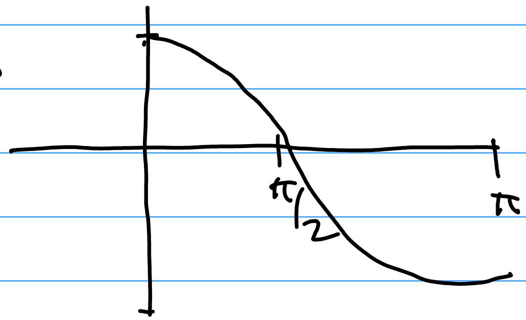
# Arcocoseno

cos




$\cos : [0,\pi] \to [-1,1]$

$\cos(0) = 1$
$\cos(\pi) = -1$
y $\cos([0,\pi])$ es un intervalo $\Rightarrow$

$[-1,1] \supset \cos([0,\pi]) \supset [-1,1] \Rightarrow \cos([0,\pi]) = [-1,1]$

Así que $\cos : [0,\pi] \to [-1,1]$ es sobre.

$\cos'(x) = -\operatorname{sen}(x) < 0$ si $x \in (0,\pi) \Rightarrow$ cos es estrictamente creciente en $[0,\pi]$ $\Rightarrow$

$\cos : [0,\pi] \to [-1,1]$ es biyectiva. La inversa es $\arccos : [-1,1] \to [0,\pi]$

$\arccos(-1) = \pi$ $\qquad$ $\arccos(0) = \pi/2$
$\arccos(1) = 0$

$y \in (-1,1)$

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

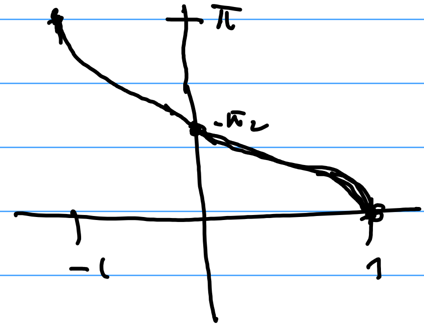
$$\arccos'(y) = \frac{1}{\cos'(\arccos(y))}$$

$$= -\frac{1}{\sin(\arccos(y))} = -\frac{1}{1-\cos^2(\arccos(y))} = -\frac{1}{1-y^2}$$

$\sin^2 x + \cos^2 x = 1 \Rightarrow \sin^2 x = 1 - \cos^2(x)$

como $x \in [0,\pi]$, $\sin(x) \geq 0$

$\rightarrow \sin x = \sqrt{1-\cos^2(x)}$




Si $y \in (-1,1)$, $\arccos'(y) < 0$

$$-\frac{1}{\underbrace{1-y^2}_{0^+}} \xrightarrow[y\to -1^+]{} -\infty \qquad (+\infty)$$

$$-\frac{1}{1-y^2} \xrightarrow[y\to 1]{} -\infty$$

## Arcotangente

$\tan(x) = \dfrac{\operatorname{sen}(x)}{\cos(x)}$ $\quad$ cos es positivo en $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\tan : \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$.

Como tan es continua, su imagen es un intervalo.

$$\tan(x) = \frac{\overbrace{\operatorname{sen}(x)}^{\to \pm 1}}{\underbrace{\cos(x)}_{\to 0^+}} \xrightarrow[x \to \pm\frac{\pi}{2}]{} \pm\infty$$

→ el intervalo es todo $\mathbb{R}$

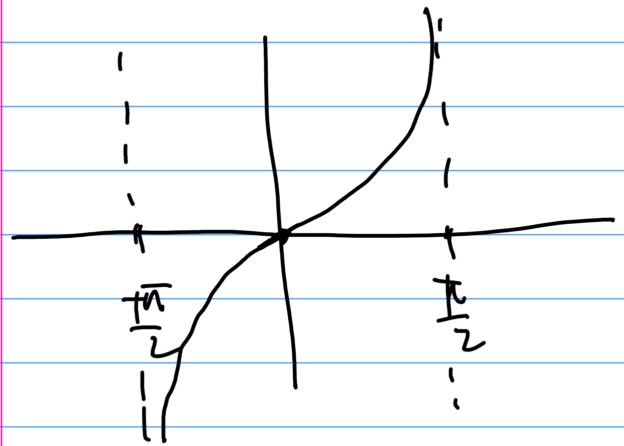
⇒ tan es sobreyectiva.

$$\tan'(x) = 1 + \underbrace{\tan^2(x)}_{\geq 0} \geq 1 > 0$$

⇒ tan es estrictamente creciente

→ tan tiene inversa

$\arctan : \mathbb{R} \to \left(-\frac{\pi}{2}, \pi/2\right)$

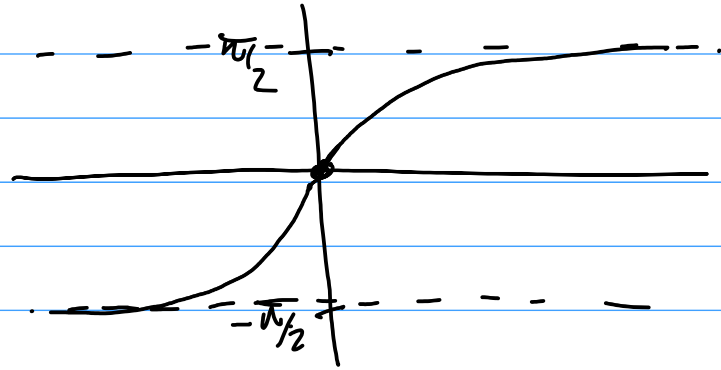


$$\arctan'(y) = \frac{1}{\tan'(\arctan(y))} = \frac{1}{1+\tan^2(\arctan(y))} = \frac{1}{1+y^2} > 0$$

$\arctan(0) = 0$ (pues $0 = \tan(0)$).

arctan es estrictamente creciente y su imagen es todo $(-\frac{\pi}{2}, \pi/2)$

Asíntotas $\arctan(y) \xrightarrow[y \to \pm\infty]{} \pm\pi/2$



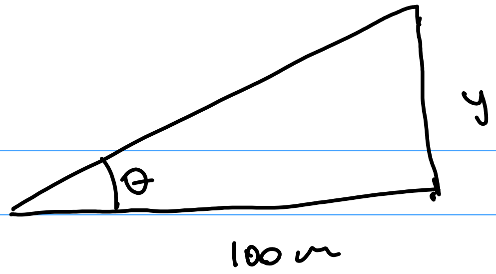

$$\arctan'(y) = (1+y^2)^{-1}$$

$$\arctan''(y) = -2y\,\underbrace{(1+y^2)^{-2}}_{>0}$$

$\arctan''(y)$ es $< 0$ si $y > 0$

$> 0$ si $y < 0$

Ej




Se suelta un globo a 100 m del observador. El globo sube a 50 m/min.

¿A qué velocidad crece $\theta$ cuando el globo alcanza 100 m?

$y(t)$ altura

$\theta(t)$ ángulo.

$y'(t) = 50.$ $\qquad \tan(\theta(t)) = \dfrac{y(t)}{100}.$

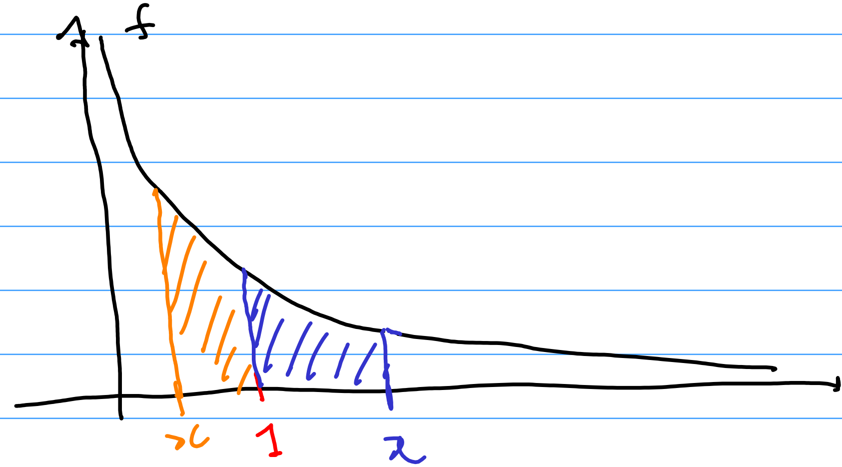
$$\theta(t) = \arctan\left(\frac{y(t)}{100}\right).$$

$$\theta'(t) = \frac{y'(t)}{100} \arctan'\left(\frac{y(t)}{100}\right) = \frac{\overbrace{y'(t)}^{50}}{100} \frac{1}{1+\left(\frac{y(t)}{100}\right)^2} = \frac{1}{2} \frac{1}{1+\left(\frac{y(t)}{100}\right)^2}$$

Si $y(t_0) = 100$, entonces $\theta'(t_0) = \dfrac{1}{2}\dfrac{1}{1+1^2} = \dfrac{1}{4}$ radianes/min.

# Logaritmos

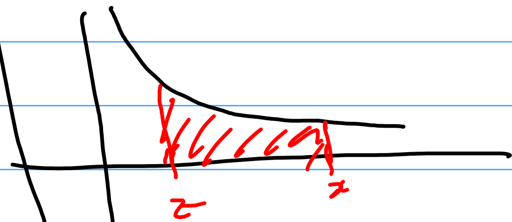
$f: (0,+\infty) \to \mathbb{R}$, $\quad f(x) = \frac{1}{x}$




Si $x \geq 1$, definimos
$\log(x)$ = área azul.

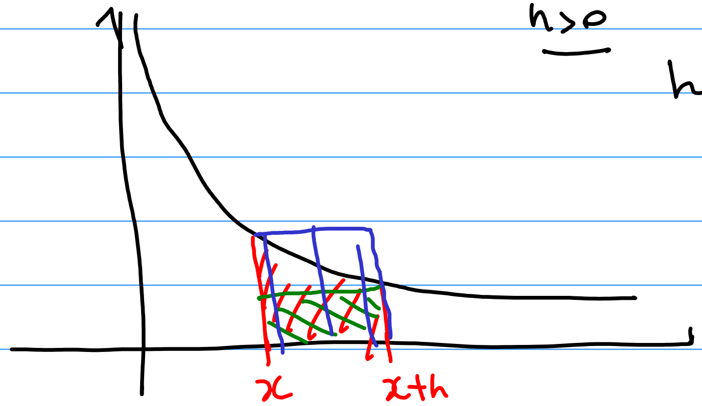
Si $x < 1$ definimos $\log(x)$ como $-$(area naranja)

Obs si $0 < z < x$, entonces $\log(x) - \log(z)$ = area en rojo.




Proposición $\log : (0,+\infty) \to \mathbb{R}$
es derivable y $\log'(x) = \frac{1}{x}$

dem




$h>0$

$$h\frac{1}{x+h} \leq \log(x+h) - \log(x) \leq h\frac{1}{x}$$

Dividiendo por $h>0$ $\quad \frac{1}{x+h} < \frac{\log(x+h) - \log(x)}{h} < \frac{1}{x}$

$\frac{1}{x+h} \xrightarrow{h\to 0^+} 1/x$

$$\Rightarrow \exists \lim_{h\to 0^+} \frac{\log(x+h) - \log(x)}{h} = 1/x$$

Similarmente si $h<0$, uno ve $\lim_{h\to 0^-} \frac{\log(x+h) - \log(x)}{h} = 1/x$

$$\Rightarrow \exists \log'(x) = \frac{1}{x}.$$

$\blacksquare$

**Prop** $\log(ab) = \log(a) + \log(b)$.

**dem** $f(x) = \log(a\cdot x)$ Queremos $f = g$.

$g(x) = \log(a) + \log(x)$.

$f'(x) = a \log'(ax) = a \frac{1}{ax} = \frac{1}{x}$

$g'(x) = \frac{1}{x}$

$\Rightarrow f' = g' \Rightarrow f(x) = g(x) + C$ para un $C \in \mathbb{R}$

$f(1) = \log(a)$.

$g(1) = \log(a) + \cancel{\log(1)}$

$\Rightarrow C = f(1) - g(1) = 0 \rightarrow f(x) = g(x)$ $\forall x$ ∎

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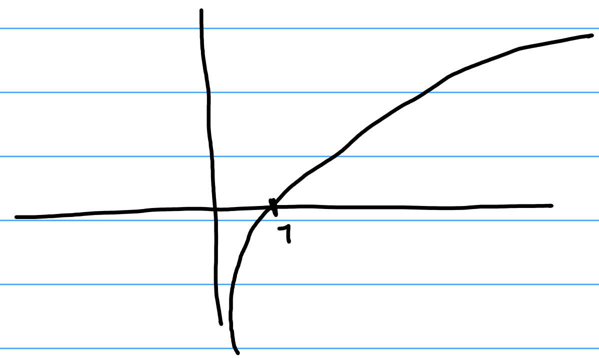
**Teorema** $\log : (0, +\infty) \to \mathbb{R}$ cumple:

- es estrictamente creciente
- $\lim_{x\to+\infty} \log(x) = +\infty$, $\lim_{x\to 0^+} \log(x) = -\infty$
- es sobreyectiva



dem $\log'(x) = \frac{1}{x} > 0 \rightarrow$ estrictamente creciente

$\Rightarrow$ existen $\lim_{x \to +\infty} \log(x)$ y $\lim_{x \to 0^+} \log(x)$

$$\lim_{x \to +\infty} \log(x) = \lim_{n \to \infty} \log(2^n) = \lim_{n \to +\infty} n \underbrace{\log(2)}_{>0} = +\infty.$$

$$\left[ \begin{array}{l} \log(a \cdot a \cdot a) = \log(a) + \log(a \cdot a) = 3\log(a) \\ \log(a^n) = \log(a \cdot a^{n-1}) = \log a + \log a^{n-1} = \cdots\cdots n \log(a). \end{array} \right]$$

Similarmente $\lim_{x \to 0^+} \log(x) = \lim_{n \to \infty} \log\left(\left(\frac{1}{2}\right)^n\right) = \lim_{n \to \infty} n \overbrace{\log(1/2)}^{<0} = -\infty.$

## Función exponencial

Como $\log : (0, +\infty) \to \mathbb{R}$ es biyectiva tiene inversa: $\exp : \mathbb{R} \to (0, +\infty)$

**Def** $e = \exp(1) \in \mathbb{R}$.

$\exp(0) = 1$ pues $0 = \log(1)$.

<u>Prop</u> $\exp(x+y) = \exp(x)\cdot\exp(y)$.

<u>dem</u> Aplicando log (que es inyectivo) la igualdad es equivalente a

$$\log\exp(x+y) = x+y$$

$$\log(\exp(x)\cdot\exp(y)) = \log\exp(x) + \log\exp(y) = x+y$$

<u>Prop</u> exp es derivable y $\exp' = \exp$

<u>dem</u> exp es derivable pues log es derivable y $\log'(x) = \frac{1}{x} > 0$.

Además $\exp'(x) = \dfrac{1}{\log'(\exp(x))} = \dfrac{1}{\frac{1}{\exp(x)}} = \exp(x)$.

<u>Notación</u> Se escribe $\exp(x) = e^x$.