Partes: $f, g: [a,b] \to \mathbb{R}$ derivables

$$\int_a^b f(x) g'(x)\, dx = f(x) g(x) \Big|_a^b - \int_a^b f'(x) g(x)\, dx$$

$$\left[(fg)' = f'g + fg' \iff fg' = (fg)' - f'g \quad \text{integrando esto} \right]$$

Ej Encontrar una primitiva de la función $h(x) = e^x \operatorname{sen}(x)$. Está definida para cualquier $x \in \mathbb{R}$.

Una primitiva es $H(x) = \int_a^x h(u)\, du = \int_a^x \underbrace{\overbrace{e^u}^{f(u)}}_{g'(u)} \underbrace{\overbrace{\operatorname{sen}(u)}^{g'(u)}}_{f(u)}\, du$

Como sen tiene primitiva $-\cos$ $\quad (-\cos' = \operatorname{sen})$

$$H(x) = \overbrace{e^u}^{f(u)} \overbrace{(-\cos(u))}^{g(u)} \Big|_a^x - \int_a^x \overbrace{e^u}^{f'(u)} \overbrace{(-\cos(u))}^{g(u)}\, du = -e^u \cos(u) \Big|_a^x + \int_a^x e^u \cos(u)\, du$$

$$H(x) = \overbrace{e^u}^{g(u)} \overbrace{\text{sen}(u)}^{f(u)} \Big|_a^x - \int_a^x \overbrace{e^u}^{g(u)} \overbrace{\cos u}^{f'(u)}\, du$$

Sumando ambos renglones: $2H(x) = \left(e^u \text{sen}(u) - e^u \cos(u) \right) \Big|_a^x$

$$H(x) = e^u \left(\text{sen}(u) - \cos(u) \right) \Big|_a^x$$

Así que $K(x) = e^x \left(\text{sen}(x) - \cos(x) \right)$ es una primitiva de h.

Algunas integrales de funciones trigonométricas

Recordar:

$$\operatorname{sen}^2(x) + \cos^2(x) = 1$$

$$\cos(2x) = \cos^2(x) - \operatorname{sen}^2(x)$$

$$\operatorname{sen}(2x) = 2\cos x \operatorname{sen} x$$

$$\overline{1} - \underbrace{\cos(2x)}_{} = \overbrace{\operatorname{sen}^2(x) + \cancel{\cos^2(x)}}^{} - (\underbrace{\cancel{\cos^2(x)} - \operatorname{sen}^2 x}_{}) = 2\operatorname{sen}^2(x)$$

Así que $\operatorname{sen}^2(x) = \dfrac{1-\cos(2x)}{2} = \dfrac{1}{2} - \dfrac{\cos(2x)}{2}$

Usamos esto para:

$$\int_a^x \operatorname{sen}^2(u)\,du = \frac{1}{2}\int_a^x 1\,du - \frac{1}{2}\underbrace{\int_a^x \cos(2u)\,du}_{} = \frac{1}{2}u\Big|_a^x - \frac{1}{4}\operatorname{sen}(v)\Big|_{2a}^{2x}$$

$$\int_a^x \overbrace{\cos(2u)}^{g(f(u))}\,du = \frac{1}{2}\int_a^x \overbrace{\cos(2u)}^{g(f(u))}\,\overbrace{2}^{f'(u)}\,du = \frac{1}{2}\int_{f(a)}^{f(x)} g(v)\,dv = \frac{1}{2}\int_{2a}^{2x} \cos(v)\,dv$$

$$= \underbrace{\frac{1}{2}\operatorname{sen}(v)\Big|_{2a}^{2x}}_{}$$

$g(v) = \cos(v)$, $f(u) = 2u$

$\int_a^x \operatorname{sen}^2(u)\,du$ es una primitiva de sen^2 ✓

y difiere de $F(x) = \frac{x}{2} - \frac{\operatorname{sen}(2x)}{4}$ en una constante

Aunque $F(x) = \frac{x}{2} - \frac{\operatorname{sen}(2x)}{4}$ es primitiva de $\underline{\operatorname{sen}^2(x)}$..

Tratemos con $\operatorname{sen}^3(x)$, Escribimos $\operatorname{sen}^3(x) = \operatorname{sen}(x)\left(1-\cos^2(x)\right)$.

$$\int_0^x \operatorname{sen}^3(u)\,du = \int_0^x \operatorname{sen}(u)\left(1-\cos^2(u)\right)du = \underbrace{\int_0^x \operatorname{sen}(u)\,du}_{-\cos(u)\big|_0^x} - \int_0^x \operatorname{sen}(u)\cos^2(u)\,du$$

$$\int_0^x \underbrace{\cos^2(u)}_{g(\cos(u))}\,\overbrace{\operatorname{sen}(u)}^{-f'(u)}\,du = -\int_0^x g(f(u))f'(u)\,du = -\int_{f(0)}^{f(x)} g(v)\,dv = -\int_{\cos(0)}^{\cos(x)} v^2\,dv$$

$$= -\frac{v^3}{3}\Bigg|_{\cos(0)}^{\cos(x)} = -\left[\frac{\cos^3(x)}{3} - \frac{1}{3}\right]$$

$g(v) = v^2$

$f(u) = \cos(u) \Rightarrow f'(u) = -\operatorname{sen}(u)$

Así que $\displaystyle\int_0^x \operatorname{sen}^3(x) = -\cos(x)\Big|_0^x + \left(\frac{\cos^3(x)}{3} - \frac{1}{3}\right)$

Así que $\displaystyle -\cos(x) + \frac{\cos^3(x)}{3}$ es primitiva de $\operatorname{sen}^3(x)$

<u>Teorema</u> Si n es un número natural >0,

$$\int_a^x \operatorname{sen}^n(u)\,du = -\frac{1}{n}\operatorname{sen}^{n-1}(u)\cos(u)\Big|_a^x + \frac{n-1}{n}\int_a^x \operatorname{sen}^{n-2}(u)\,du$$

$$\int_a^x \cos^n(u)\,du = \frac{1}{n}\cos^{n-1}(u)\operatorname{sen}(u)\Big|_a^x + \frac{n-1}{n}\int_a^x \cos^{n-2}(u)\,du$$