -20 \log \omega$

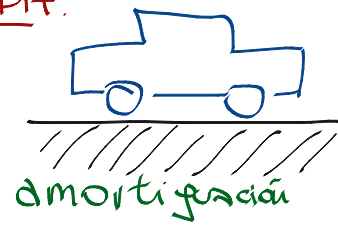
* Orden: la caída del filtro

* Orden: cantidad de derivadas o diferencias/retardos



SISTEMAS (FILTROS) BASADOS EN EC. DIF.

* en la realidad el filtrado se realiza
la implementación física de un SLIT

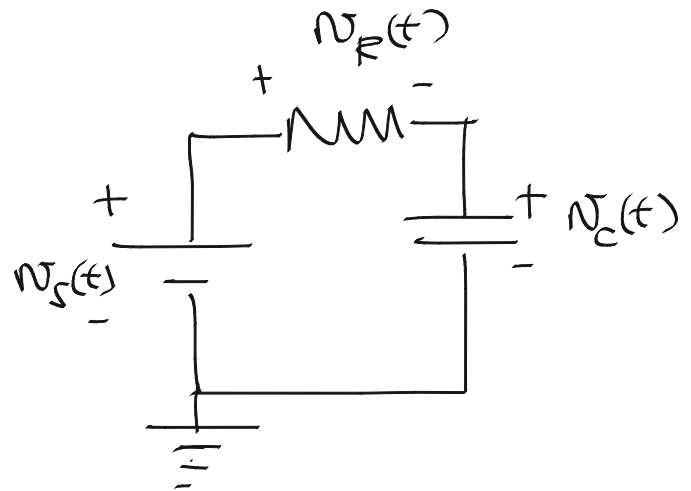


* los ponemos en práctica con circuitos eléctricos

* los SLITs son muy flexibles y hay muchos sistemas
que se pueden aproximar por SLITs.

Ejemplo: "pasabajos real"

a) Cálculo (alternativo)
de la resp. en frecuencia.



$$\frac{S}{RC \cdot v_c'(t) + v_c(t) = v_s(t)}$$

$$v_s(t) = e^{j\omega t} \rightarrow v_c(t) = H(j\omega) \cdot e^{j\omega t}$$

$$\boxed{RC = \tau}$$

$$\tau \cdot j\omega \cdot H(j\omega) \cdot e^{j\omega t} + H(j\omega) \cdot e^{j\omega t} = e^{j\omega t}$$

$$(j\omega\tau + 1) \cdot H(j\omega) = 1 \Rightarrow H(j\omega) = \frac{1}{1 + j\omega\tau}$$

$$e.g. \tau = 1 \rightarrow \frac{1}{1 + j\omega}$$

se parece

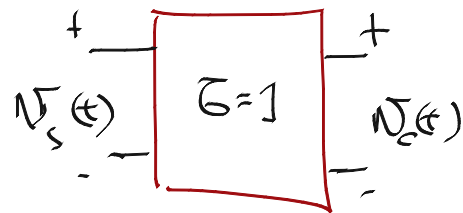


$$(\tau = 1)$$

b) Análisis en frecuencia

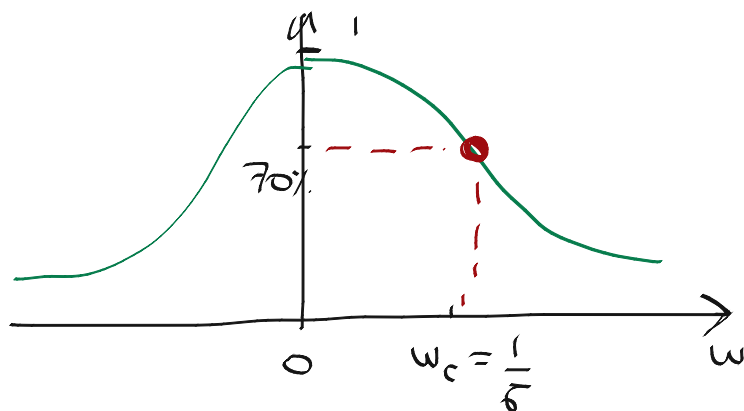
* cálculo ω_c : $|H(j\omega_c)|^2 = 0.5 \Rightarrow \frac{1}{1 + \omega_c^2 \tau^2} = 0.5$

$\Rightarrow \omega_c^2 \tau^2 + 1 = 2 \Rightarrow \boxed{\omega_c = \frac{1}{\tau}}$



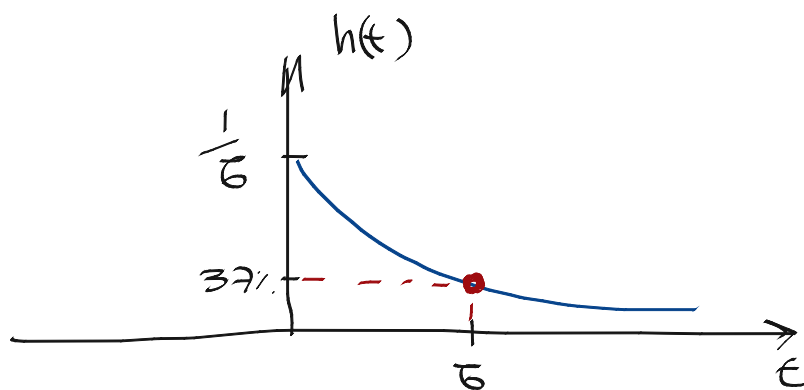
* $\omega_c = 1 \text{ Hz} \Rightarrow \tau = 1 \text{ s} \rightarrow R = 1 \Omega, C = 1 \text{ F}$

$\rightarrow R = 1 \text{ k}\Omega, C = 1 \mu\text{F}$

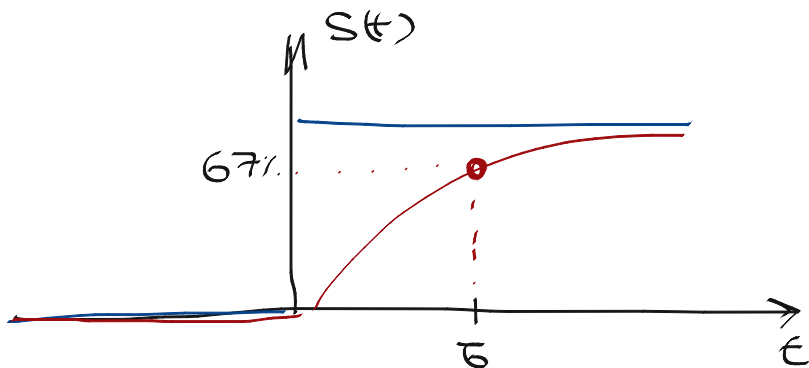


c) Análisis en el tiempo

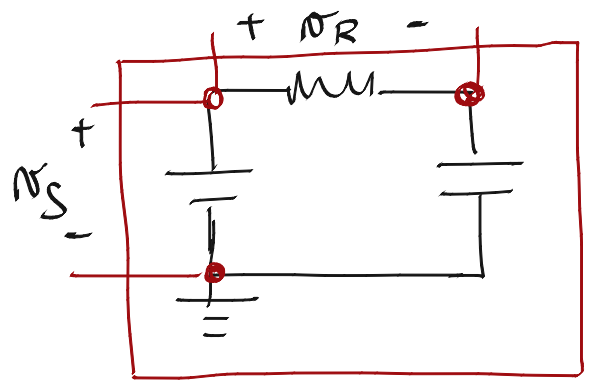
i) $h(t) = u(t) \cdot \frac{1}{\tau} \cdot e^{-\frac{t}{\tau}}$
 $h(\tau) = \frac{1}{\tau} \cdot \underbrace{e^{-1}}_{0.37} = 0.37$



ii) $s(t) = u(t) \left(1 - e^{-\frac{t}{\tau}}\right)$



Ejemplo: "pasajitos real"
 a) Resp. frecuencia.



$$N_S(t) = N_R(t) + N_C(t)$$

$$N'_S(t) = N'_R(t) + N'_C(t)$$

$$N'_S(t) = N'_R(t) + \frac{i(t)}{C} = N'_R(t) + \frac{N_R(t)}{RC} \quad \text{RC } \tau$$

$$N_R(t) = R \cdot i(t)$$

$$C \cdot N'_C(t) = i(t)$$

$$\tau \cdot N'_R(t) + N_R(t) = \tau \cdot N'_S(t)$$

$$N_S(t) = e^{j\omega t}$$

$$\tau \cdot j\omega \cdot H(j\omega) \cdot e^{j\omega t} + H(j\omega) \cdot e^{j\omega t} = \tau \cdot j\omega \cdot e^{j\omega t}$$

$$N_R(t) = H(j\omega) \cdot e^{j\omega t}$$

$$H(j\omega) \cdot (j\omega\tau + 1) = j\omega\tau \Rightarrow$$

$$H(j\omega) = \frac{j\omega\tau}{1 + j\omega\tau}$$

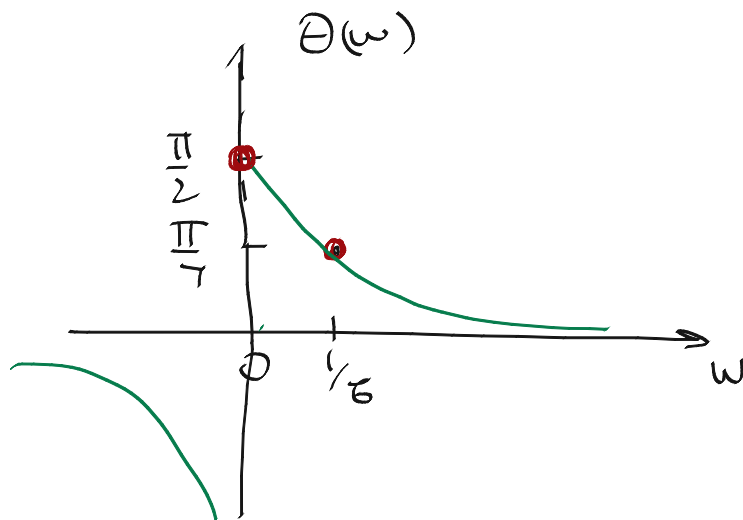
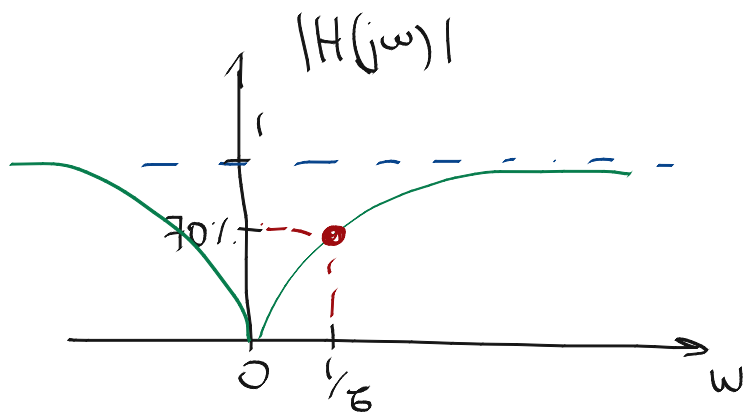
b)

$$|H(j\omega)|^2 = \frac{\omega^2 \tau^2}{1 + \omega^2 \tau^2} = 0.5 \rightarrow \omega_c$$

$$|H(j\frac{1}{\tau})|^2 = \frac{\frac{1}{\tau^2} \cdot \tau^2}{1 + \frac{1}{\tau^2} \cdot \tau^2} = \frac{1}{1+1} = \frac{1}{2}$$

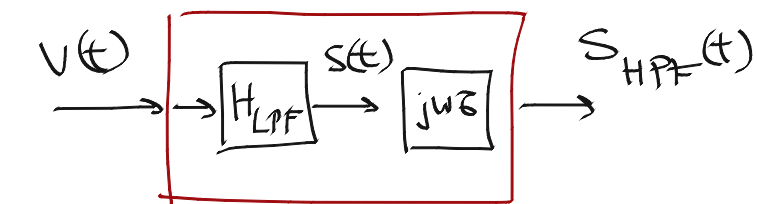
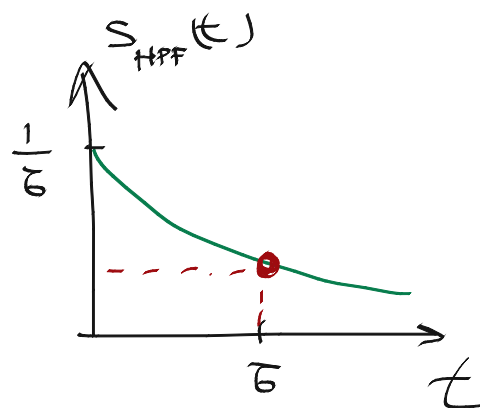
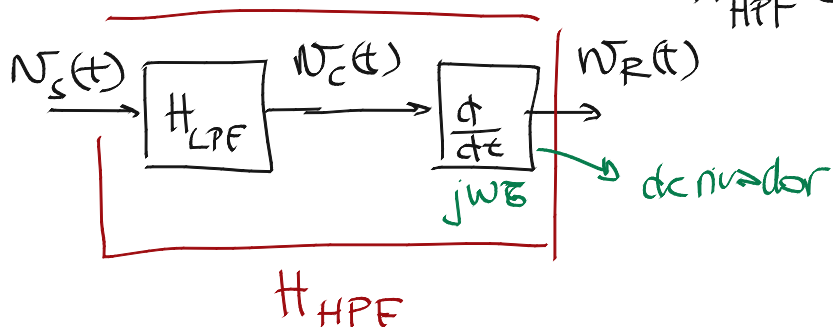
$$\Theta(\omega) = \frac{\pi}{2} - \arctan(\omega\tau)$$

$$\Theta(\frac{1}{\tau}) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$



c) Análisis en el tiempo.

$$H_{HPF}(\omega) = j\omega\tau \cdot H_{LPF}(\omega)$$



$$S_{HPF}(t) = S'_{LPF}(t) = \boxed{V(t) \cdot \frac{1}{\tau} \cdot e^{-\frac{t}{\tau}}}$$

Practico

Series de Fourier en t. discreto

Sistemas basados en ecuaciones
en diferencias

FILTROS DISCRETOS BASADOS EN E.C. DIF.

Ejemplo IIR (p 244)

$$y[n] - a y[n-1] = x[n]$$

$$X[n] = e^{j\omega n}$$

$$y[n] = H(e^{j\omega}) \cdot e^{j\omega n}$$

$$H(e^{j\omega}) e^{j\omega n} - a H(e^{j\omega}) e^{j\omega(n-1)} = e^{j\omega n}$$

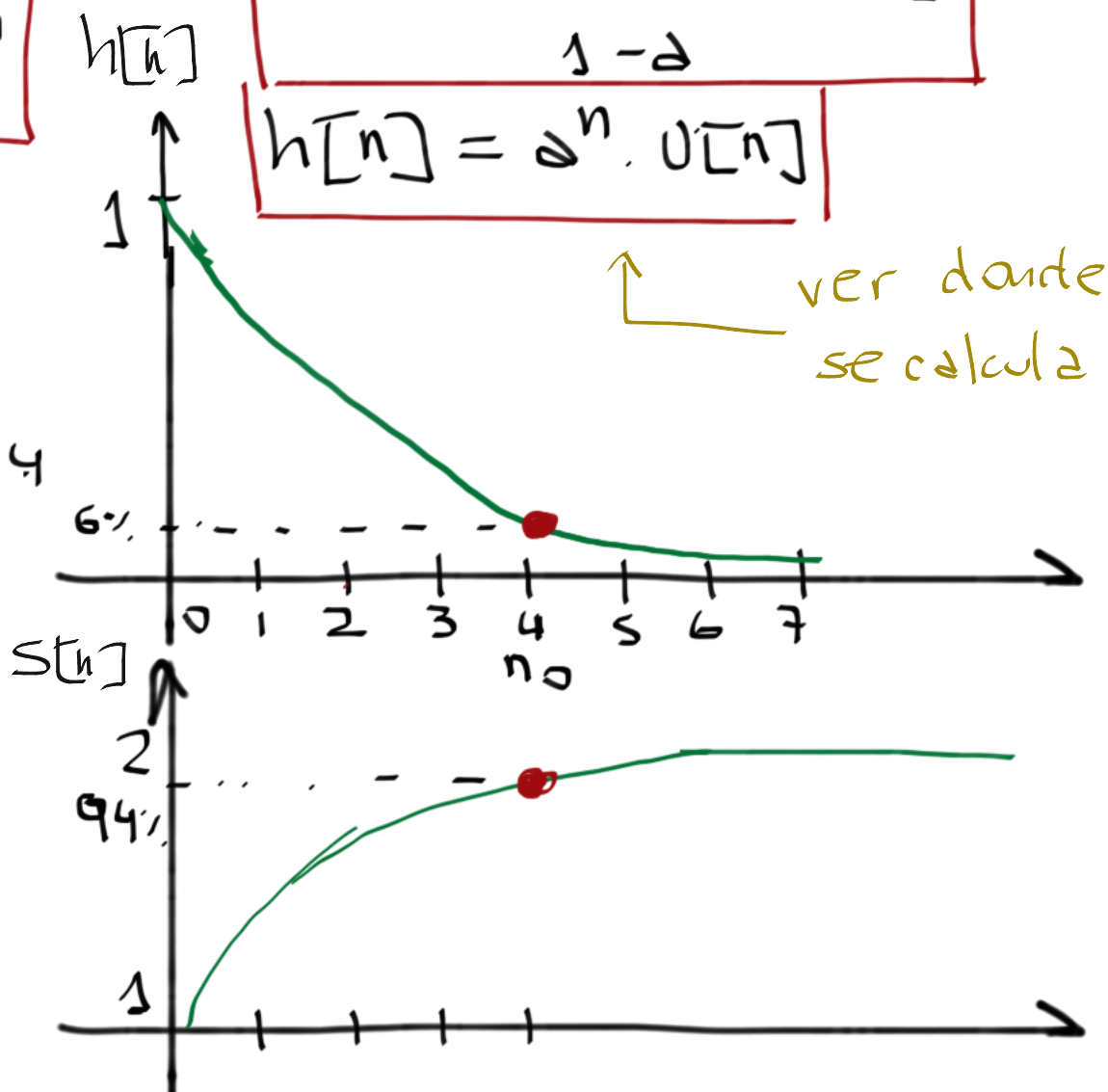
$$(1 - a e^{-j\omega}) H(e^{j\omega}) = 1$$

$$H(e^{j\omega}) = \frac{1}{1 - a e^{j\omega}}$$

$$S[n] = \frac{1 - a^{n+1}}{1 - a} \cdot u[n]$$

$$h[n] = a^n \cdot u[n]$$

$$n_0 = \left\lceil \frac{\log(0.1)}{\log(a)} \right\rceil = \lceil 3.3 \rceil = 4$$
$$a^4 = 0.0625$$



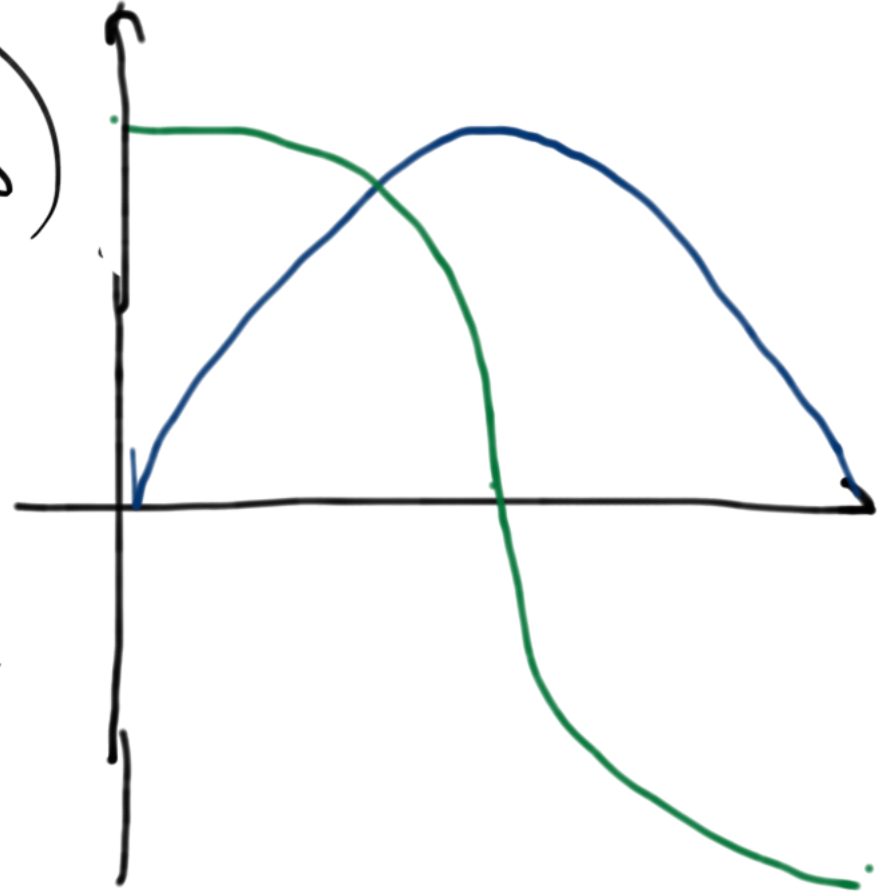
$$|H(e^{j\omega})| = \frac{1}{\sqrt{(1 - a \cos \omega)^2 + a^2 \sin^2 \omega}}$$

$$\angle H(e^{j\omega}) = -a \tan\left(\frac{a \sin \omega}{1 - a \cos \omega}\right)$$

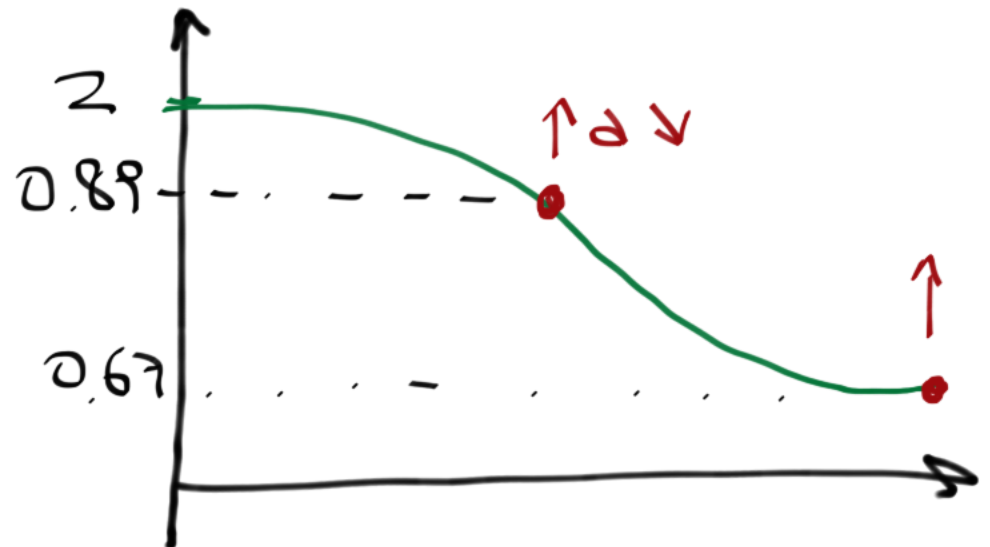
$$|H(e^{j0})| = 2$$

$$|H(e^{j\frac{\pi}{2}})| = \frac{1}{\sqrt{1+a}} = 0.89$$

$$|H(e^{j\pi})| = \frac{1}{\sqrt{(1+a)^2}} = 0.67$$



obs: * pasabajas: 070
 * pasadibos: 250
 * ver 3.17 matlab



$$S[n] = u[n] * h[n]$$

$$S[n] = \sum_{k=-\infty}^{\infty} u[k] \cdot d^k v[n-k] = \sum_{k=0}^n d^k = \frac{1-d^{n+1}}{1-d}$$

$$S[n] = \frac{1-d^{n+1}}{1-d} u[n]$$

Anexo

Material extra