$F' = f \qquad \int_a^b f = F(b) - F(a)$

$\int_a^b f+g = \int_a^b f + \int_a^b f$;

Método de sustitución

Recordar "regla de la cadena". Si $[a,b] \xrightarrow{f} [c,d] \xrightarrow{H} \mathbb{R}$ son funciones derivables, entonces $\boxed{(H \circ f)'(x) = f'(x) \cdot H'(f(x))}$

Teorema $[a,b] \xrightarrow{f} [c,d] \xrightarrow{g} \mathbb{R}$ f derivable y g continua.

Entonces $\boxed{\int_{f(x_0)}^{f(x)} g = \int_{x_0}^{x} f' \cdot (g \circ f)}$

También se puede escribir:

$$\boxed{\int_{f(x_0)}^{f(x)} g(u)\,du = \int_{x_0}^{x} g(f(v))\cdot f'(v)\,dv}$$

dem $g:[c,d]\to\mathbb{R}$ es continua $\Rightarrow$ tiene una primitiva $G:[c,d]\to\mathbb{R}$

$$\boxed{\int_{f(x_0)}^{f(x)} g} = G\Big|_{f(x_0)}^{f(x)} = G(f(x)) - G(f(x_0))$$

$$(G\circ f)'(x) = f'(x)\cdot G'(f(x)) \quad \text{por la regla de la cadena}$$

$$= f'(x)\cdot g(f(x))$$

$$\boxed{\int_{x_0}^{x} f'\cdot(g\circ f)} = \int_{x_0}^{x} (G\circ f)' = (G\circ f)\Big|_{x_0}^{x} = G(f(x)) - G(f(x_0))$$

$$=$$

Ej:

$$\int_{y_0}^{y} \underbrace{(\overbrace{x^2}^{f(x)}+1)^3}_{g(f(x))} \overbrace{2x}^{f'(x)}\, dx$$

$g(f(x))$
$\parallel$
$g(x^2)$

$f(x) = x^2$

$g(u) = (u+1)^3$

$$\parallel$$

$$\int_{f(y_0)}^{f(y)} g(u)\, du = \int_{y_0^2}^{y^2} \underbrace{\underbrace{(u+1)}_{h(u)}{}^3}_{k(h(u))} 1\, du =$$

$k(h(u))$
$\parallel$
$(u+1)^3$

$h(u) = u+1$

$h'(u) = 1$

$k(v) = v^3$

$$= \int_{y_0^2}^{y^2} k(h(u)) \cdot \overbrace{h'(u)}^{1}\, du$$

$$= \int_{h(y_0^2)}^{h(y^2)} k(v)\, dv = \int_{y_0^2+1}^{y^2+1} v^3\, dv = \frac{v^4}{4}\Bigg|_{y_0^2+1}^{y^2+1}$$

Caso de cambio de variable $v = u+\lambda$ $\lambda \in \mathbb{R}$

$f: [a,b] \to [c,d]$ $f(u) = u+\lambda$

En este caso, $f'(u) = 1.$

$$\underbrace{\int_{x_0+\lambda}^{x+\lambda} g(v)\,dv}_{} = \int_{x_0}^{x} g(\underbrace{f(u)}_{u+\lambda})\ \underbrace{f'(u)}_{=1}\,du = \underbrace{\int_{x_0}^{x} g(u+\lambda)\,du}_{}.$$

Caso $v = \alpha u+\lambda$ $\alpha, \lambda \in \mathbb{R}$.

o sea $f(u) = \alpha u+\lambda$. $\Rightarrow$ $f'(u) = \alpha$.

$$\int_{\alpha x_0+\lambda}^{\alpha x+\lambda} g(v)\,dv = \alpha \int_{x_0}^{x} g(\alpha u+\lambda)\cdot du.$$

Ej

$$\int_a^b \overbrace{\text{sen}}^{g}(\overbrace{2x}^{f(x)})\,dx.$$

$f(x) = 2x$, $f'(x) = 2$

$g(v) = \text{sen}(v).$

$$" \quad \frac{1}{2}\int_a^b \overbrace{\text{sen}}^{g}(\overbrace{2x}^{f(x)})\,\overbrace{2}^{f'}\,dx$$

$$= \frac{1}{2}\int_{2a}^{2b} \text{sen}(v)\,dv = \frac{1}{2}\; -\cos(v)\Big|_{2a}^{2b}$$

Ej.

$$\int_a^b \underbrace{(x^3+x)^9}_{g(f(x))}\underbrace{(3x^2+1)}_{f'(x)}\,dx$$

$f(x) = x^3+x$

$g(v) = v^9$

$$= \int_{a^3+a}^{b^3+b} v^9\,dv = \frac{v^{10}}{10}\Bigg|_{a^3+a}^{b^3+b}$$

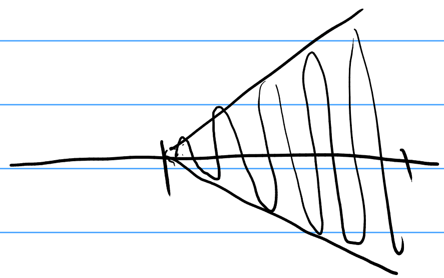
Ej $\displaystyle\int_0^{\sqrt{\pi}} \underbrace{x}_{\frac{f'(x)}{2}} \overbrace{\mathrm{sen}(\underbrace{x^2}_{f(x)})}^{g} \, dx$

$f(x) = x^2$

$g(v) = \mathrm{sen}(v)$

$$= \frac{1}{2} \int_0^{\sqrt{\pi}} f'(x)\, g(f(x)) \, dx = \frac{1}{2} \int_{f(0)}^{f(\sqrt{\pi})} g(v)\, dv$$

$$= \frac{1}{2} \int_0^{\pi} \mathrm{sen}(v)\, dv = \frac{1}{2} \left(-\cos(v)\right)\Big|_0^{\pi} = \frac{1}{2} \overbrace{\left(1 - (-1)\right)}^{2} = 1$$



Ej

$$\int_a^b \overbrace{\text{sen}(x)}^{g(f(x))}\overbrace{\cos(x)}^{f'(x)}\,dx$$

$g(v) = v$

$f(x) = \text{sen}(x)$

$$= \int_{f(a)}^{f(b)} g = \frac{v^2}{2}\Bigg|_{\text{sen}\,a}^{\text{sen}\,b} = \frac{1}{2}\left(\text{sen}^2 b - \text{sen}^2 a\right).$$

Ej Hallar una primitiva de $h(x) = \dfrac{x}{x+1}$ $\quad h : (-1, +\infty) \to \mathbb{R}$

$$H(y) = \int_0^y h = \int_0^y \underbrace{\frac{x}{x+1}}_{g(f(x))}\,dx = \int_{f(0)}^{f(y)} \frac{v-1}{v}\,dv$$

$f(x) = x+1$

$f'(x) = 1$

$g(v) = \dfrac{v-1}{v}$

$$= \int_1^{y+1}\left(1 - \frac{1}{v}\right)dv = \int_1^{y+1} 1\,dv - \int_1^{y+1} \frac{1}{v}\,dv$$

$$= v\Big|_1^{y+1} - \log(v)\Big|_1^{y+1} = y + \cancel{1} - \cancel{1} - \Big(\log(y+1) - \overbrace{\log(1)}^{0}\Big)$$

$$= y - \log(y+1)$$

Integración por partes

Recordemos que $\underline{(f\cdot g)' = f'g + fg'}$ (cuando f' y g' existen)

O sea $\underbrace{fg' = (fg)' - f'g}$

Integramos :

$$\underline{\int_a^b f\cdot g' = \int_a^b (fg)' - \int_a^b f'g} = fg\Big|_a^b - \int_a^b f'g$$

Así que:

$$\boxed{\int_a^b f\cdot g' = (fg)\Big|_a^b - \int_a^b f'g \, .}$$

ó

$$\boxed{\int_a^b f(x)\, g'(x)\, dx = fg\Big|_a^b - \int_a^b f'(x) g(x) \, .}$$

$a>0, b>0$ $\qquad g(x)=x$

<u>Ej</u> $\displaystyle \int_a^b \log(x)\,dx = \int_a^b \underbrace{\log(x)}_{f(x)} \cdot \underbrace{1}_{g'(x)}\,dx$

$$= \log(x)\,x \Big|_a^b - \int_a^b \overbrace{\underbrace{\log'(x)}_{1/x} \cdot x}^{1}\,dx$$

$$= \log(x)\,x \Big|_a^b - x\Big|_a^b =$$

$$= \left(x\cdot\log(x) - x\right)\Big|_a^b .$$

Así que $L(x) = x\log(x) - x$ es una primitiva de $\log(x)$.

$$L(x) = \int_a^x \log(u)\,du = u\log(u) - u \Big|_a^x = x\log(x) - x - \underbrace{(\text{número})}_{\text{no importa para la primitiva.}}$$

Ej: Hallar una primitiva de $h(x) = e^x \operatorname{sen}(x)$.

$$H(y) = \int_0^y h = \int_0^y \underbrace{e^x}_{g'} \underbrace{\operatorname{sen}(x)}_{f} \, dx = e^x \operatorname{sen} x \Big|_0^y - \int_0^y \underbrace{e^x}_{g} \underbrace{\cos(x)}_{f'} \, dx$$

$$\int_0^y \underbrace{e^x}_{f} \underbrace{\cos(x)}_{g'} \, dx = e^x \operatorname{sen}(x) \Big|_0^y - \int_0^y e^x \operatorname{sen}(x) \, dx$$

$$H(y) = e^y \operatorname{sen}(y) - \left(e^y \operatorname{sen}(y) - H(y) \right)$$