

Clase 18

Transformada de Fourier

Introducción

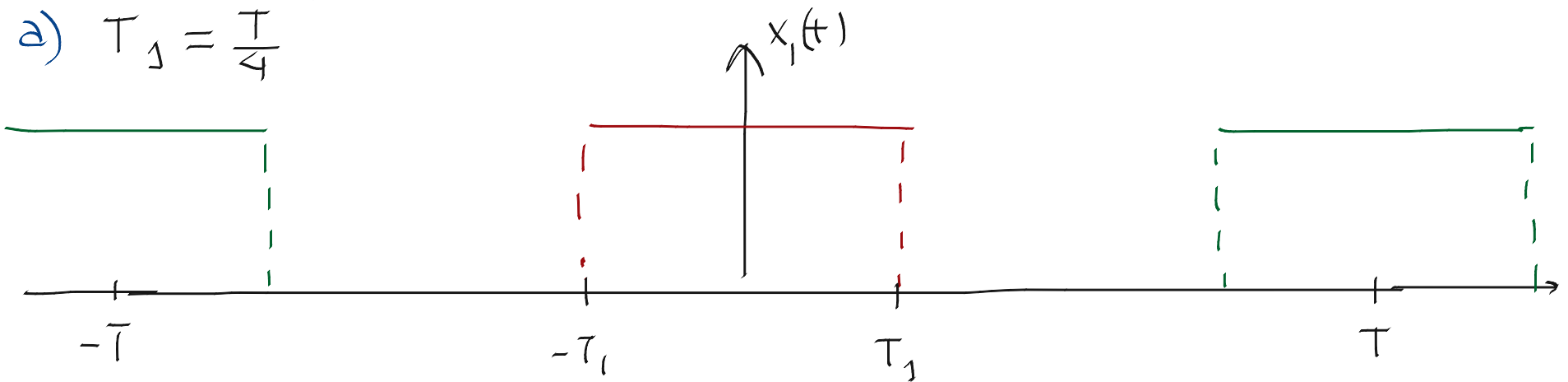
RESUMEN FOURIER

- * Descripción alternativa de sistemas
- * Exp. son funciones propias de SLIT
- * Toda señal periódica admite una desc. como s.f.
- * descomposición y superposición
- * entrada periódica $\Rightarrow$ salida periódica
- * Resp. en frecuencia
- * Estudiamos sistemas : filtros
- * Aplicaciones:
 - $\rightarrow$ conformadores vs selectivos
 - $\rightarrow$ def. parámetros
 - $\rightarrow$ sistemas comunes (op. matemáticas)
 - $\rightarrow$ sistemas reales (basados en ec.dif)
 - $\rightarrow$ Aplicaciones reales

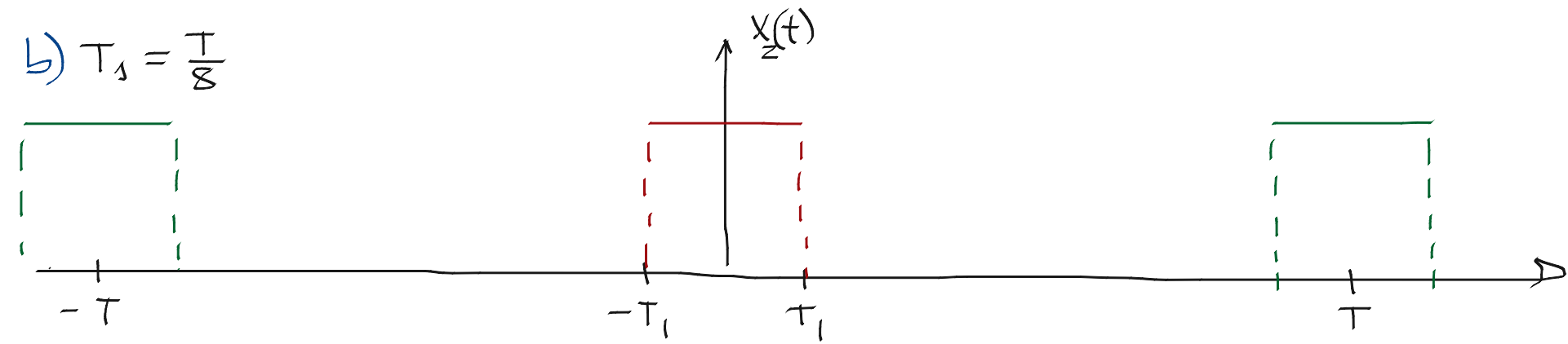
REPRESENTACIÓN DE SEÑALES APERIÓDICAS CONTINUAS

Repaso ej. 3.5 "T fijo, T_1 variable"

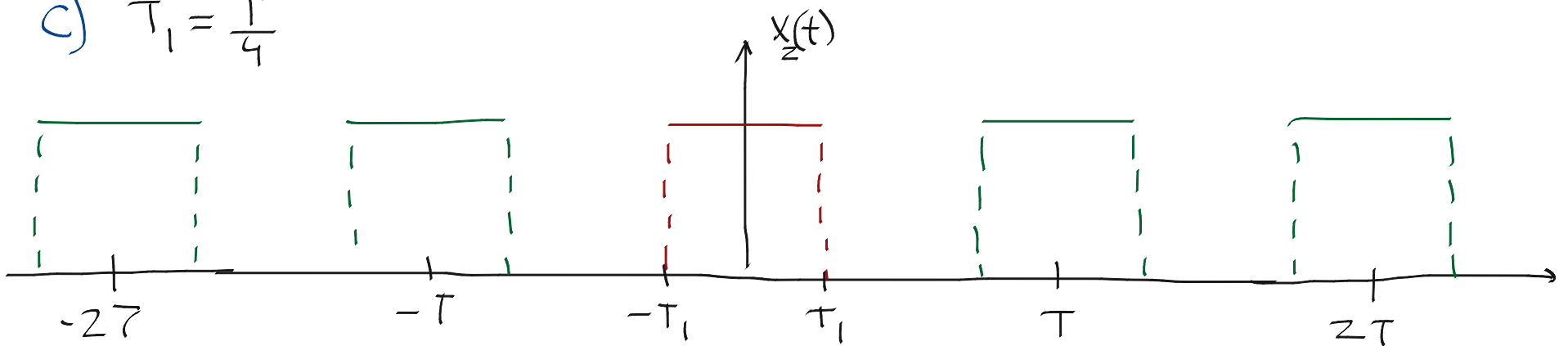
a) $T_1 = \frac{T}{4}$



b) $T_1 = \frac{T}{8}$



c) $T_1 = \frac{T}{4}$



Ejemplo 3.5 (cont)

$$d_k = \begin{cases} \frac{2}{T} \frac{\sin[k\omega_0 T_1]}{k\omega_0} & , k \neq 0 \\ \frac{2T_1}{T} & , k=0 \end{cases}$$

$$\omega_0 = \frac{2\pi}{T}$$

a) $T_1=2, T=8 \Rightarrow \omega_0 = \frac{\pi}{4}$

$$d_k = \frac{1}{4} \frac{\sin\left[\frac{\pi}{2}k\right]}{k \cdot \frac{\pi}{4}} = \frac{\sin\left[\frac{\pi}{2}k\right]}{\pi k}, \quad d_0 = \frac{1}{2}$$

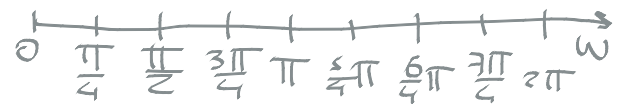
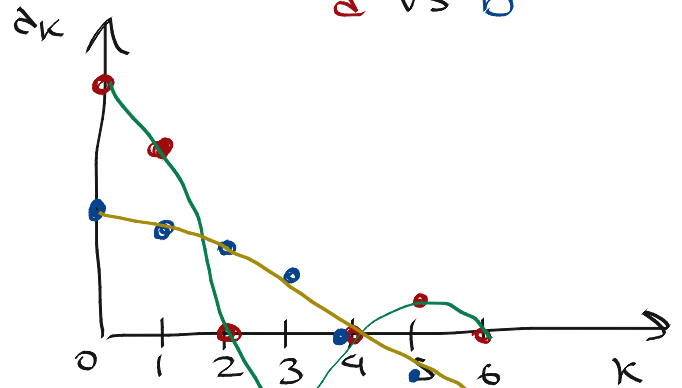
b) $T_1=1, T=8 \Rightarrow \omega_0 = \frac{\pi}{8}$

$$d_k = \frac{1}{4} \frac{\sin\left[\frac{\pi}{4}k\right]}{\frac{\pi}{4}k} = \frac{\sin\left[\frac{\pi}{4}k\right]}{\pi k}, \quad d_0 = \frac{1}{4}$$

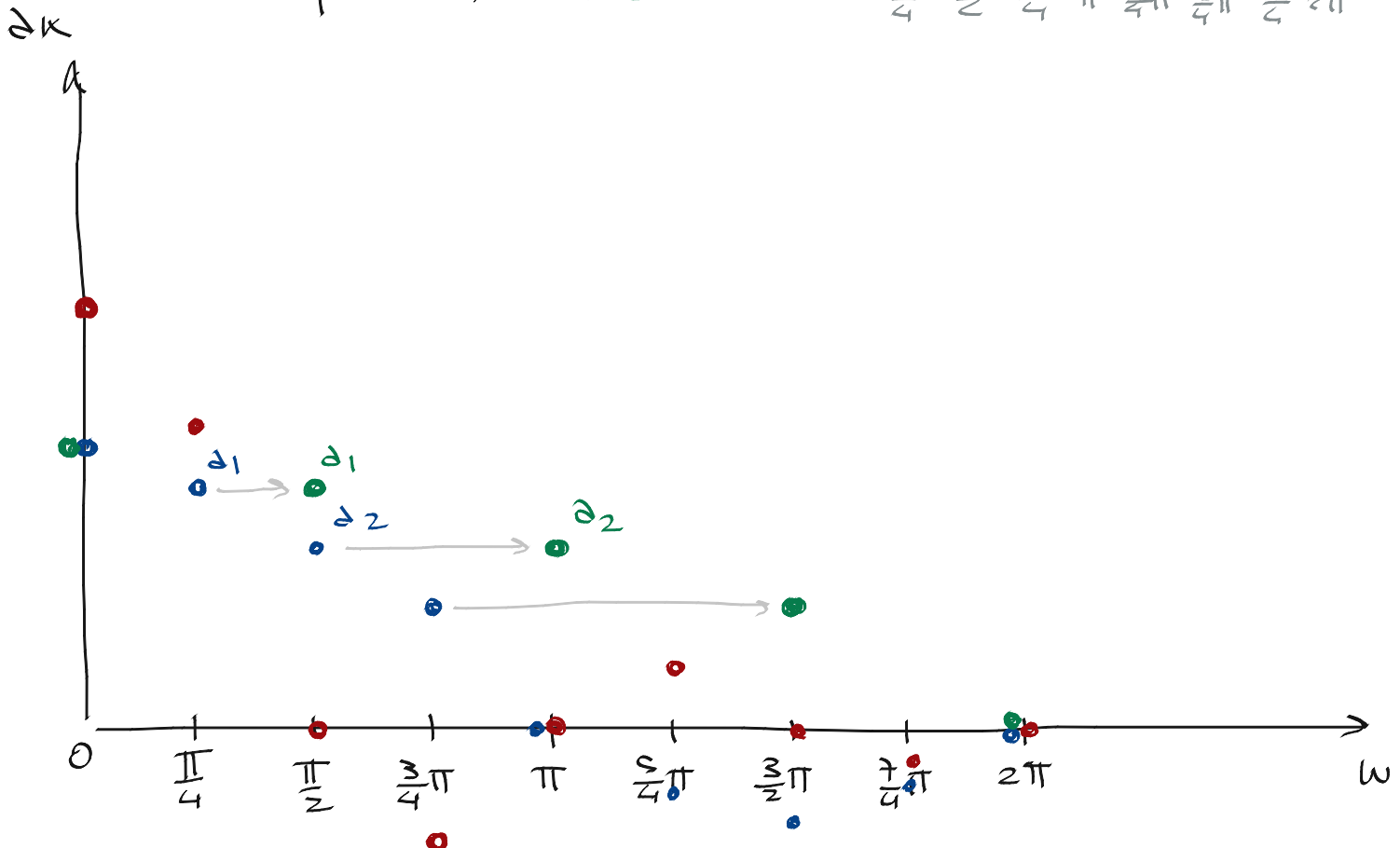
c) $T_1=1, T=4 \Rightarrow \omega_0 = \frac{\pi}{2}$

$$d_k = \frac{\sin\left[\frac{\pi}{2}k\right]}{\pi k}, \quad d_0 = \frac{1}{2}$$

a vs b



Comparison: a b c



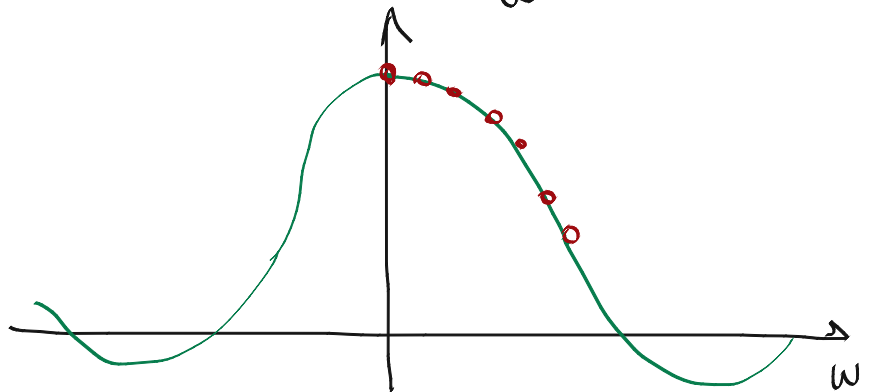
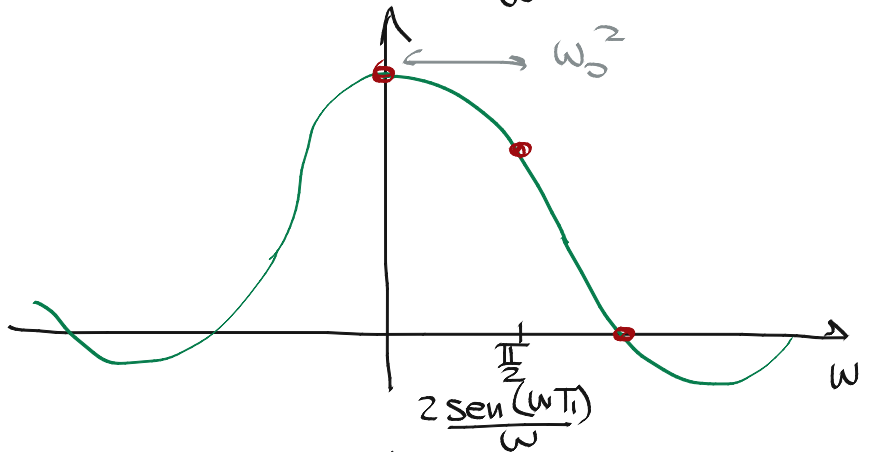
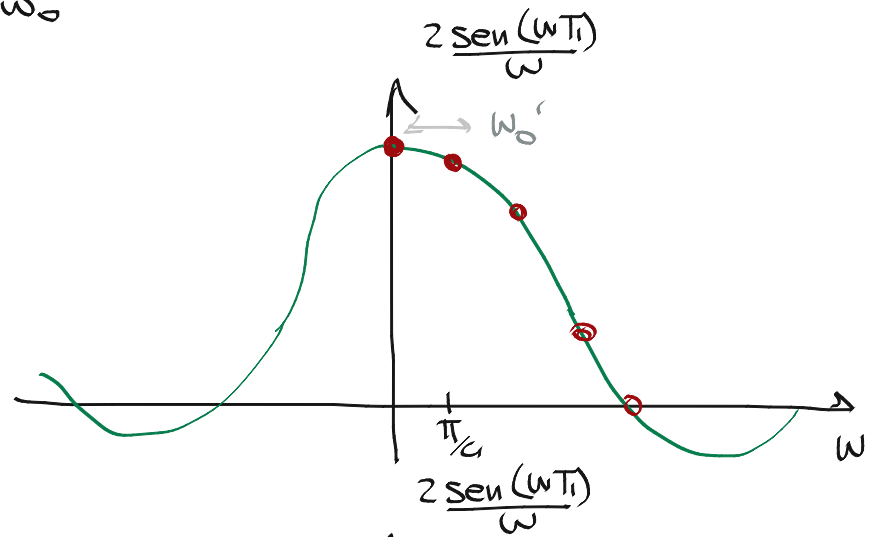
$$d_k = \frac{2}{T} \frac{\sin[k\omega_0 T_1]}{k\omega_0} \rightarrow T a_k = 2 \cdot \frac{\sin[k\omega_0 T_1]}{\omega_0} \quad \omega$$

$$T a_k = 2 \frac{\sin(\omega T_1)}{\omega} \Big|_{\omega = k \cdot \omega_0}$$

b) $T=8$, $\omega_0' = \frac{\pi}{4}$

c) $T=4$, $\omega_0^2 = \frac{\pi}{2}$

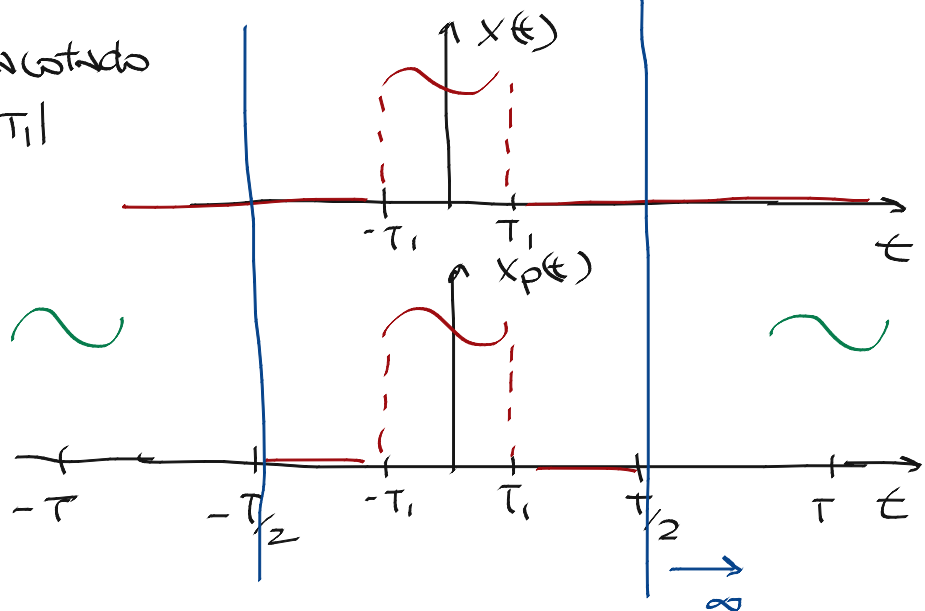
$T \rightarrow \infty$, $\omega_0 \rightarrow 0$



Idea: $x(t)$ soporte acotado
 $x(t) = 0, \forall t > |T_1|$

$x_p(t) = x(t), |t| < T/2$
 periódica T

$\lim_{T \rightarrow \infty} x_p(t) = x(t)$



2)

$$x_p(t) = \sum_{k=-\infty}^{\infty} d_k \cdot e^{jk\omega_0 t}$$

$$d_k = \frac{1}{T} \int_{-T/2}^{T/2} x_p(t) \cdot e^{-jk\omega_0 t} dt$$

$$T_{dk} = \int_{-T/2}^{T/2} x_p(t) \cdot e^{-j \overset{\sim \omega}{k\omega_0} t} dt \Rightarrow \lim_{T \rightarrow \infty} T_{dk} = \underbrace{\int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt}_{\text{función de } \omega} = X(j\omega)$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt$$

ec. análisis

$$d_k = \frac{1}{T} \cdot X(j\omega) \Big|_{\omega = k\omega_0}$$

S.F. es un caso particular de la T.F.

b)

$$x_p(t) = \sum_{k=-\infty}^{\infty} a_k \cdot e^{j k \omega_0 t}$$

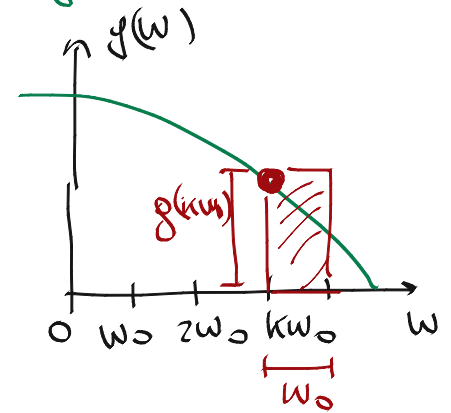
$$\omega_0 = \frac{2\pi}{T} \rightarrow \frac{1}{T} = \frac{\omega_0}{2\pi}$$

$$\lim_{T \rightarrow \infty} \sum_{k=-\infty}^{\infty} \frac{1}{T} \cdot X(j k \omega_0) \cdot e^{j k \omega_0 t} = \lim_{T \rightarrow \infty} \frac{1}{2\pi} \cdot \sum_{k=-\infty}^{\infty} \underbrace{X(j k \omega_0)}_{g(\omega)} \cdot e^{j k \omega_0 t} \cdot \omega_0$$

$$\lim_{T \rightarrow \infty} \frac{1}{2\pi} \cdot \sum_{k=-\infty}^{\infty} \underbrace{g(k \omega_0)}_{\text{área}} \cdot \omega_0 = \int_{-\infty}^{\infty} g(\omega) \cdot d\omega$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) \cdot e^{j\omega t} \cdot d\omega$$

ec. síntesis



Definición

* ec. análisis:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \cdot dt$$

* ec. síntesis:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) \cdot e^{j\omega t} \cdot d\omega$$

Ejemplo 4.1

$$x(t) = v(t) \cdot e^{-at}, \quad a > 0$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt$$

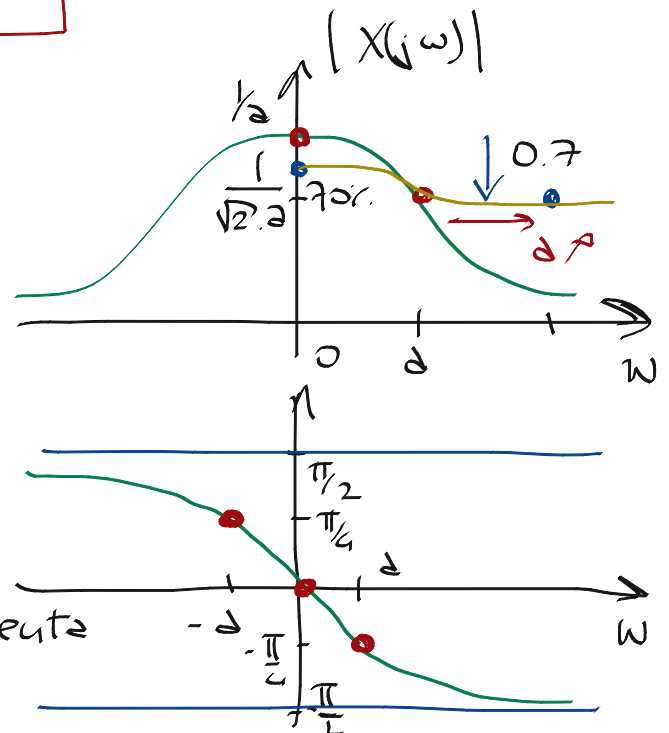
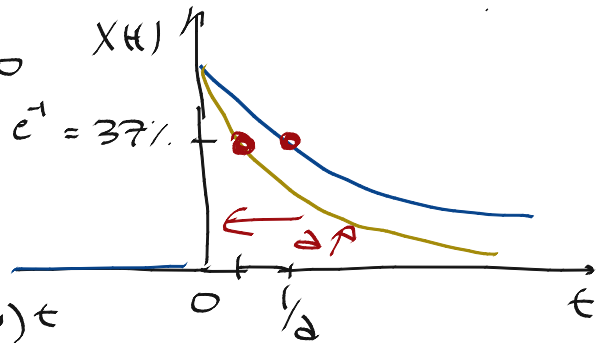
$$X(j\omega) = \int_0^{\infty} e^{-at} \cdot e^{-j\omega t} dt = \int_0^{\infty} e^{-(a+j\omega)t} dt$$

$$X(j\omega) = - \frac{e^{-(a+j\omega)t}}{a+j\omega} \Big|_0^{\infty} = \boxed{\frac{1}{a+j\omega}}$$

$$\frac{1}{1+j\omega\tau} \quad \tau \sim \frac{1}{a}$$

$$|X(j\omega)| = \boxed{\frac{1}{\sqrt{a^2 + \omega^2}}}$$

$$\Theta(\omega) = \angle X(j\omega) = \boxed{-\arctan\left(\frac{\omega}{a}\right)}$$



obs:

- * es la T.F. de una función
- * si fuera señal diría que es lenta
- * si fuera sistema es:
 - pasabajas
 - causal

CONVERGENCIA

Condición simple

$$\hat{X}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) \cdot e^{j\omega t} d\omega$$

$$e(t) = \hat{X}(t) - X(t)$$

Si $X(t)$ de energía finita

$$\int_{-\infty}^{\infty} |X(t)|^2 dt < \infty \Rightarrow \int_{-\infty}^{\infty} |e(t)|^2 dt = 0$$

→ $\hat{X}(t)$ converge.

→ $\hat{X}(t) = X(t)$ en casi todo punto

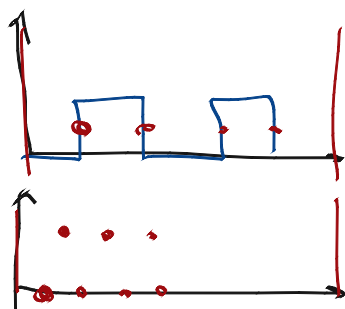
ds: $\int_1^{\infty} \frac{1}{t^2} dt > \int_1^{\infty} \frac{1}{t^a} dt$ $a > 2$

Condiciones de Dirichlet

1) $X(t)$ absolutamente integrable: $\int_{-\infty}^{\infty} |X(t)| dt < \infty$

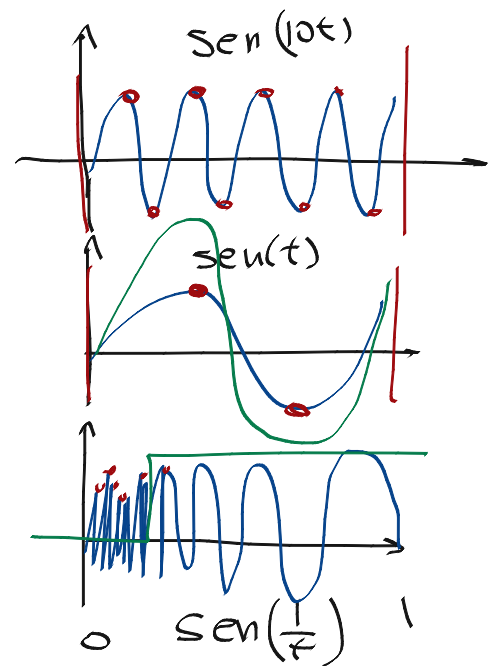
2) $X(t)$ tenga un número finito de extremos en un intervalo finito

3) $X(t)$ tenga un número finito de discontinuidades finitas en un intervalo finito.



función dirichlet.

$$\begin{cases} 1 & t \in \mathbb{Q} \\ 0 & t \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$



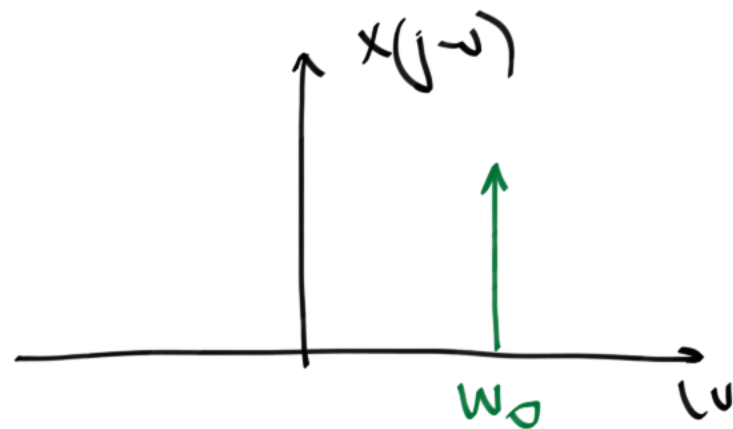
Clase 19

Transformada de Fourier

Señales periódicas y propiedades

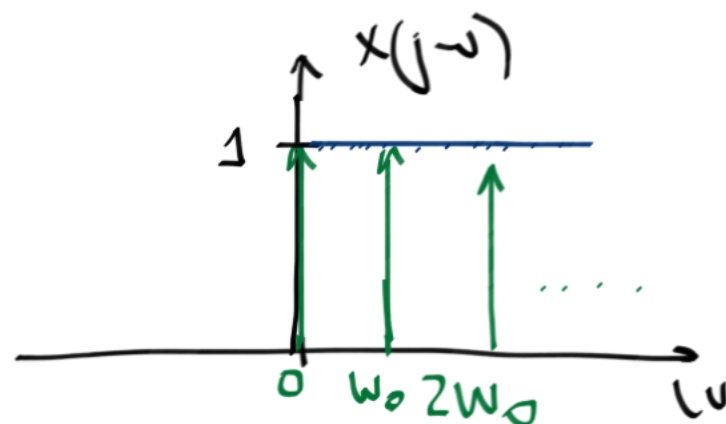
TRANSFORMADA DE SEÑALES PERIÓDICAS.

a) $X(j\omega) = 2\pi \cdot \delta(\omega - \omega_0)$



$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi \cdot \delta(\omega - \omega_0) e^{j\omega t} d\omega$$

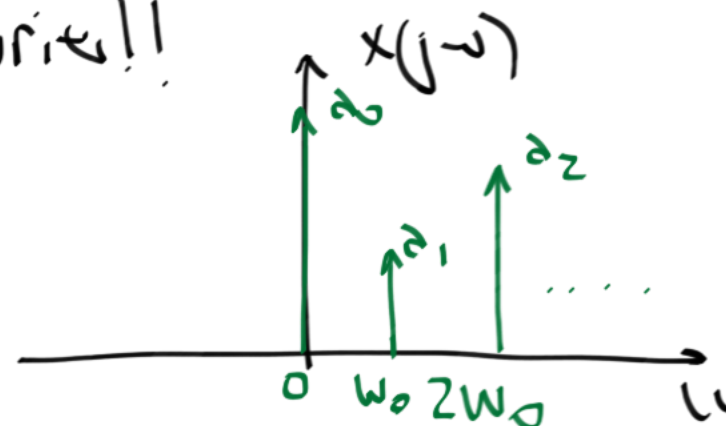
$$x(t) = e^{j\omega_0 t} \Big|_{\omega = \omega_0} = e^{j\omega_0 t}$$



b) $X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi \cdot \delta(\omega - k\omega_0)$

$$x(t) = \sum_{k=-\infty}^{\infty} 1 \cdot e^{jk\omega_0 t}$$

← es una
Serie de
Fourier!!



b) $X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \cdot \delta(\omega - k\omega_0)$

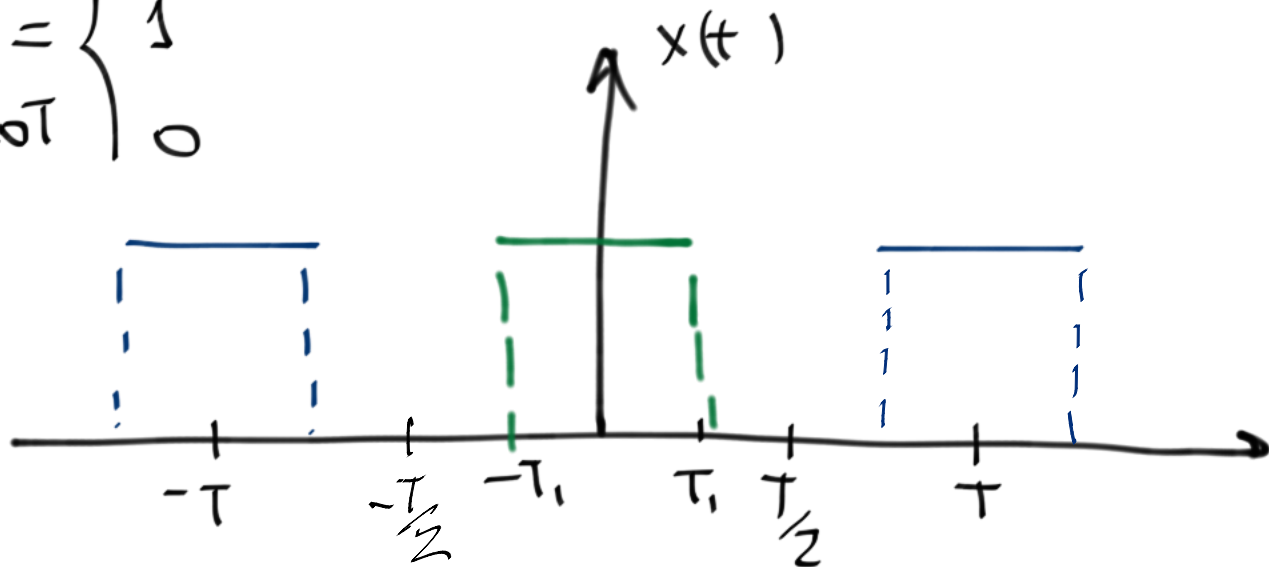
$$x(t) = \sum_{k=-\infty}^{\infty} a_k \cdot e^{jk\omega_0 t}$$

← es una
Serie de
Fourier!!

ec. síntesis

Ejemplo 4.6 $x(t) = \begin{cases} 1 & \text{periodo } T \\ 0 & \end{cases}$

$$a_k = \frac{\text{sen}(k\omega_0 T_1)}{\pi k}$$



$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2 \frac{\text{sen}(k\omega_0 T_1)}{k} \cdot \delta(\omega - k\omega_0)$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2 \frac{\text{sen}(\frac{\pi}{2} k)}{k} \cdot \delta(\omega - k\omega_0)$$

completa
con discretización

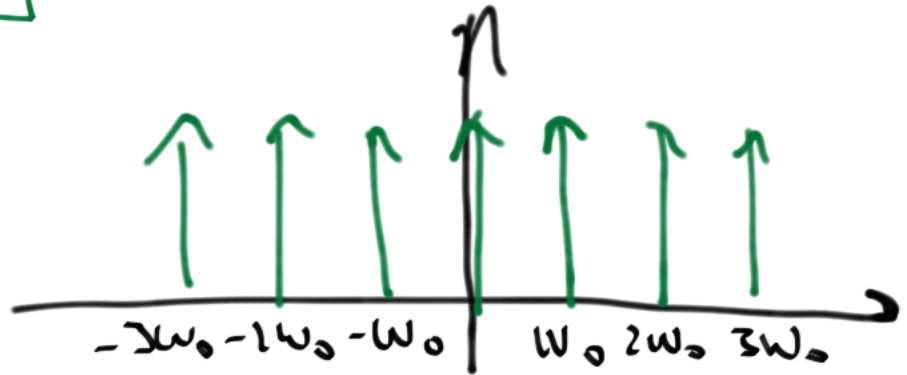
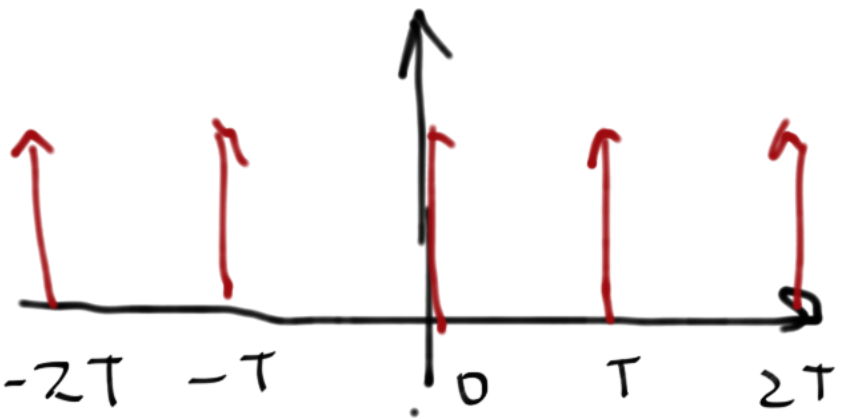
Ejemplo 4.8

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

$$d_k = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) \cdot e^{-j\omega_0 t} dt = \boxed{\frac{1}{T}}$$

$$X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} \frac{1}{T} \cdot \delta(\omega - k\omega_0)$$

$$X(j\omega) = \omega_0 \cdot \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0)$$



Obs: * los anchos tienen comportamientos opuestos

PROPIEDADES DE LA TRANSFORMADA

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega), \quad y(t) \xleftrightarrow{\mathcal{F}} Y(j\omega), \quad z(t) \xleftrightarrow{\mathcal{F}} Z(j\omega)$$

Linealidad

$$a x(t) + b y(t) \xleftrightarrow{\mathcal{F}} a X(j\omega) + b Y(j\omega)$$

Desplazamiento temporal

$$x(t-t_0) \xleftrightarrow{\mathcal{F}} e^{-j\omega t_0} X(j\omega)$$

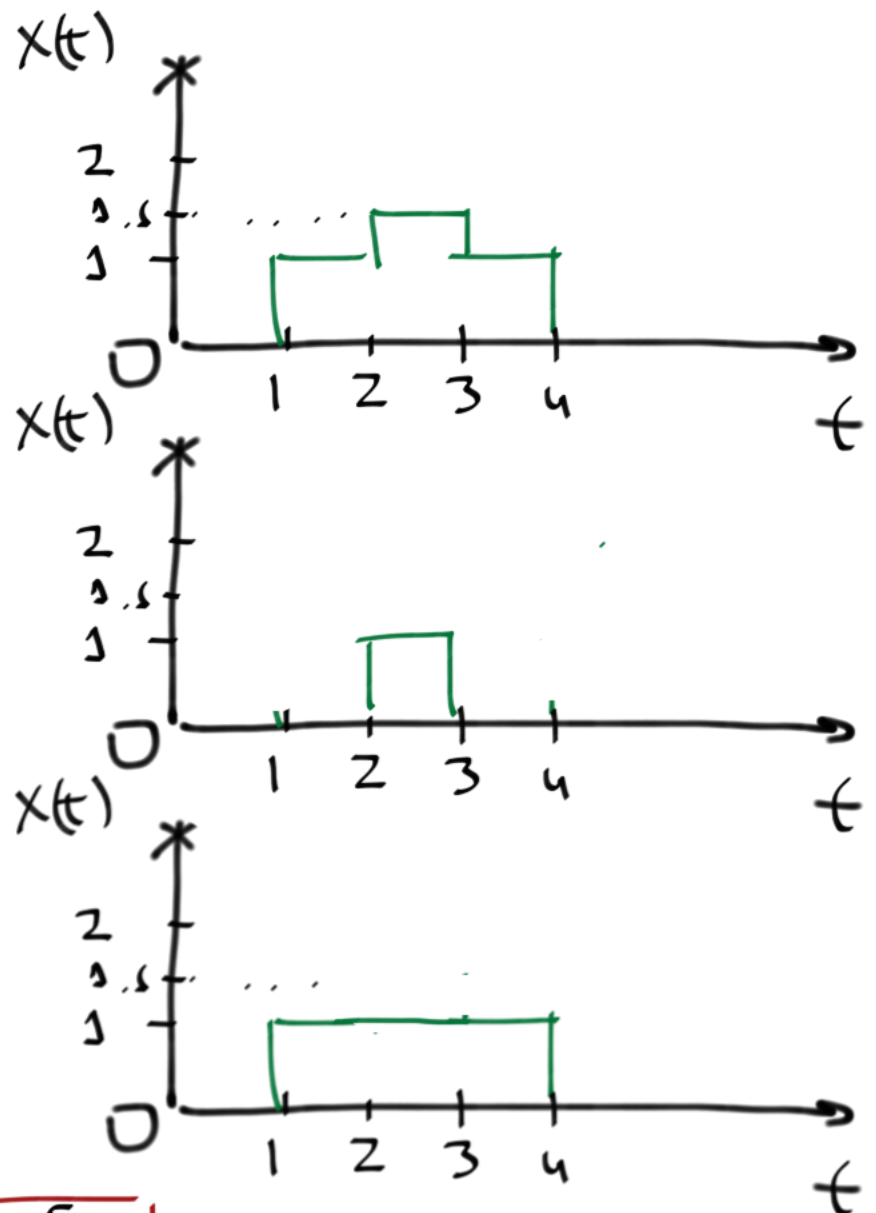
Ejemplo 4.9 (p. 302)

$$x(t) = \frac{1}{2} x_1(t - \frac{\pi}{2}) + x_2(t - \frac{\pi}{2})$$

$$P(j\omega) = \frac{2 \cdot \text{sen}(\omega T_1)}{\omega}$$

$$X_1(j\omega) = \frac{2 \cdot \text{sen}(\frac{\omega}{2})}{\omega}$$

$$X_2(j\omega) = \frac{2 \cdot \text{sen}(\frac{3}{2}\omega)}{\omega}$$



$$X(j\omega) = \frac{2}{\omega} \left(\frac{1}{2} \text{sen}\left(\frac{\omega}{2}\right) + \text{sen}\left(\frac{3}{2}\omega\right) \right) e^{j\frac{\pi}{2}\omega}$$

Conjugation

$$a) x^*(t) \xleftrightarrow{\mathcal{F}} X^*(-j\omega)$$

$$X^*(j\omega) = \int_{-\infty}^{\infty} x^*(t) \cdot e^{j\omega t} dt \Rightarrow X(-j\omega) = \int_{-\infty}^{\infty} x^*(t) \cdot e^{-j\omega t} dt = \mathcal{F}\{x^*(t)\}$$

$$b) x(t) \text{ real} \quad X(j\omega) = X^*(-j\omega)$$

$$c) * \operatorname{Re}\{X(j\omega)\} = \operatorname{Re}\{X^*(-j\omega)\} = \operatorname{Re}\{X(-j\omega)\} \Rightarrow \boxed{\operatorname{Re}\{X(j\omega)\} \text{ PAR}}$$

$$* \operatorname{Im}\{X(j\omega)\} = \operatorname{Im}\{X^*(-j\omega)\} = -\operatorname{Im}\{X(-j\omega)\} \Rightarrow \boxed{\operatorname{Im}\{X(j\omega)\} \text{ IMPAR}}$$

$$* |X(j\omega)| \text{ PAR}$$

$$* \angle X(j\omega) \text{ IMPAR}$$

$$d) x(t) \text{ real y PAR} \quad X(-j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{+j\omega t} dt \quad \begin{matrix} \downarrow \tau = -t \\ \int_{-\infty}^{\infty} x(\tau) \cdot e^{-j\omega \tau} d\tau \end{matrix}$$

$$X(-j\omega) = \int_{-\infty}^{\infty} x(\tau) \cdot e^{-j\omega \tau} d\tau = X(j\omega) \Rightarrow \boxed{X(j\omega) \text{ PAR}}$$

$$X(j\omega) = X^*(-j\omega) = X^*(j\omega) \Rightarrow \boxed{X(j\omega) \text{ REAL}}$$

$$e) x(t) \text{ real y PAR} \Rightarrow \boxed{X(j\omega) \text{ imag e impar}}$$

$$f) \quad x(t) = x_{pa}(t) + x_{imp}(t)$$

$$\begin{aligned} x(t) &\xleftrightarrow{\mathcal{F}} X(j\omega) \\ x_{pa} &\xleftrightarrow{\mathcal{F}} \operatorname{Re}\{X(j\omega)\} \\ x_{imp} &\xleftrightarrow{\mathcal{F}} \operatorname{Im}\{X(j\omega)\} \end{aligned}$$

Ejemplo 4.10

$$x(t) = e^{-a|t|}, \quad a > 0 \quad (\text{ident. 4.2})$$

$$4.1 \Rightarrow e^{-at} u(t) \xleftrightarrow{\mathcal{F}} \frac{1}{a + j\omega}$$

$$x(t) \text{ es par} \Rightarrow x(t) = u(t) \cdot e^{-at} + u(-t) \cdot e^{at}$$

$$x(t) = \frac{1}{2} \cdot \left(\overbrace{u(t) \cdot e^{-at}}^{x_1(t)} + \overbrace{u(-t) \cdot e^{at}}^{x_2(t)} \right) = 2 \operatorname{Par}\{u(t) \cdot e^{-at}\}$$

$$X(j\omega) = 2 \operatorname{Par}\{x_1(j\omega)\} = \boxed{2 \cdot \frac{a}{a^2 + \omega^2}}$$

Clase 20

Transformada de Fourier

Propiedades (cont)

Diferenciación en t

$$\frac{d x(t)}{dt} = \frac{d}{dt} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) \cdot e^{j\omega t} d\omega \right)$$

ec. síntesis

$$\frac{d x(t)}{dt} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{j\omega \cdot X(j\omega)}_{G(j\omega)} \cdot \underline{e^{j\omega t}} d\omega$$

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$$

$$\frac{d x(t)}{dt} \xleftrightarrow{\mathcal{F}} ?$$

$$\boxed{\frac{d x(t)}{dt} \xleftrightarrow{\mathcal{F}} j\omega \cdot X(j\omega)}$$

Integración

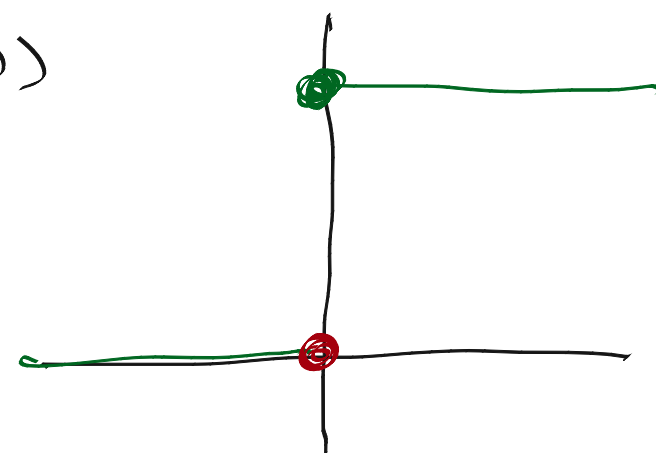
$$\boxed{\int_{-\infty}^t x(\tau) d\tau \xleftrightarrow{\mathcal{F}} \frac{1}{j\omega} \cdot X(j\omega) + \pi \cdot \underbrace{X(j \cdot 0)}_{\text{continua}} \cdot \delta(\omega)}$$

Ejemplo 4.11 $y(t) = \delta(t) \xleftrightarrow{\mathcal{F}} G(j\omega) = 1$

$$v(t) = \int_{-\infty}^t \delta(\tau) d\tau \Rightarrow \mathcal{F}\{v(t)\} = \frac{1}{j\omega} \cdot 1 + \pi \cdot 1 \cdot \delta(\omega)$$

$\swarrow \quad \searrow$
 $\quad \quad \quad \delta(j \cdot 0)$

$$\boxed{v(t) \xleftrightarrow{\mathcal{F}} \frac{1}{j\omega} + \pi \cdot \delta(\omega)}$$



verif:

$$\delta(t) = \frac{d v(t)}{dt} \xleftrightarrow{\mathcal{F}} j\omega \left(\frac{1}{j\omega} + \pi \cdot \delta(\omega) \right) = 1 + \pi \cdot j\omega \cdot \delta(\omega)$$

$$= 1 + \pi \cdot \cancel{j \cdot 0} \cdot \delta(\omega)$$

$$\boxed{\delta(t) \xleftrightarrow{\mathcal{F}} 1}$$

Ejemplo 4.12 $X(j\omega)$? $y(t) = \frac{d}{dt} x(t)$

$$G(j\omega) = \underbrace{2 \frac{\sin(\omega)}{\omega}}_{G_1} - \underbrace{e^{-j\omega} - e^{-j(-1)\omega}}_{G_2}$$

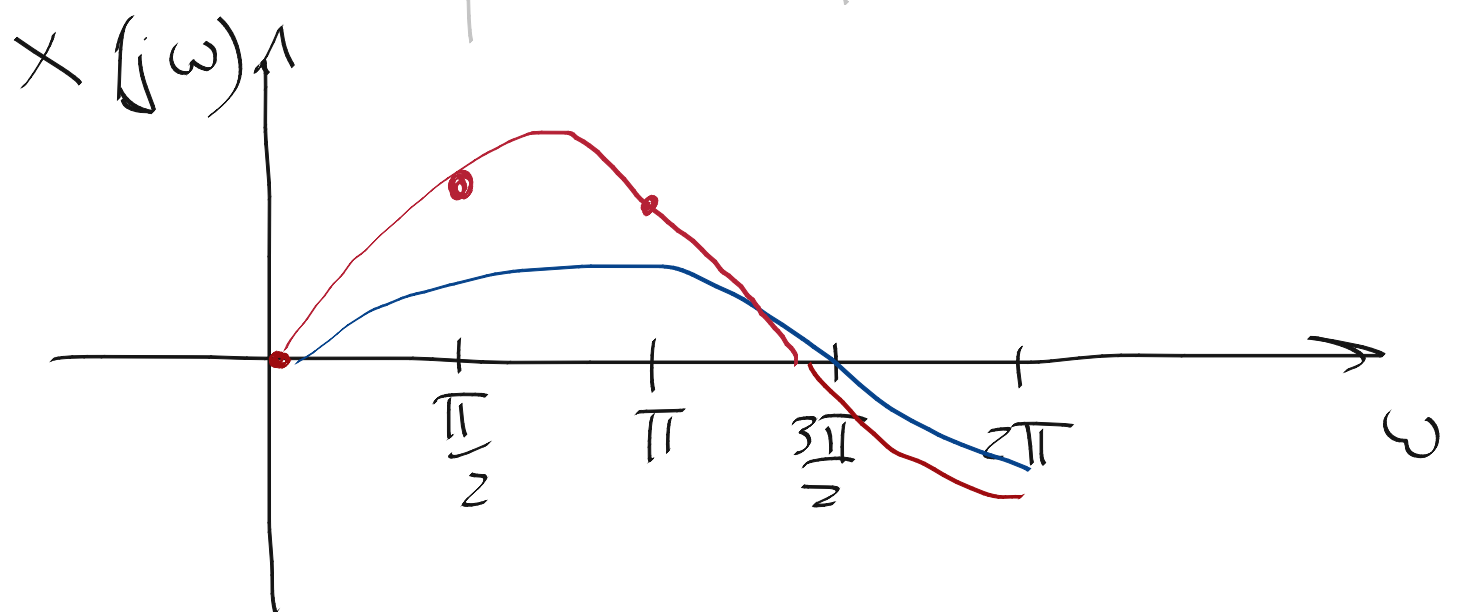
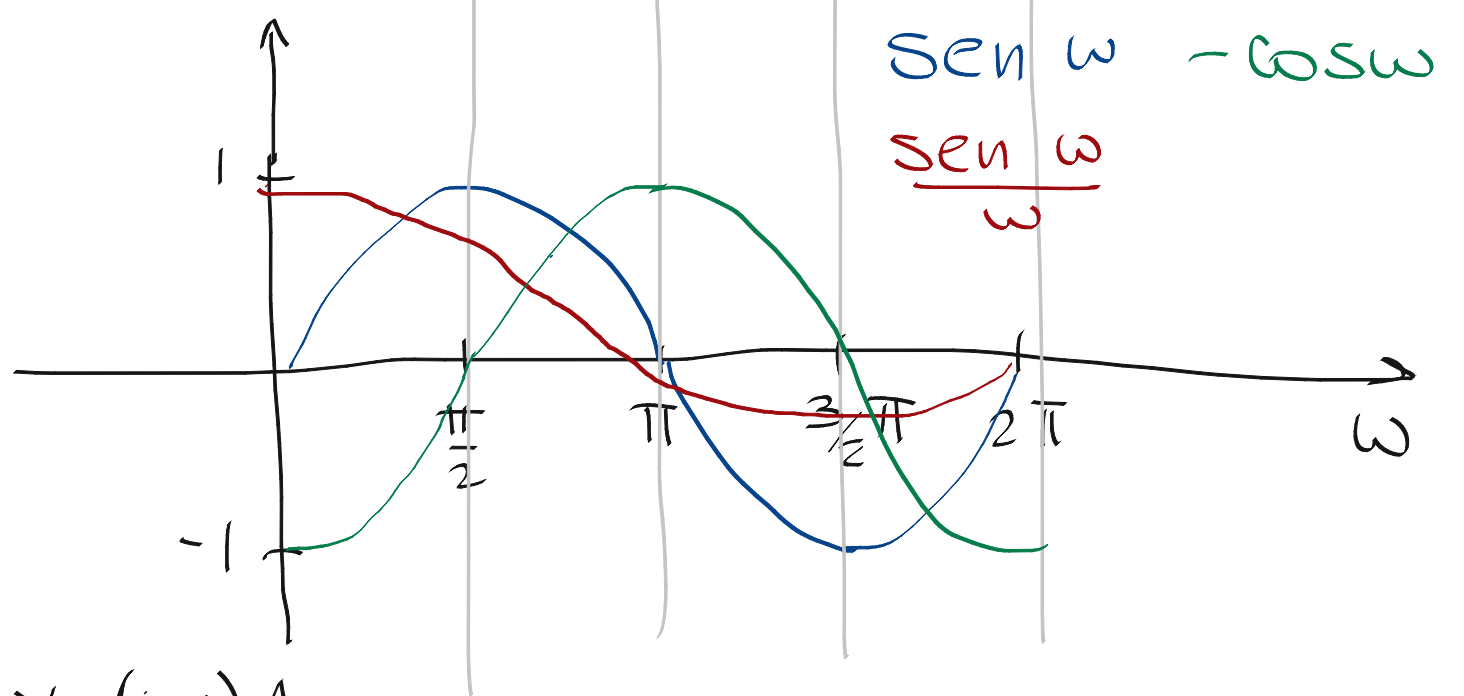
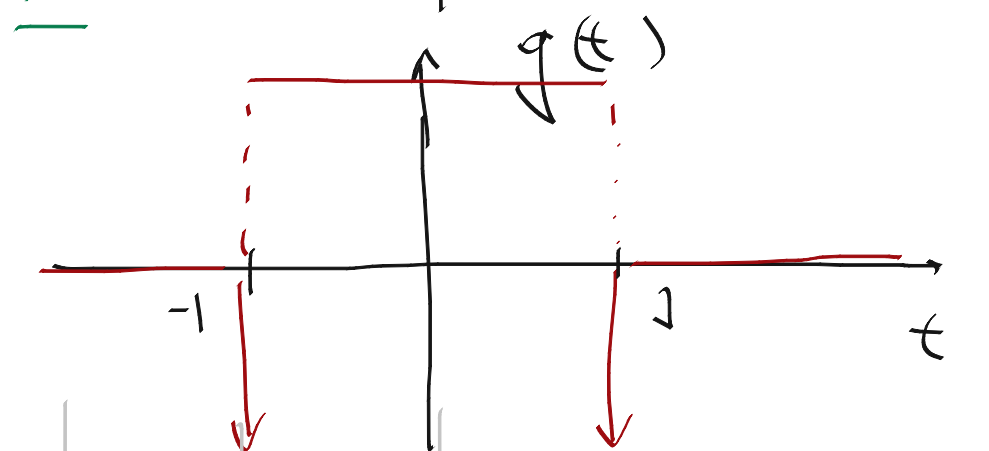
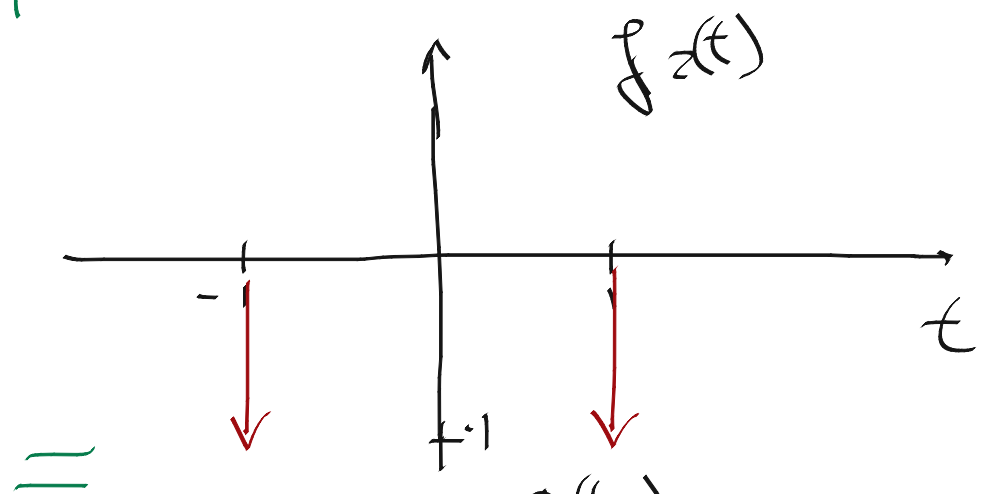
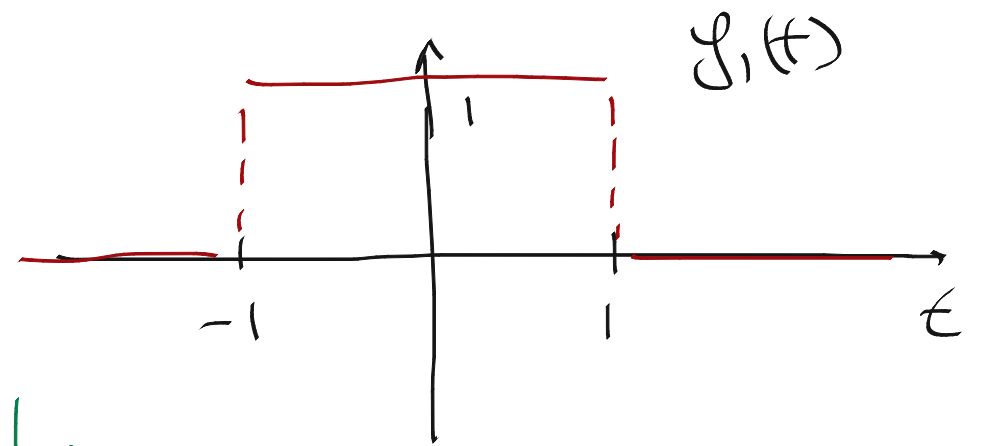
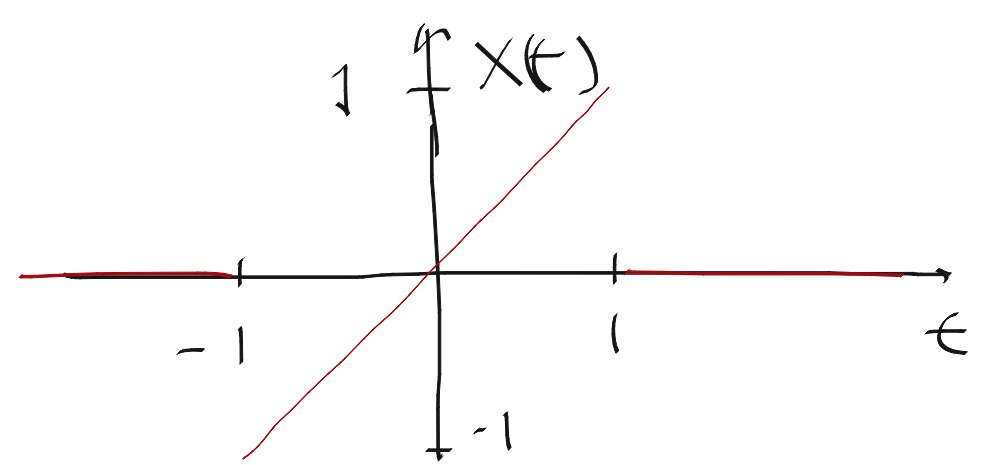
$$G(j\omega) = 2 \frac{\sin(\omega)}{\omega} - 2 \cos(\omega)$$

$$X(j\omega) = \frac{1}{j\omega} \cdot 2 \left(\frac{\sin(\omega)}{\omega} - \cos(\omega) \right) + \pi \cdot G(j0) \cdot \delta(\omega) +$$

$$X(j\omega) = \frac{2}{j\omega} (\sin(\omega) - \cos(\omega))$$

$$|X(\frac{\pi}{2})| = \frac{2}{\pi/2} \left(\frac{1}{\pi/2} - 0 \right) = \frac{8}{\pi^2} \approx 0.81$$

$$|X(\pi)| = \frac{2}{\pi} (0 - (-1)) = \frac{2}{\pi} \approx 0.64$$



Escala miento temporal (compresión/expansión)

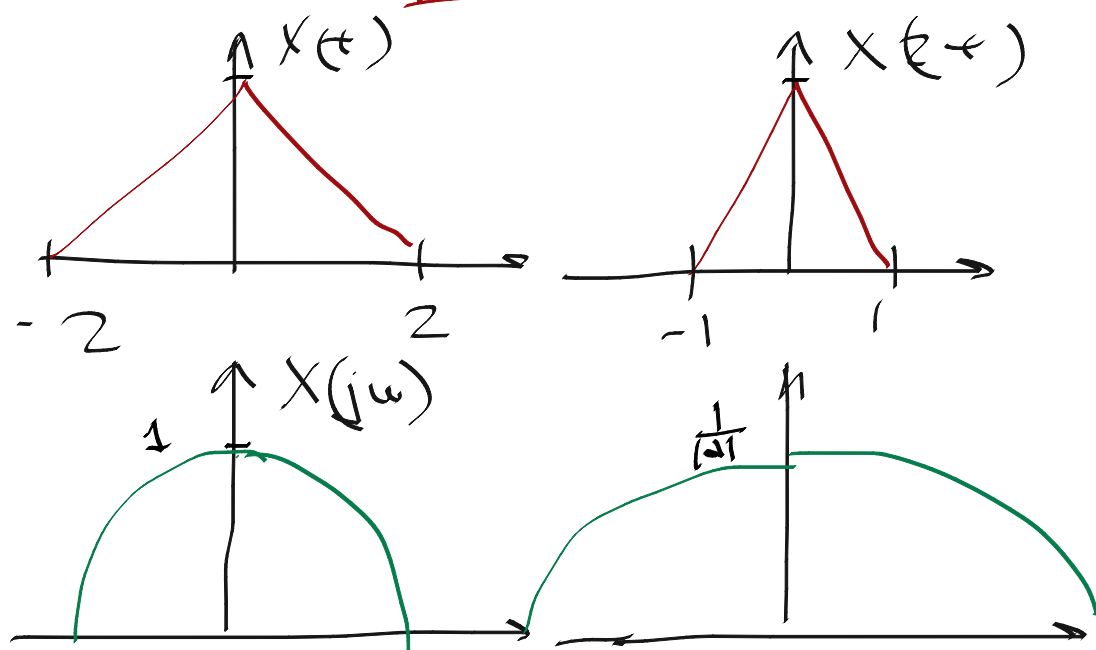
$$\mathcal{F}\{x(at)\} = \int_{-\infty}^{\infty} x(at) e^{-j\omega t} dt \stackrel{\text{c.v. } \tau=at}{=} \int_{-\infty}^{\infty} x(\tau) e^{-j\omega \frac{\tau}{a}} \frac{d\tau}{a}$$

ec. análoga

a) $a > 0$, $\mathcal{F}\{x(at)\} = \frac{1}{a} \int_{-\infty}^{\infty} x(\tau) e^{-j\frac{\omega}{a}\tau} d\tau = \frac{1}{a} X(j\frac{\omega}{a})$

b) $a < 0$, $\mathcal{F}\{x(at)\} = -\frac{1}{a} \int_{-\infty}^{\infty} x(\tau) e^{-j\frac{\omega}{a}\tau} d\tau = -\frac{1}{a} X(j\frac{\omega}{a})$

$$X(t) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|} X(j\frac{\omega}{a})$$



Corolario: $a = -1 \rightarrow$ inversión temporal

$$\mathcal{F}\{x(-t)\} = X(-j\omega)$$

Dualidad

ej 4.4:

$$x_1(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & \text{e.o.c.} \end{cases}$$

$\longleftrightarrow$

$$X_1(j\omega) = 2T_1 \frac{\text{sen}(T_1 \omega)}{T_1 \omega}$$

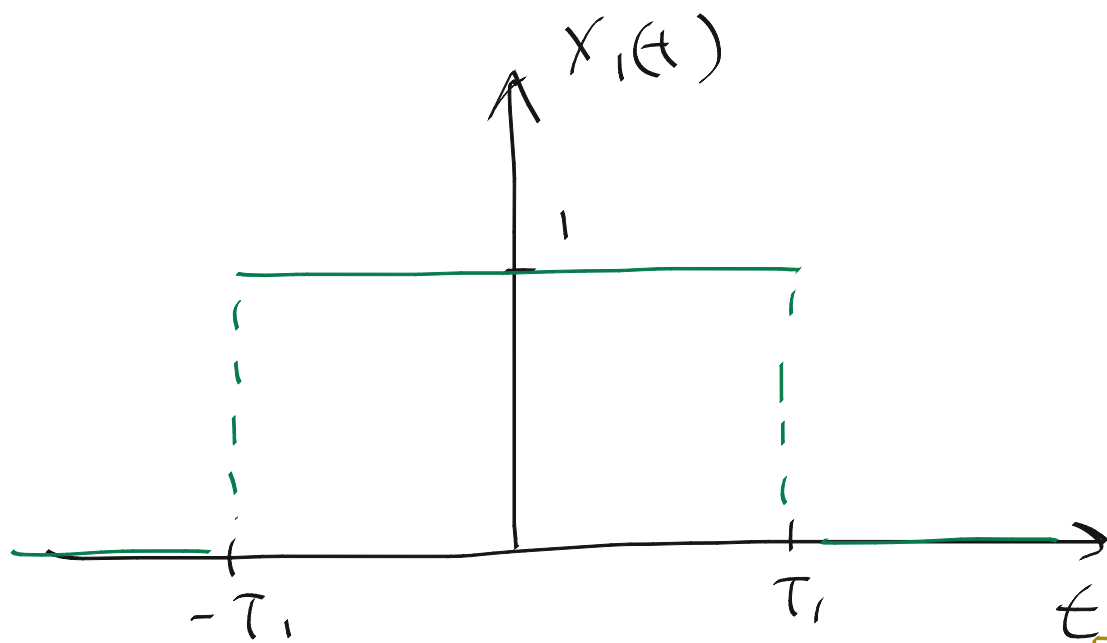
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ej 4.5:

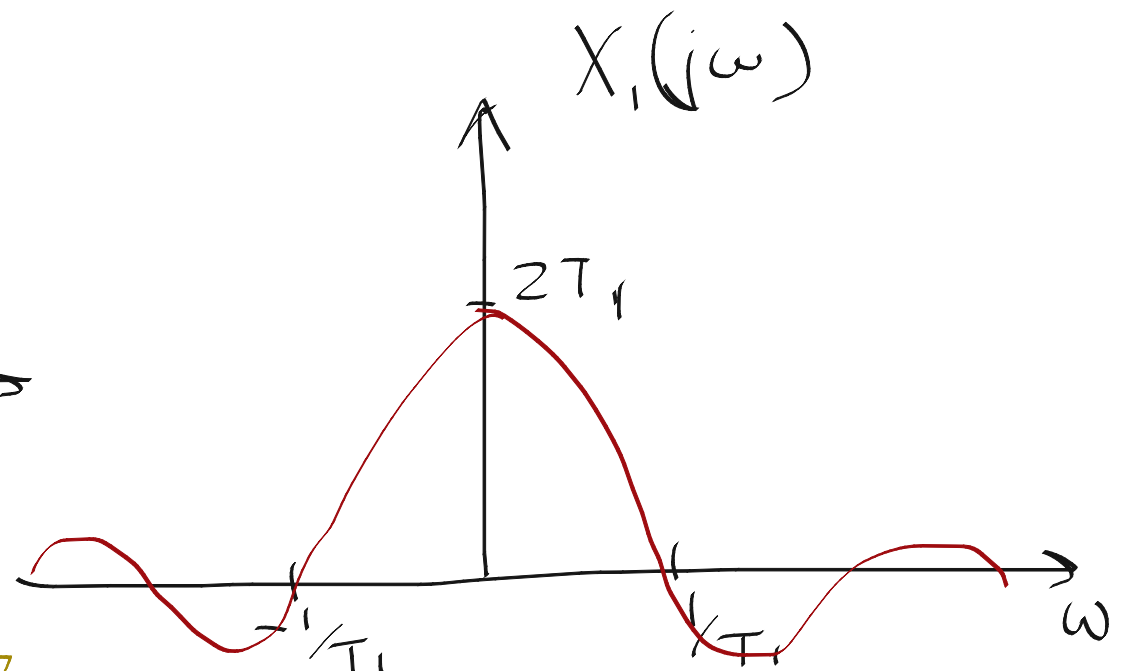
$$x_2(t) = \frac{W_1}{\pi} \frac{\text{sen}(W_1 t)}{W_1 t}$$

$\longleftrightarrow$

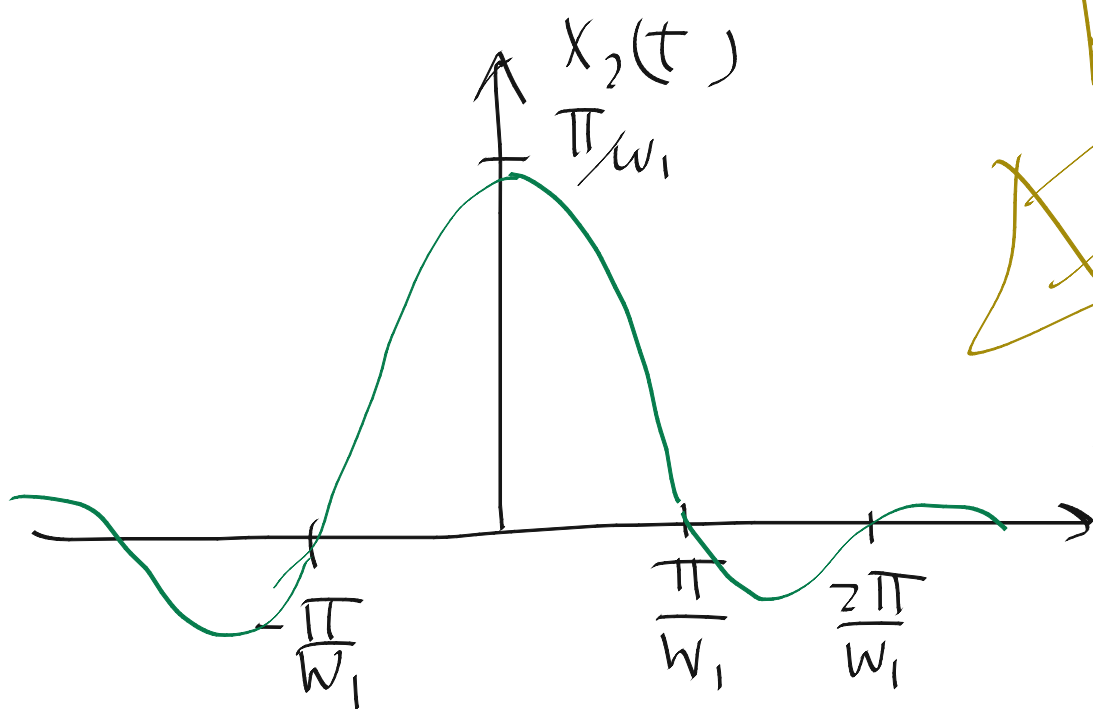
$$X_2(j\omega) = \begin{cases} 1 & |\omega| \leq W_1 \\ 0 & \text{e.o.c.} \end{cases}$$



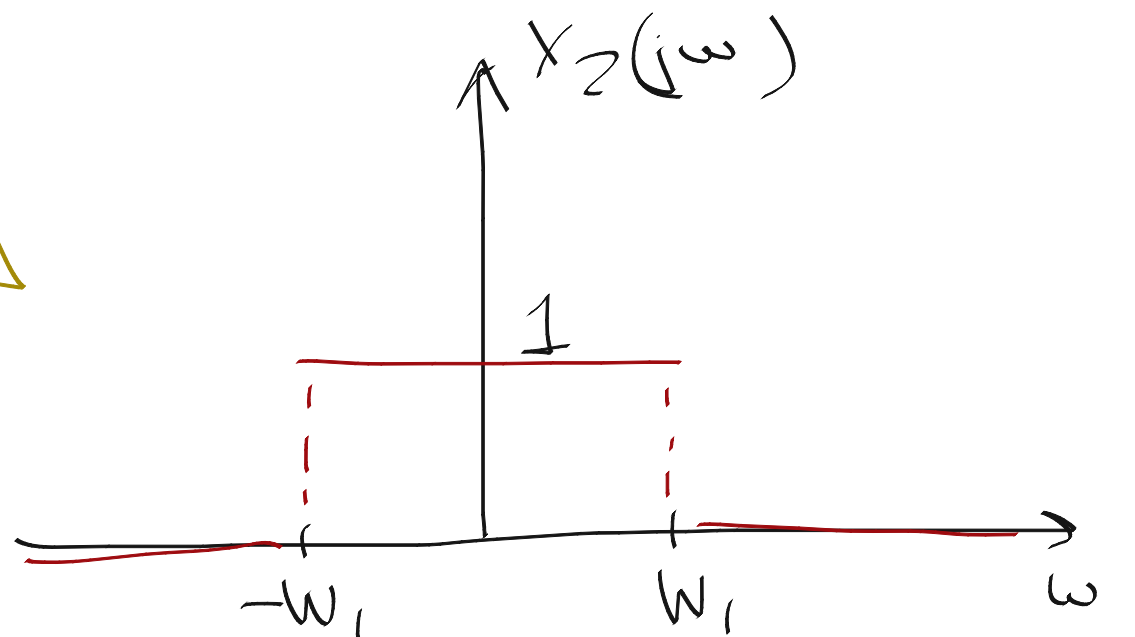
$\longleftrightarrow$



$\longleftrightarrow$



$\longleftrightarrow$



Ejemplo 4.13: (p 310)

$$g(t) = \frac{1}{1+t^2}, \quad G(j\omega) = ?$$

$$\textcircled{1} \rightarrow X(j\omega) = \frac{1}{1+\omega^2} \quad \mathcal{F}^{-1}$$

$$\textcircled{2} \rightarrow X(t) = \frac{1}{2} e^{-|t|}$$

$e^{j4.2} \uparrow$
 $\Delta=1$

$$\textcircled{3} \text{ síntesis: } X(t) \stackrel{\textcircled{2}}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

$$\pi \cdot \frac{1}{2} e^{-|t|} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1+\omega^2} e^{j\omega t} d\omega \quad \pi$$

C.V. $\tau = -t$

$$\pi \cdot e^{-|\tau|} = \int_{-\infty}^{\infty} \frac{1}{1+\omega^2} e^{-j\omega \tau} d\omega$$

C.V. $\tau \leftrightarrow \omega$

$$\pi \cdot e^{-|\omega|} = \int_{-\infty}^{\infty} \frac{1}{1+\tau^2} e^{-\tau \omega} d\tau$$

$\mathcal{F}$

$$\boxed{\pi \cdot e^{-|\omega|} \stackrel{\mathcal{F}^{-1}}{\longleftrightarrow} \frac{1}{1+\tau^2}}$$

$$g(t) = \frac{1}{1+t^2} \xrightarrow{\mathcal{F}} G(j\omega) = ?$$

$$\textcircled{3} \xrightarrow{\mathcal{F}^{-1}} \frac{1}{1+\omega^2} = X(j\omega)$$

$$X(t) =$$

obs:

$$\begin{array}{ccc} g(t) = \frac{1}{1+t^2} & \xleftrightarrow{\text{①}} & \pi \cdot e^{-|\omega|} = G(j\omega) \\ & \searrow & \\ x(t) = \frac{1}{2} e^{-|t|} & \xleftrightarrow{\text{②}} & \frac{1}{1+\omega^2} = X(j\omega) \end{array}$$

Diagram illustrating the Fourier transform pairs and their relationships:

- Top pair: $g(t) = \frac{1}{1+t^2} \xleftrightarrow{\text{①}} \pi \cdot e^{-|\omega|} = G(j\omega)$
- Bottom pair: $x(t) = \frac{1}{2} e^{-|t|} \xleftrightarrow{\text{②}} \frac{1}{1+\omega^2} = X(j\omega)$
- Relationships: A curved arrow labeled ③ points from $G(j\omega)$ to $X(j\omega)$, and another curved arrow labeled ③ points from $X(j\omega)$ to $g(t)$.

- * Sirve para aprovechar cuentas
- * Sirve para deducir nuevas propiedades
 - puedo demostrar "la mitad"

Diferenciación (en t)

$$j\omega X(\omega) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{dX(\omega)}{d\omega}$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt$$

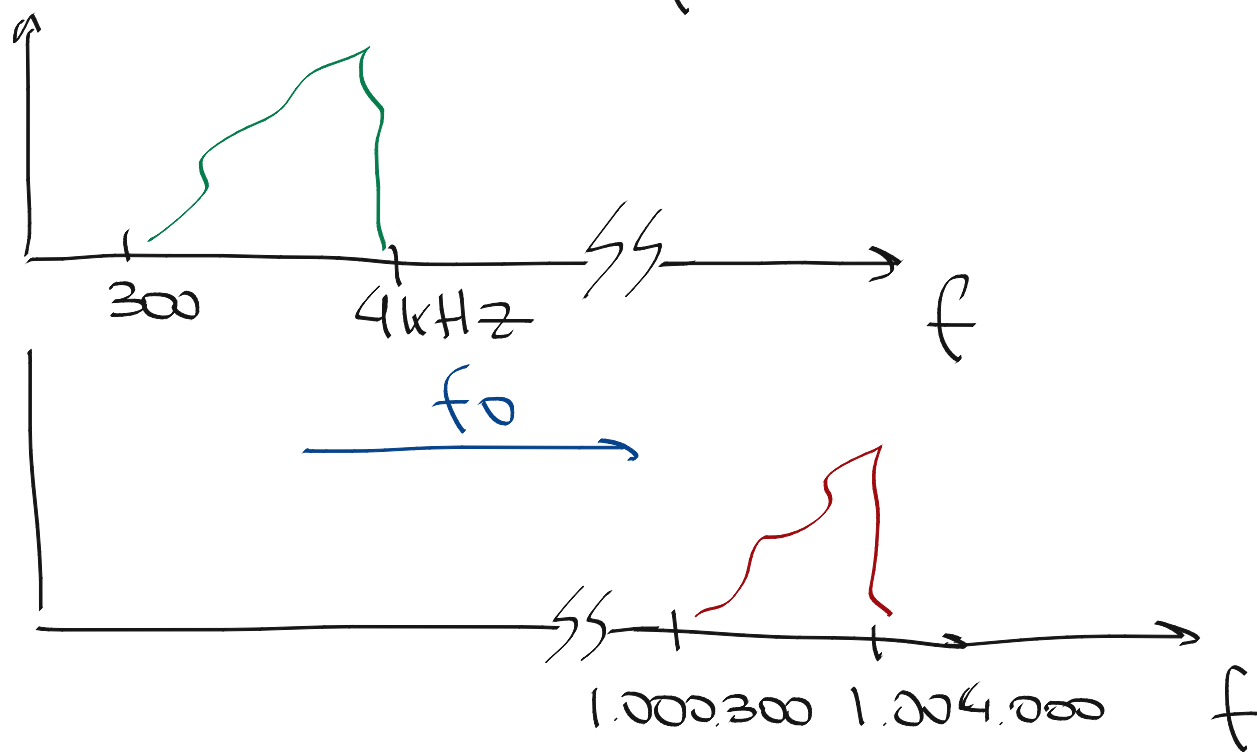
$$\frac{dX(\omega)}{d\omega} = \int_{-\infty}^{\infty} \underbrace{-jt \cdot x(t)}_{\mathcal{F}} \cdot \underbrace{e^{-j\omega t}}_{\mathcal{F}} dt$$

$$\boxed{-jt x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{dX(\omega)}{d\omega}}$$

Desplazamiento en frecuencia

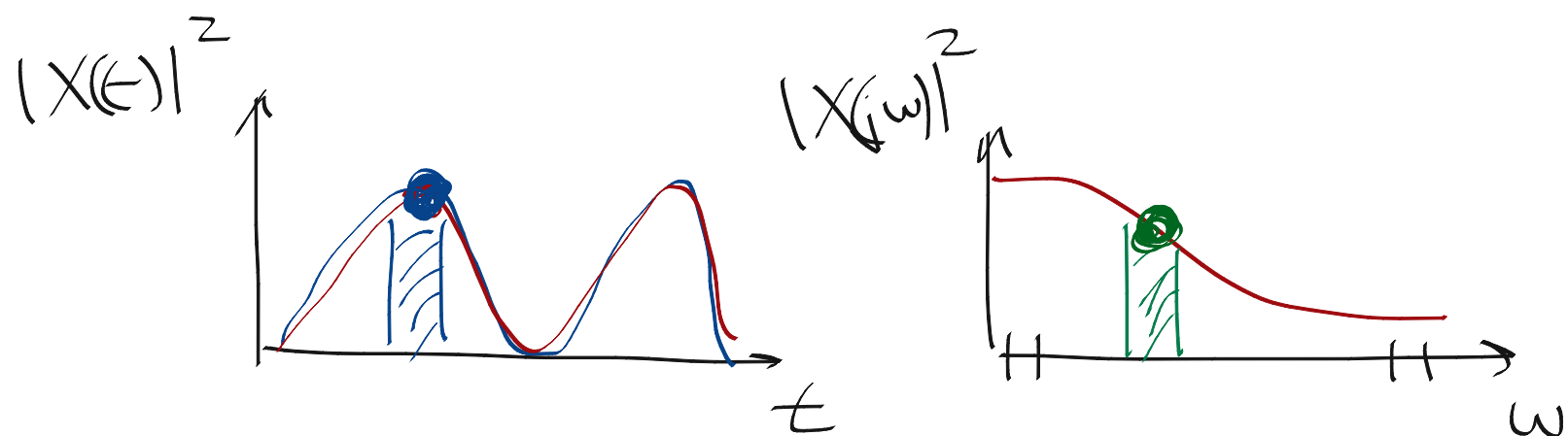
$$e^{j\omega_0 t} x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j(\omega - \omega_0))$$

Ejemplo: "modulación" $f_0 = 1 \text{ MHz}$

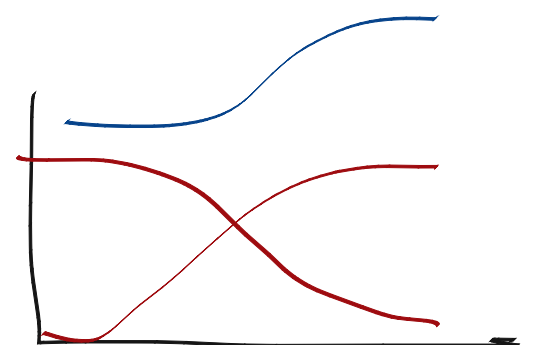


Relación de Parseval

$$E = \int_{-\infty}^{\infty} \underbrace{|x(t)|^2}_{\substack{\downarrow \\ \text{potencia} \\ \downarrow \\ \text{Energía por} \\ \text{unidad de} \\ \text{tiempo}}} dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{|X(j\omega)|^2}_{\substack{\downarrow \\ \text{energía por} \\ \text{unidad de frecuencia} \\ \downarrow \\ \text{densidad espectral} \\ \text{de potencia}}} d\omega$$

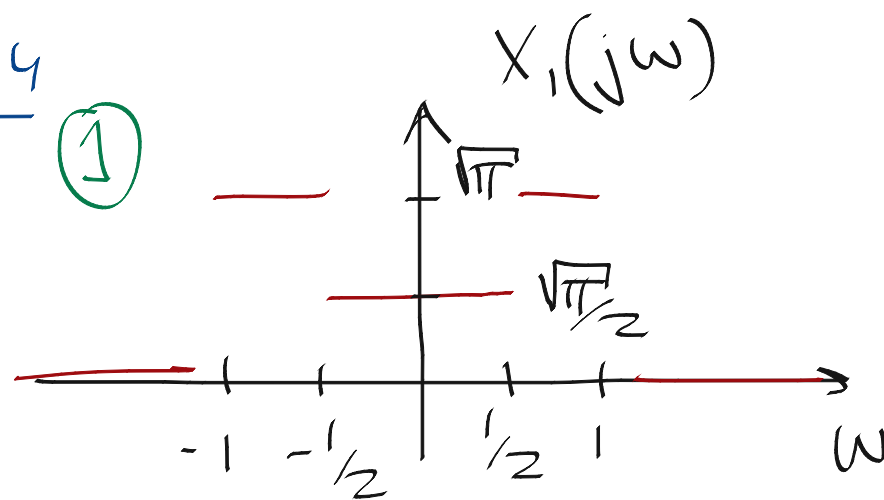


- Obs:
- * $|x(t)|^2$: Potencia (densidad temporal de potencia)
 - * $|X(j\omega)|^2$: Densidad espectral de potencia
 - * En el práctico η hicimos reconstrucción parcial usando esto

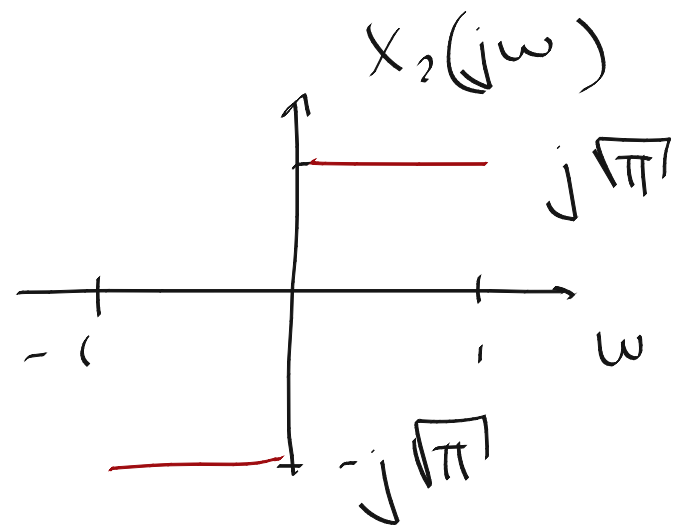


Ejemplo 4.14

①



②



$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$D = \left. \frac{d}{dt} x(t) \right|_{t=0}$$

d) energías

$$E_1 = \frac{1}{2\pi} \left[\frac{1}{2} \cdot \pi + 1 \cdot \frac{\pi}{4} + \frac{1}{2} \cdot \pi \right] = \frac{1}{2\pi} \cdot \frac{5\pi}{4} = \frac{5}{8}$$

$$E_2 = \frac{1}{2\pi} \cdot 2 \cdot 1 \cdot \pi = 1$$

Obs: * Puedo calcular cantidades en el tiempo sin la fórmula explícita (sin reconstruir)

b) derivadas

$$g(t) = \frac{d x(t)}{dt} \rightarrow D = g(0)$$

$$g(t) \xleftrightarrow{\mathcal{F}} j\omega X(j\omega) = G(j\omega)$$

ec. síntesis de $g(t)$

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} j\omega X(j\omega) \cdot e^{j\omega t} d\omega$$

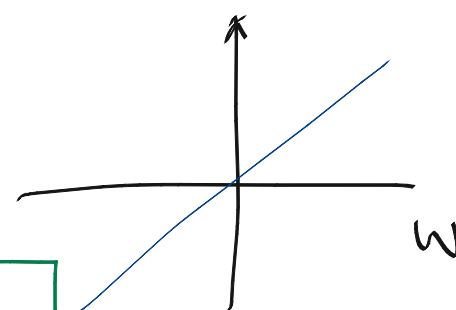
$t=0$

$$g(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} j\omega X(j\omega) d\omega$$

$$D_1 = 0 \quad (\text{impar})$$

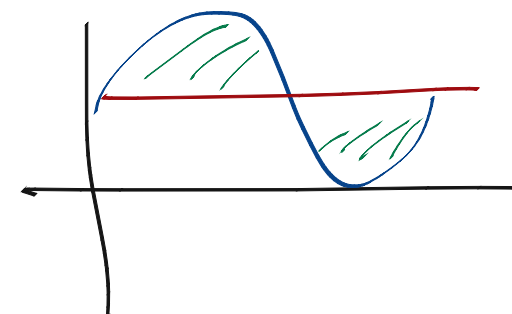
$$D_2 = \textcircled{2} \cdot \frac{1}{2\pi} \int_0^1 j\omega \cdot \frac{1}{\sqrt{\pi}} d\omega = -\frac{\sqrt{\pi}}{\pi} \cdot \frac{\omega^2}{2} \Big|_0^1 = \boxed{-\frac{1}{2\sqrt{\pi}}}$$

par $\uparrow$



OLS: * repaso del concepto de continua.

$$X(j0) = \int_{-\infty}^{\infty} x(t) dt$$



* valores puntuales en un dominio

se transforman en integrales en el otro.

Clase 21

Transformada de Fourier

Propiedades (cont)

Propiedades aplicadas a sistemas

Convolution

$$y(t) = h(t) * x(t) \quad \xrightarrow{x(t)} \boxed{h(t)} \rightarrow y(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau) \cdot d\tau$$

$$\mathcal{F}\{y(t)\} = \int_{-\infty}^{\infty} y(t) \cdot e^{-j\omega t} \cdot dt = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau) \cdot d\tau \right) \cdot e^{-j\omega t} \cdot dt$$

$$= \int_{-\infty}^{\infty} x(\tau) \underbrace{\int_{-\infty}^{\infty} h(t-\tau) \cdot e^{-j\omega t} \cdot dt}_{\textcircled{A}} \cdot d\tau$$

$$H(j\omega)$$

C.V: $r = t - \tau$

$$\textcircled{A} = \int_{-\infty}^{\infty} h(r) \cdot e^{-j(r+\tau)\omega} \cdot dr = e^{-j\omega\tau} \int_{-\infty}^{\infty} h(r) \cdot e^{-j\omega r} \cdot dr$$
$$= \boxed{H(j\omega) \cdot e^{-j\omega\tau}}$$

$$Y(j\omega) = \int_{-\infty}^{\infty} x(\tau) \cdot \underbrace{H(j\omega)}_{\textcircled{A}} \cdot e^{-j\omega\tau} \cdot d\tau = H(j\omega) \cdot \underbrace{\int_{-\infty}^{\infty} x(\tau) \cdot e^{-j\omega\tau} \cdot d\tau}_{X(j\omega)}$$

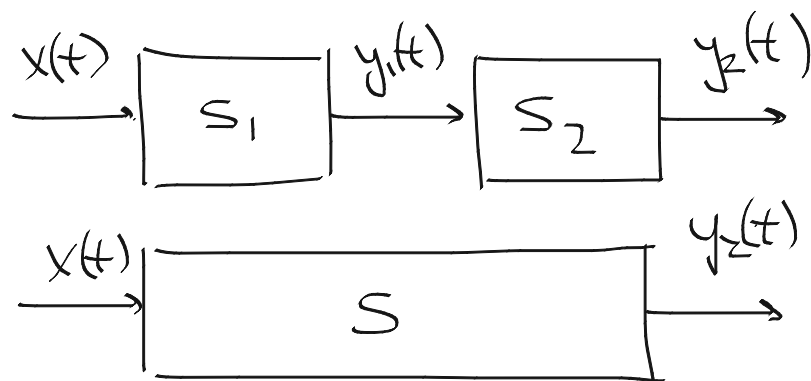
$$\boxed{Y(j\omega) = H(j\omega) \cdot X(j\omega)}$$

$$y(t) = h(t) * x(t) \quad \xleftrightarrow{\mathcal{F}} \quad Y(\omega) = H(\omega) \cdot X(j\omega)$$

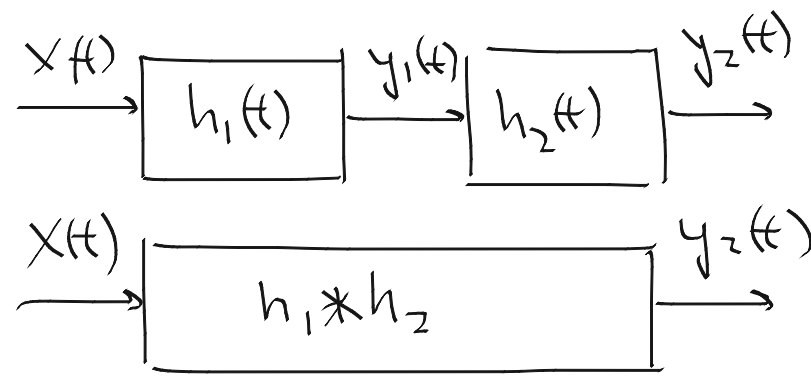
- obs: *
- * nos permite interpretar los sistemas como filtros y ver como modifican el espectro de la entrada
 - * Como $h(t)$ caracteriza completamente el sistema, entonces $H(j\omega)$ también
 - * puedo ver propiedades del sistema en $H(j\omega)$

Ejemplo:

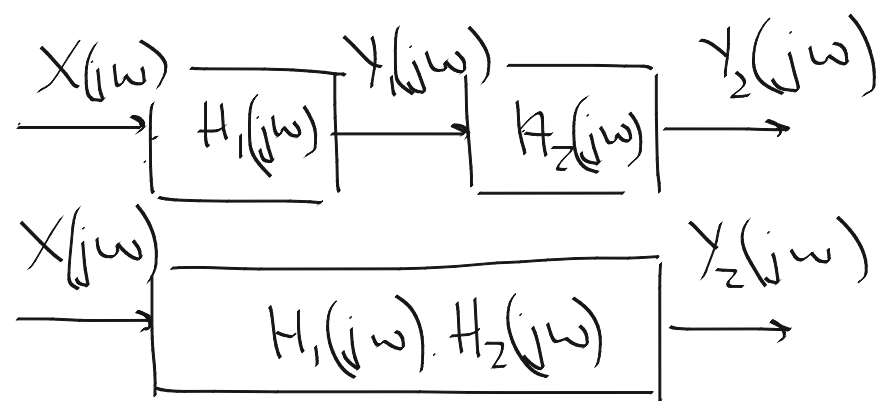
SLIT
Def. tiempo
Ec. transf.



SLIT
tiempo
convolución



SLIT
Espectral
T. Fourier

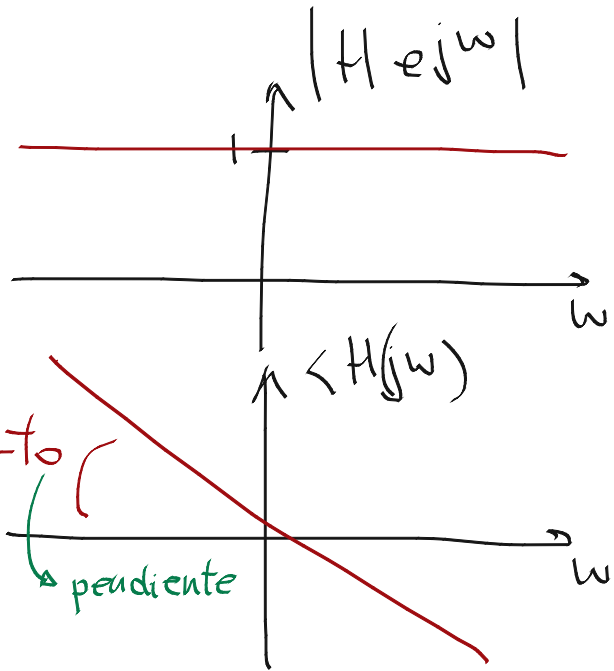


obs: * todas las propiedades se vuelven más evidentes
* por ejemplo la asociativa y conmutativa.

Ejemplo 4.15 (p. 317)

$$h(t) = \delta(t - t_0) \quad H(j\omega) = 1 \cdot e^{-j\omega t_0}$$

$$Y(j\omega) = e^{-j\omega t_0} X(j\omega)$$



$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$$

↓ S

$$x(t) \xleftrightarrow{\mathcal{F}} e^{-j\omega t_0} X(j\omega)$$

$$x(t - t_0)$$

$$y(t) = x(t - t_0)$$

$$\tau(\omega) = - \frac{d \Theta(j\omega)}{d\omega} = t_0$$

Ejemplo 4.16 (Diferenciador)

$$y(t) = \frac{d x(t)}{dt} \rightarrow Y(j\omega) = \underline{j\omega} X(j\omega)$$

$$H(j\omega) = j\omega$$

Obs: *

- uso la propiedad al revés
- es difícil obtener $h(t) = \delta'(t)$

Ejemplo 4.19 (p. 320)

$$h(t) = u(t) \cdot e^{-at} \quad X(t) = u(t) \cdot e^{-bt}$$

$$H(j\omega) = \frac{1}{b + j\omega} \quad X(j\omega) = \frac{1}{a + j\omega}$$

↗ salida en frecuencia ↘ salida en el tiempo

$$\textcircled{1} Y(j\omega) = H(j\omega) \cdot X(j\omega) = \frac{1}{b + j\omega} \cdot \frac{1}{a + j\omega} \stackrel{b \neq a}{=} \frac{A}{a + j\omega} + \frac{B}{b + j\omega}$$

$$Y(j\omega) = \frac{1}{b-a} \left(\frac{1}{a + j\omega} - \frac{1}{b + j\omega} \right)$$

$$y(t) = \frac{1}{b-a} u(t) \cdot (e^{-at} - e^{-bt})$$

$$\textcircled{2} b=a \quad Y(j\omega) = \frac{1}{(a + j\omega)^2} = \frac{1}{-j} \cdot \frac{d}{d\omega} \left(\frac{1}{a + j\omega} \right)$$

$$Y(j\omega) = j \cdot \frac{d}{d\omega} \left(\frac{1}{a + j\omega} \right)$$

$$y(t) = j(-j) u(t) \cdot e^{-at} \Rightarrow$$

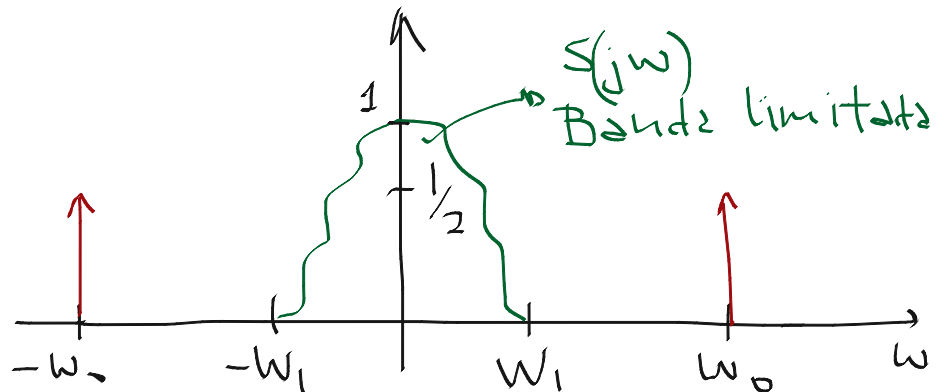
$$y(t) = u(t) \cdot t \cdot e^{-at}$$

Multiplicación (modulación)

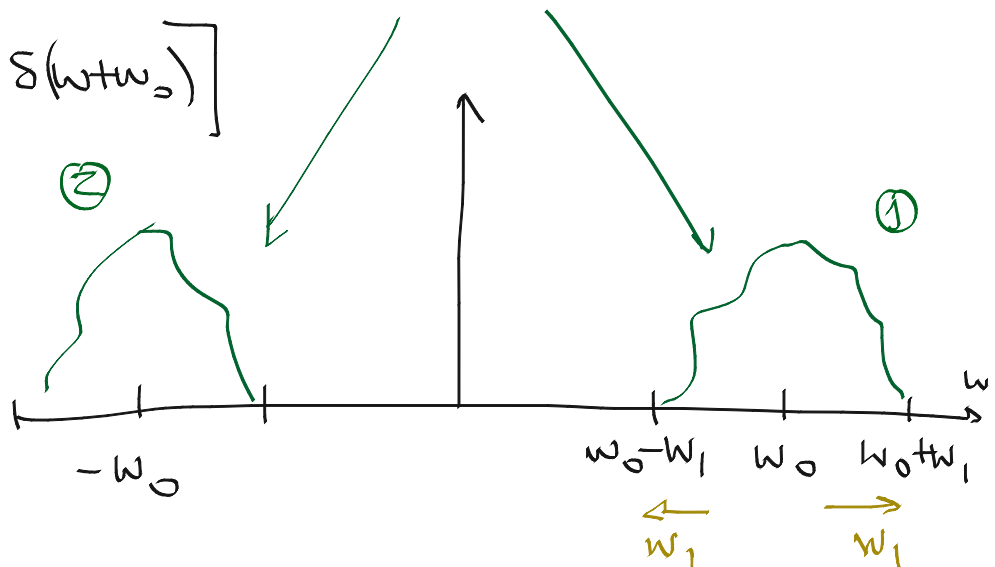
$$\underbrace{m(t)}_{\substack{\downarrow \\ \text{modulada}}} = \underbrace{s(t)}_{\substack{\downarrow \\ \text{señal}}} \cdot \underbrace{p(t)}_{\substack{\downarrow \\ \text{portadora}}} \quad \xleftrightarrow{\mathcal{F}} \quad \frac{1}{2\pi} [S(j\omega) * P(j\omega)] = M(j\omega)$$

Ejemplo 4.21

$$p(t) = \cos(\omega_0 t)$$

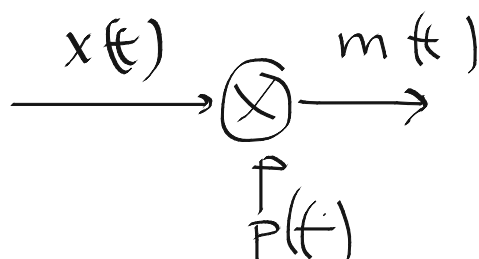


$$P(j\omega) = \frac{1}{2} [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$



$$Y(j\omega) = S(j\omega) * \frac{1}{2} [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

$$Y(j\omega) = \frac{1}{2} \cdot \underbrace{S(j(\omega - \omega_0))}_{\textcircled{1}} + \frac{1}{2} \underbrace{S(j(\omega + \omega_0))}_{\textcircled{2}}$$



Ejemplo: "modulación en amplitud"

→ voz: 300Hz - 4kHz.

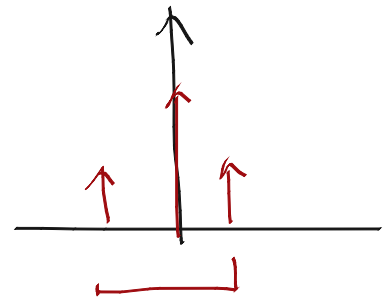
S: $x_1(t) = \text{sen}(2\pi f_1 \cdot t)$, $f_1 = 400\text{Hz}$ $f_1 \ll f_2$

P: $x_2(t) = \text{sen}(2\pi f_2 \cdot t)$, $f_2 = 5000\text{Hz}$

Obs: * ver en simulación



* $x_1(t) = \text{sen}(\omega_1 t) + k$



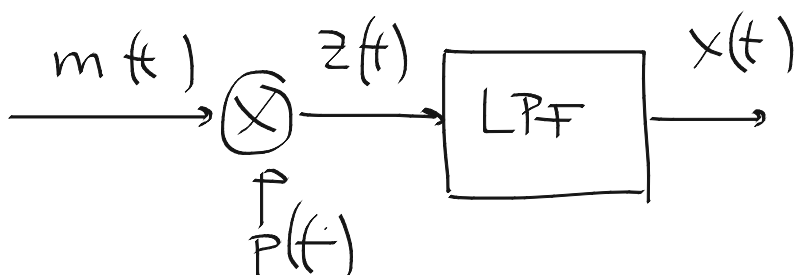
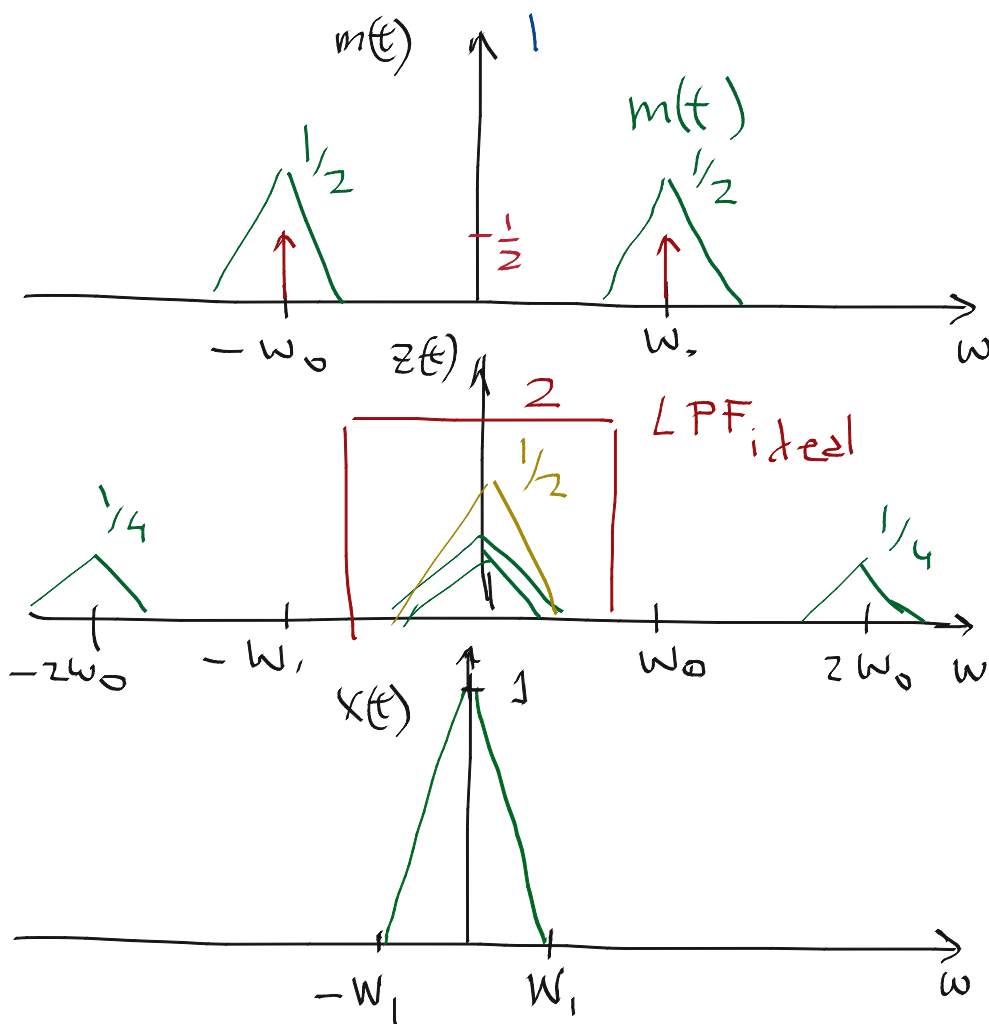
Ejemplo: "demodulator"

$$z(t) = \overbrace{[s(t) \cdot p(t)]}^{m(t)} \cdot \cos(\omega_0 t)$$

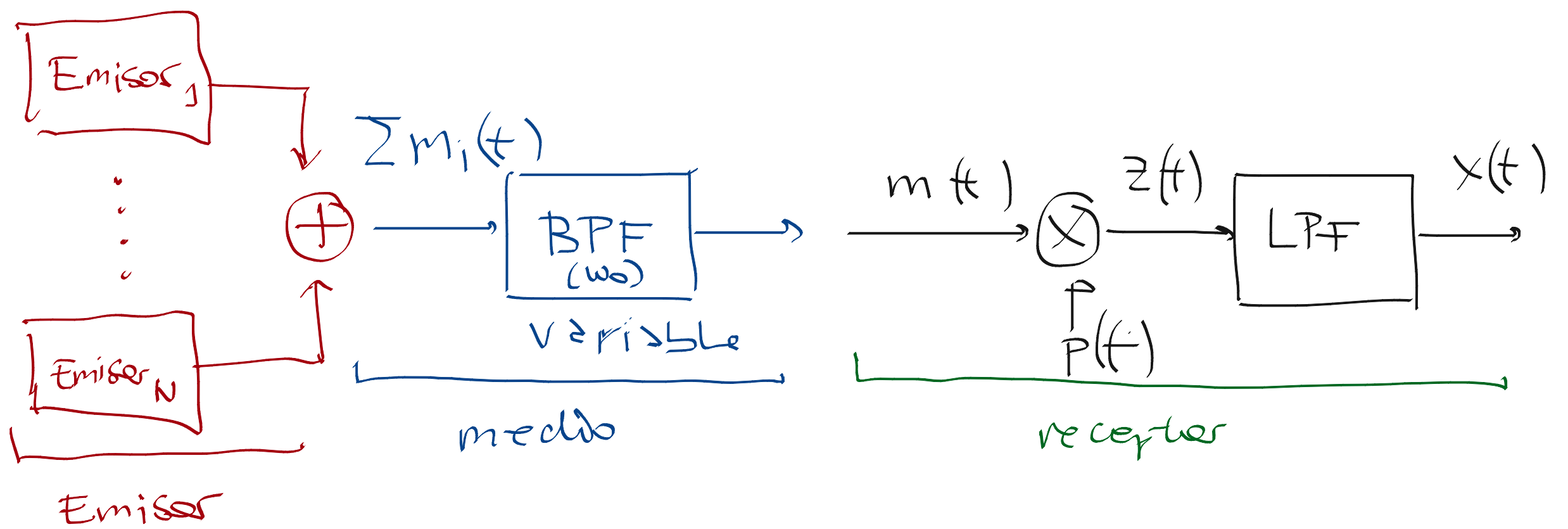
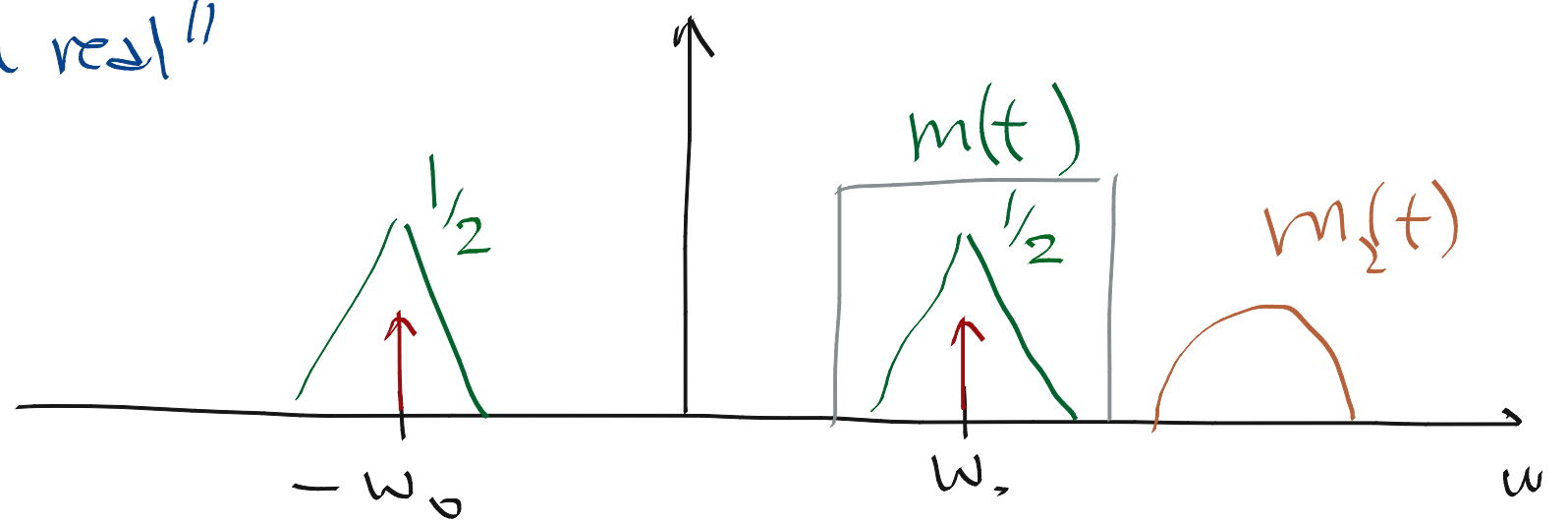
$$Z(j\omega) = \frac{1}{2} [S(j(\omega - \omega_0)) + S(j(\omega + \omega_0))] * \frac{1}{2} [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

$$Z(j\omega) = \frac{1}{4} S(j(\omega - 2\omega_0)) + \frac{1}{4} S(j\omega) + \frac{1}{4} S(j\omega) + \frac{1}{4} S(j(\omega + 2\omega_0))$$

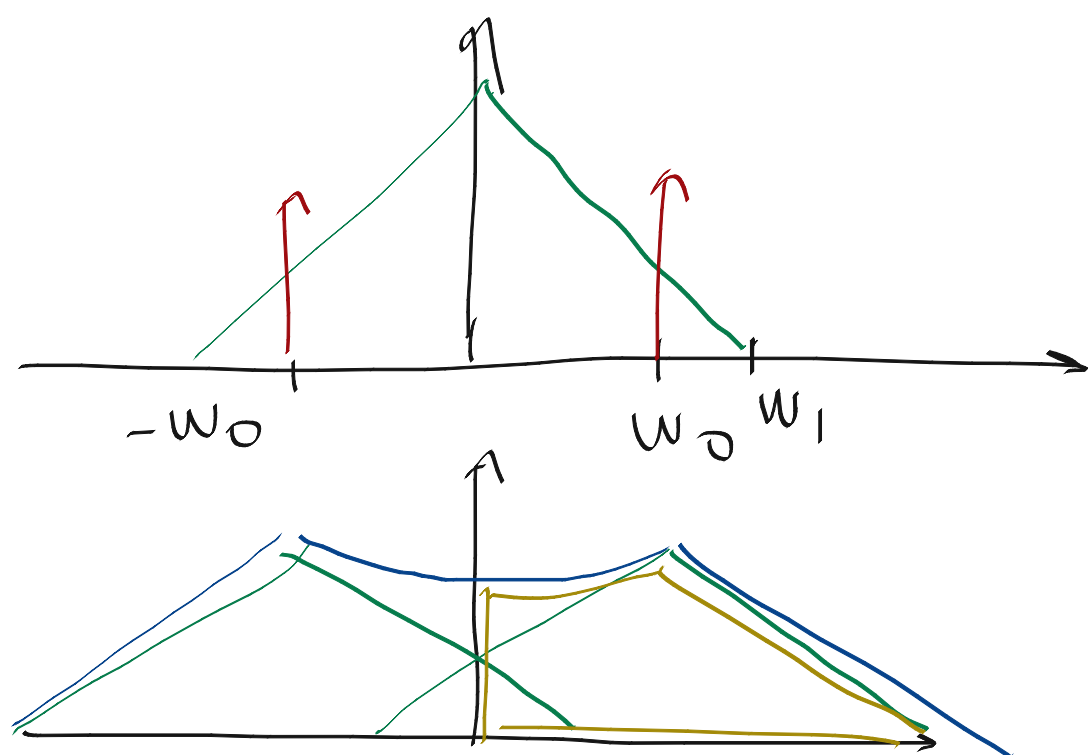
$$Z(j\omega) = \frac{1}{2} \underline{S(j\omega)} + \frac{1}{4} S(j(\omega - 2\omega_0)) + \frac{1}{4} S(j(\omega + 2\omega_0))$$

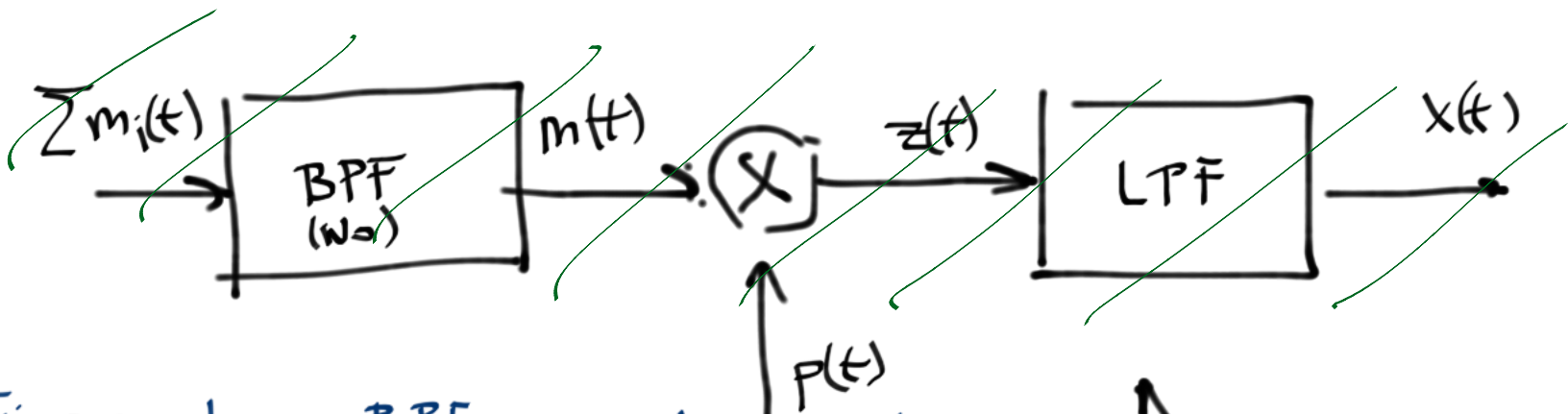


Ejemplo: "reconstrucción real"



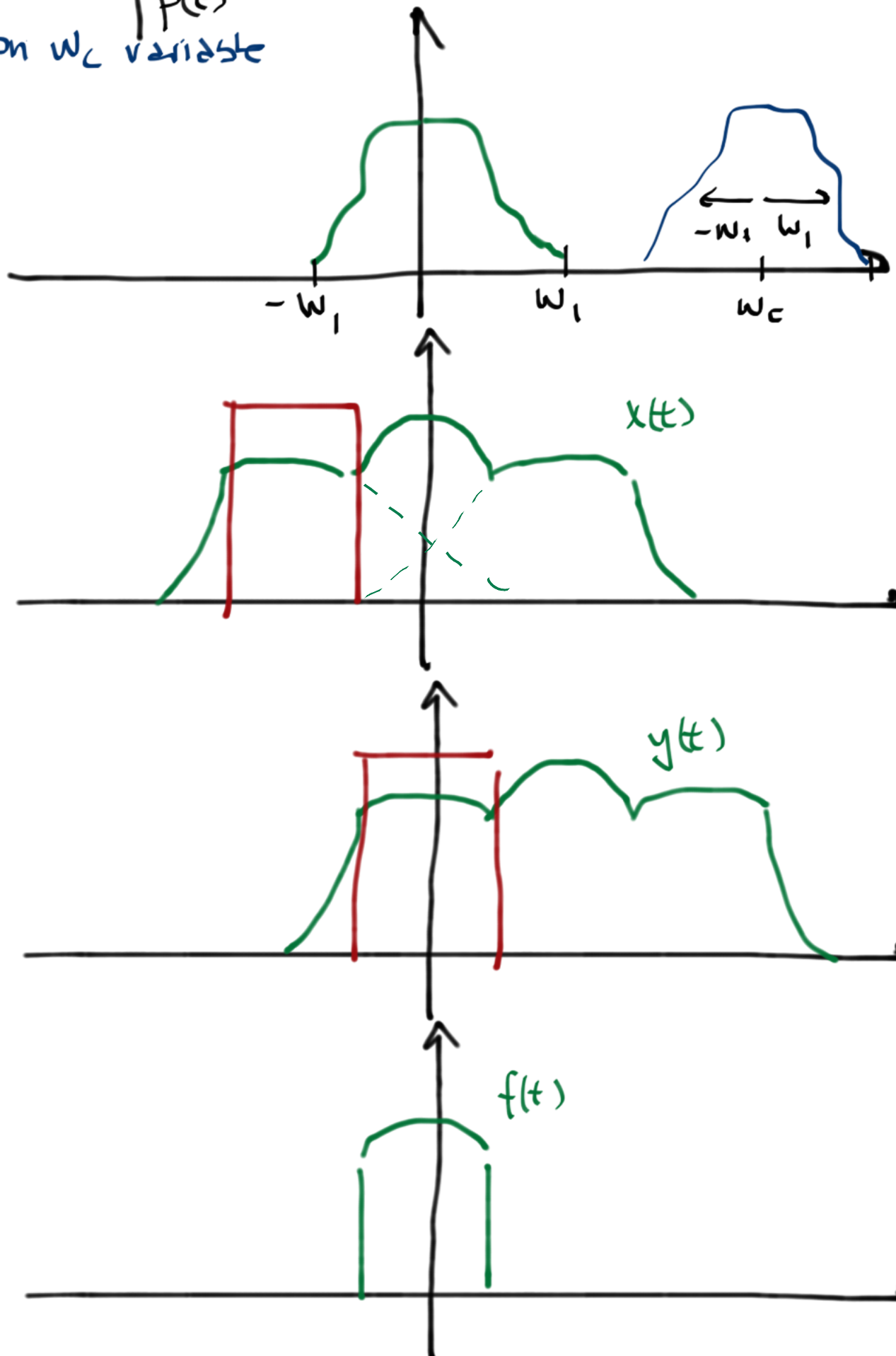
obs: "banda limitada"
 $w_1 < \frac{w_0}{2}$

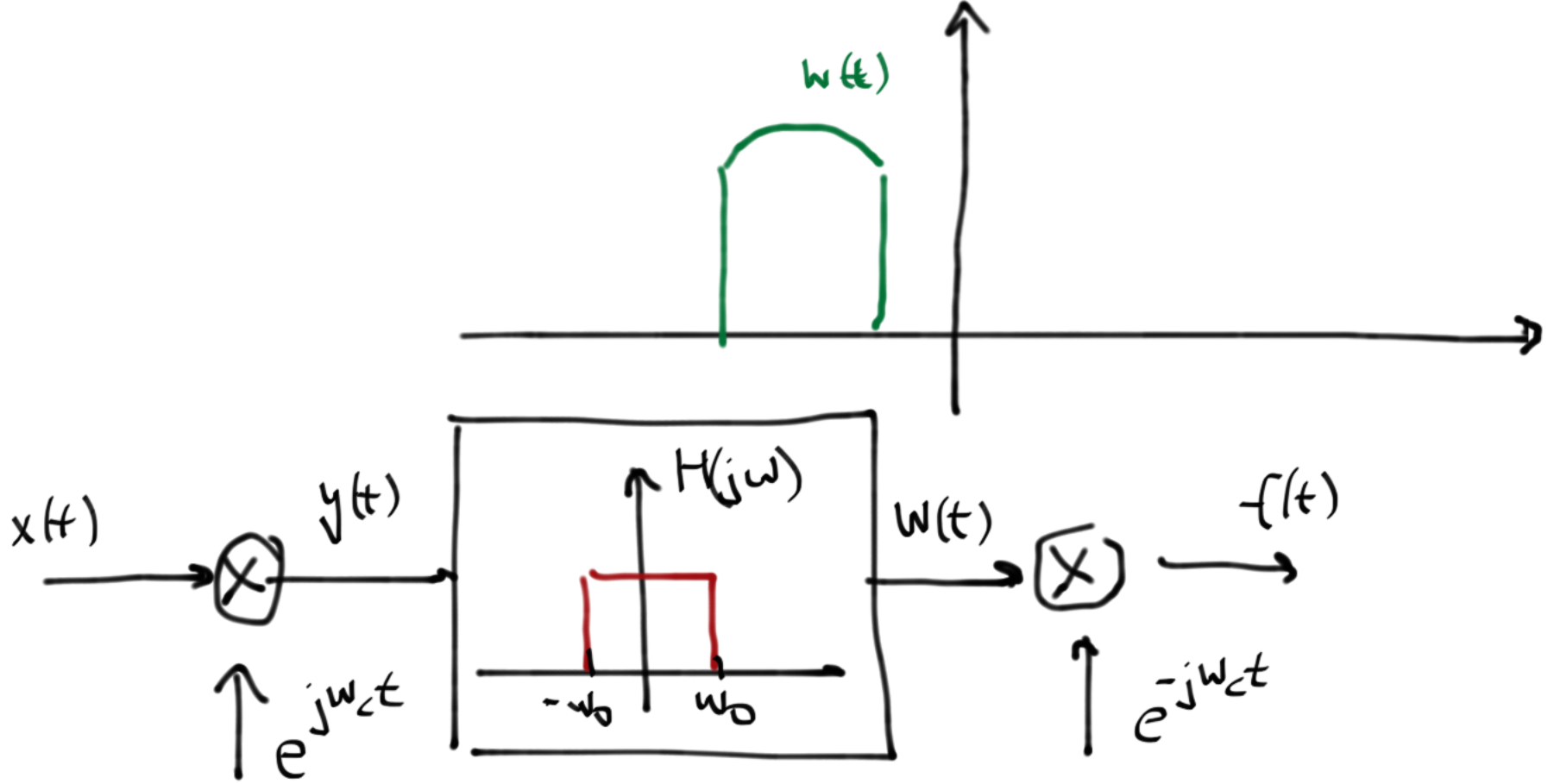




Ejemplo: BPF con ω_c variable

Hay que ver
que hacer con
este.





Practico 11

Transformada de Fourier

Ejemplos integradores

Ejemplo integrador

* limitaciones del análisis: oculograma vs batido

Clase 22

Transformada de Fourier

T. Fourier en tiempo discreto

T. FOURIER EN TIEMPO DISCRETO (DTFT)

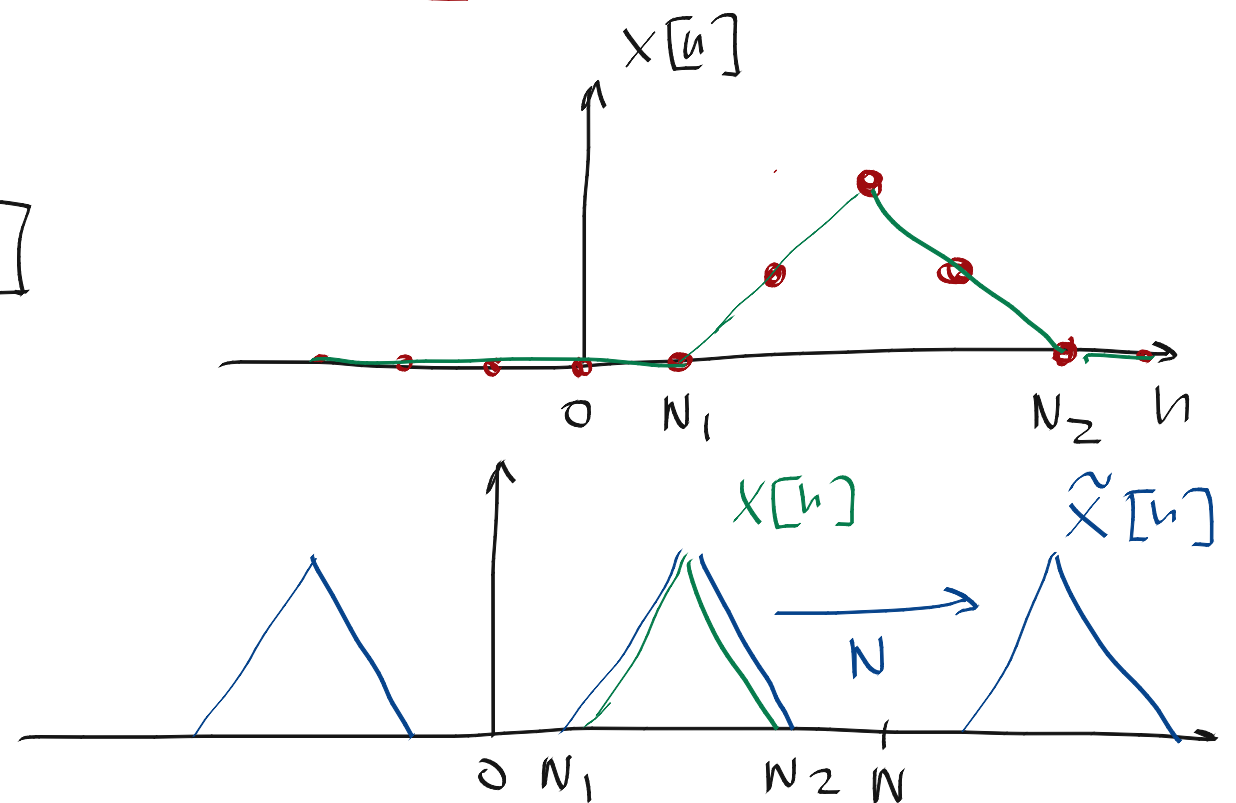
Aproximación periódica

$x[n]$ de soporte acotado $[N_1, N_2]$

$\tilde{x}[n]$ periódica N tal que:

$$\tilde{x}[n] = x[n] \text{ en } [N_1, N_2]$$

$$\lim_{N \rightarrow \infty} \tilde{x}[n] = x[n]$$



Serie de Fourier de $\tilde{x}[n]$ (DTFS)

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} a_k \cdot e^{j k \omega_0 n}, \quad \omega_0 = \frac{2\pi}{N}$$

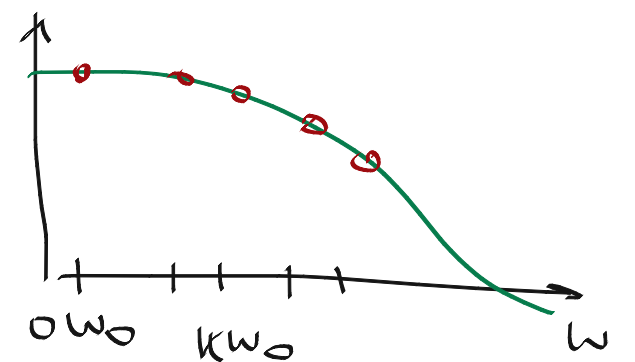
Análisis: $a_k = \frac{1}{N} \sum_{n=\langle N \rangle} \tilde{x}[n] \cdot e^{-j k \omega_0 n} = \frac{1}{N} \sum_{n=N_1}^{N_2} x[n] \cdot e^{-j k \omega_0 n}$

$x[n]$ sop. acotado
 $\downarrow$
 $\Rightarrow$

$$a_k = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j k \omega_0 n} \Rightarrow N \cdot a_k = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j k \omega_0 n}$$

$$f(k \omega_0) = f(\omega) \big|_{\omega = k \omega_0}$$

$$\lim_{N \rightarrow \infty} N a_k = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j \omega n}$$



cc. análisis

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j \omega n}$$

$$a_k = \frac{1}{N} \cdot X(e^{j\omega}) \big|_{\omega = k \omega_0}$$

$$N \rightarrow \infty$$

$$\omega_0 \rightarrow 0$$

Síntesis

$$\tilde{X}[n] = \sum_{k=\langle N \rangle} a_k \cdot e^{j k \omega_0 n}$$

$$\omega_0 = \frac{2\pi}{N}, \quad \frac{1}{N} = \frac{\omega_0}{2\pi}$$

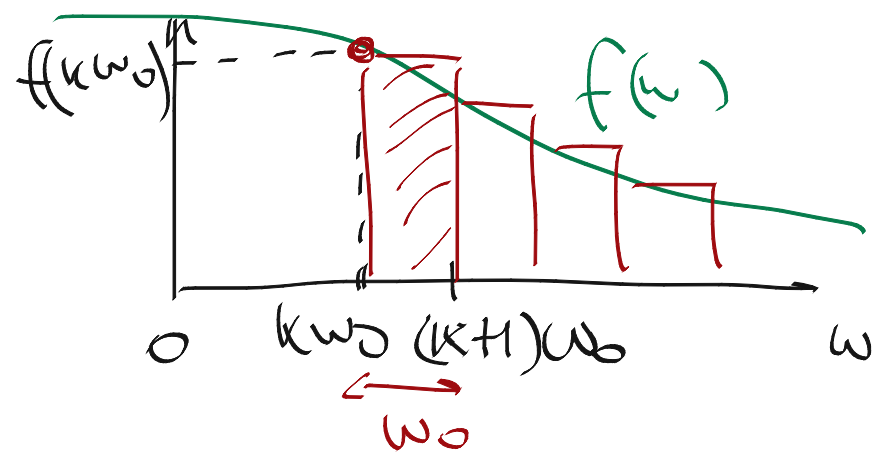
$$\tilde{X}[n] = \sum_{k=\langle N \rangle} \underbrace{\frac{\omega_0}{2\pi}}_{\frac{1}{N}} x(e^{j\omega}) \Big|_{\omega=k\omega_0} e^{j k \omega_0 n} = \frac{1}{2\pi} \sum_{k=\langle N \rangle} x(e^{j\omega}) \Big|_{\omega=k\omega_0} e^{j k \omega_0 n} \cdot \omega_0$$

$$f(k\omega_0) = f(\omega) \Big|_{\omega=k\omega_0}$$

$$X[n] = \lim_{N \rightarrow \infty} \tilde{X}[n] \Rightarrow$$

$$X[n] = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} x(e^{j\omega}) \cdot e^{j\omega n} d\omega$$

ec. síntesis



Transformada de Fourier de Tiempo Discreto

Análisis:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\omega n}$$

$$a_k = \frac{1}{N} \cdot X(e^{j\omega}) \Big|_{\omega = k\omega_0}$$

Coefs. Fourier

Síntesis:

$$x[n] = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(e^{j\omega}) \cdot e^{j\omega n} d\omega$$

Obs: * Terminamos con Fourier (continuaciones)

* Descomposición como C.L de exponenciales

* DTFS son muestras de la DTFT

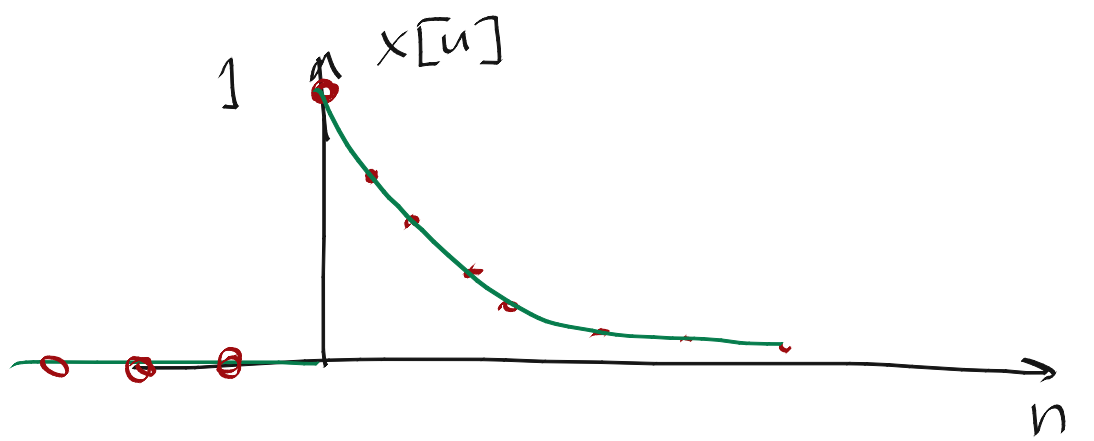
* DIFERENCIA con CTFT: $\rightarrow X(e^{j\omega})$ periódica

$\rightarrow$ Ec. síntesis finita,

Ejemplo 5.1 (p. 362)

ej 244

$$x[n] = u[n] \cdot a^n, \quad |a| < 1$$



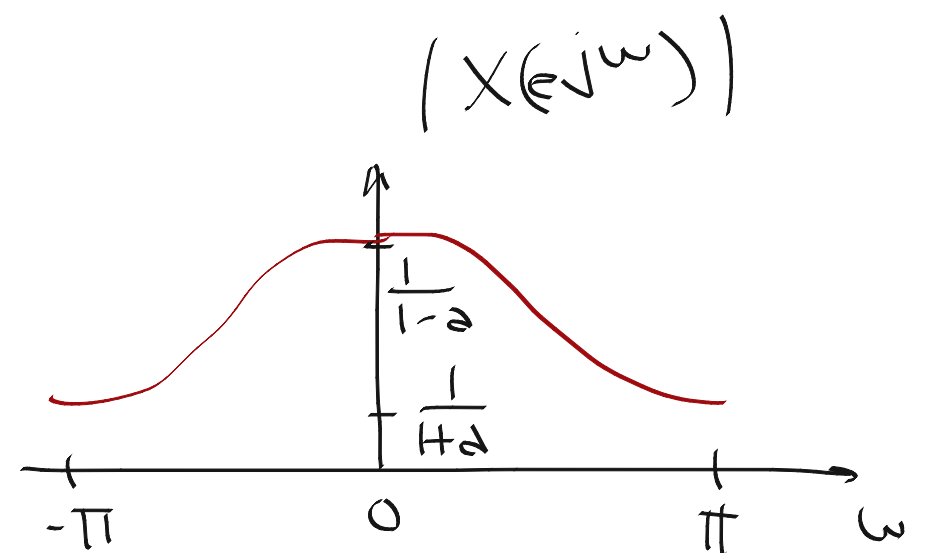
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} u[n] \cdot a^n \cdot e^{-j\omega n} = \sum_{k=0}^{\infty} (ae^{-j\omega})^k \stackrel{a \neq 1}{=} \frac{1 - (ae^{-j\omega})^{N+1}}{1 - ae^{-j\omega}} \rightarrow 0$$

$$X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

$$|X(e^{j\omega})| = \frac{1}{\sqrt{(1 - a\cos\omega)^2 + a^2 \sin^2\omega}}$$

$$|X(e^{j0})| = \frac{1}{\sqrt{(1-a)^2}} = \frac{1}{1-a}$$

$$|X(e^{j\pi})| = \frac{1}{\sqrt{(1+a)^2}} = \frac{1}{1+a}$$

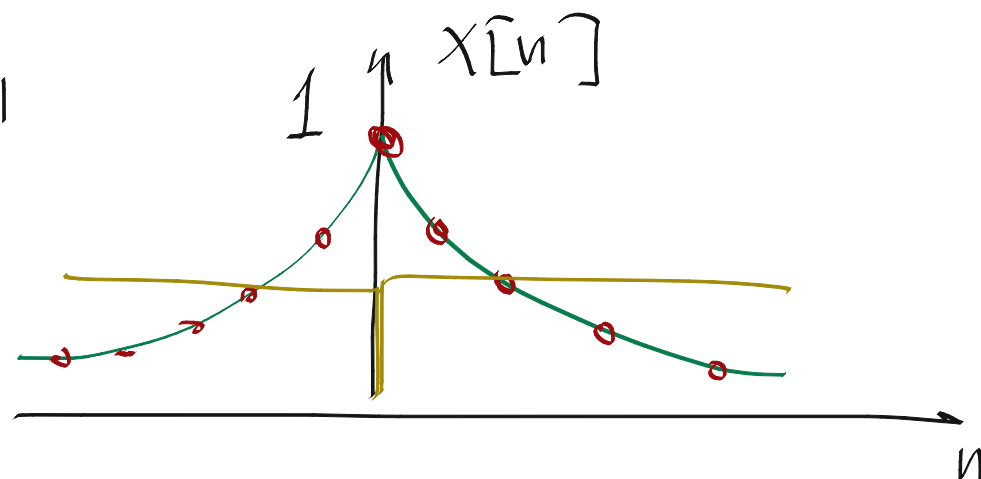


b) Código

- * formas de calcular el módulo
- * jugar con parámetros (a)
- * preguntar por periodicidad (en f y ω)
- * mostrar el save $\rightarrow$ npy

Ejemplo 5.2 (p. 364)

$$|a| < 1$$



$$a) \quad x[n] = u[n] \cdot a^n + u[-n] \cdot a^{-n}$$

$$x[n] = a^{|n|}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\omega n} = \underbrace{\sum_{n=0}^{\infty} a^n \cdot e^{-j\omega n}}_{\text{S.1}} + \underbrace{\sum_{n=-\infty}^{-1} a^{|n|} \cdot e^{-j\omega n}}_{\text{S.2}}$$

$$\text{S.2} = \sum_{n=-\infty}^{-1} a^{-n} \cdot e^{-j\omega n} \xrightarrow{m=-n} \sum_{m=1}^{\infty} a^m \cdot e^{j\omega m} = \sum_{m=0}^{\infty} a^m \cdot e^{j\omega m} - 1$$

$$= \frac{1 - (a e^{j\omega})}{1 - (a e^{j\omega})} \rightarrow 0$$

$$X(e^{j\omega}) = \frac{1 - a^2}{1 - 2a \cos \omega + a^2}$$

b) ptos. característicos

$$X(e^{j0}) = \frac{1 - a^2}{1 - 2a + a^2} = \frac{(1-a)(1+a)}{(1-a)^2} = \left[\frac{1+a}{1-a} \right] > 1$$

c) código.

* comparación

Tamara

$$\frac{2}{1 - e^{-j\omega}} - a$$

Manuel

$$\frac{1}{1 - ae^{-j\omega}} + \frac{1}{1 - \frac{e^{-j\omega}}{a}} - 1$$

Lucía

$$\frac{1}{1 - ae^{-j\omega}} + \frac{1}{1 - ae^{j\omega}} - 1$$

Ejemplo 5.2 (p. 364)

$$x[n] = a^{|n|}, |a| < 1$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-jn\omega}$$

$$= \sum_{n=-\infty}^{\infty} a^{|n|} e^{-jn\omega}$$

$$= \underbrace{\sum_{n=0}^{\infty} a^n e^{-jn\omega}}_{n=0} + \sum_{n=-\infty}^{-1} a^{-n} e^{-jn\omega} = \frac{1}{1 - ae^{-j\omega}} + \sum_{n=1}^{\infty} a^n e^{jn\omega}$$

$$= \frac{1}{1 - ae^{-j\omega}} + \sum_{n=0}^{\infty} a^n e^{jn\omega} - 1 = \frac{1}{1 - ae^{-j\omega}} + \frac{1}{1 - ae^{+j\omega}} - 1$$

$$= \frac{1 - ae^{j\omega} + 1 - ae^{-j\omega}}{1 - ae^{-j\omega} - ae^{j\omega} + a^2}$$

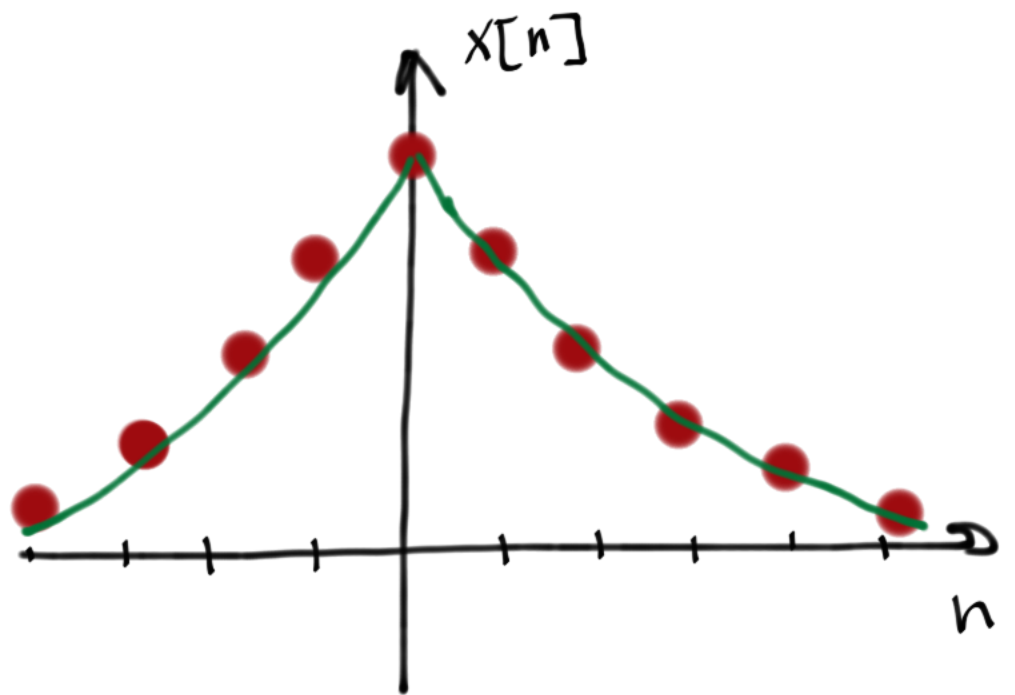
$$= \boxed{\frac{1 - a^2}{1 - 2a\cos\omega + a^2}}$$

obs: * $X(e^{j\omega}) \in \mathbb{R}$

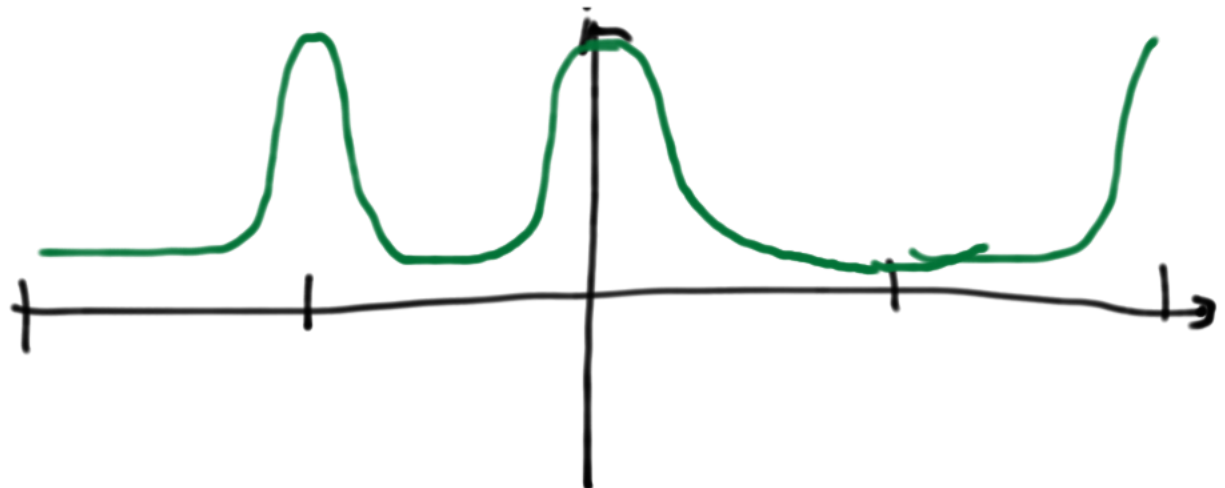
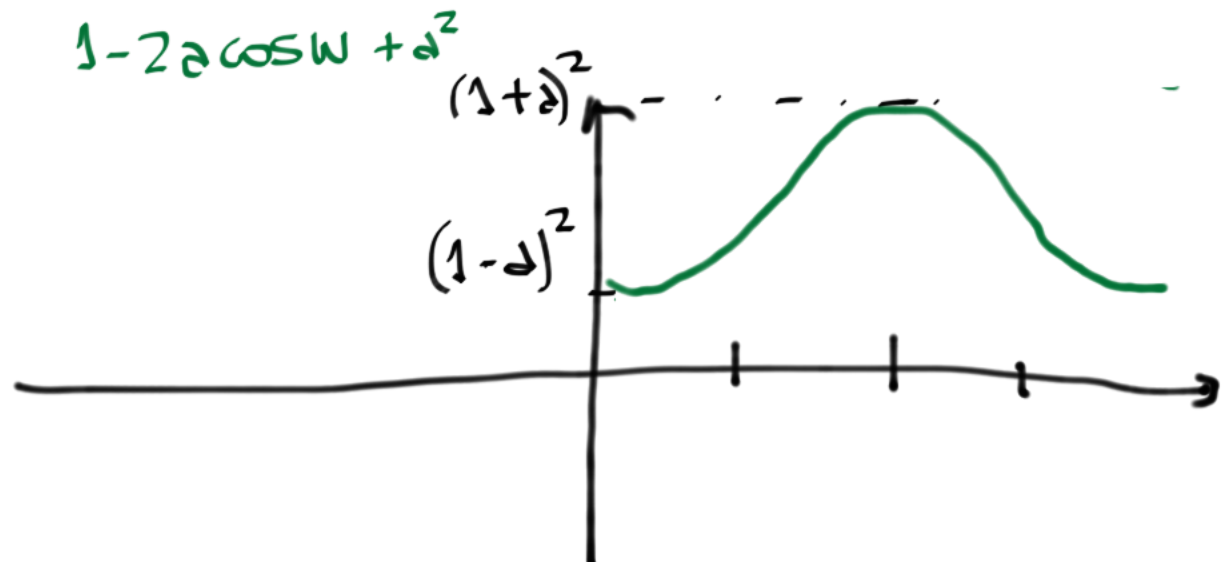
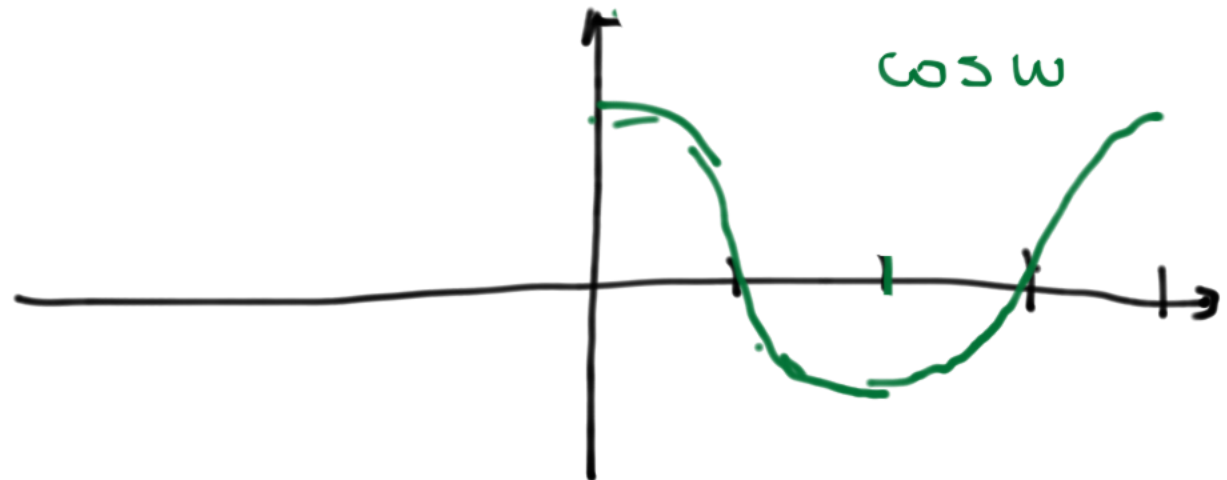
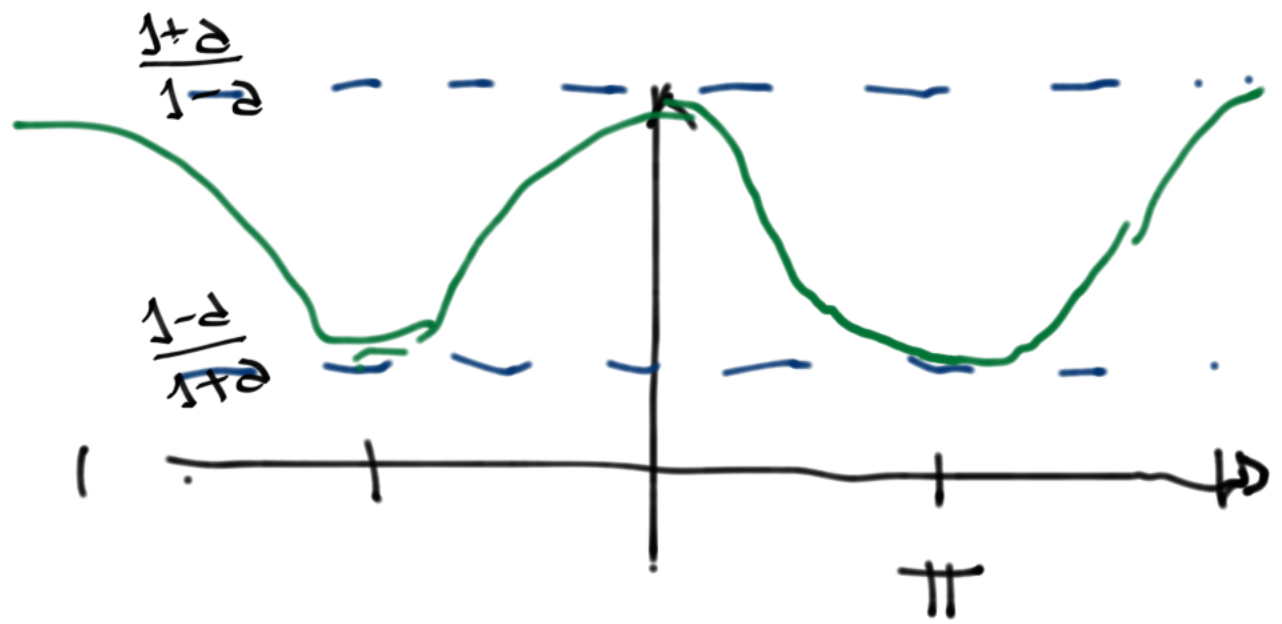
$$X(e^{j0}) = \frac{1 - a^2}{1 - 2a + a^2} = \frac{(1-a)(1+a)}{(1-a)^2} = \boxed{\frac{1+a}{1-a}}$$

$$X(e^{j\pi}) = \frac{1 - a^2}{1 + 2a + a^2} = \frac{(1-a)(1+a)}{(1+a)^2} = \boxed{\frac{1-a}{1+a}}$$

Para
pasar



conjugiv



Clase 23

Transformada de Fourier

T. Fourier en tiempo discreto

CONVERGENCIA

Obs:

* análisis:

$$X(e^{j\omega}) = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j n \omega}$$

serie X.

* síntesis:

$$x[n] = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(e^{j\omega}) \cdot e^{j n \omega} d\omega$$

integral finita ✓

Condiciones

1) Energía: $\sum_{k=-\infty}^{\infty} |x[k]|^2 < \infty$

2) Dirichlet: * $\sum_{k=-\infty}^{\infty} |x[k]| < \infty$

* otras: máximos y discontinuidades

Ejemplo 5.4: "reconstrucción parcial"

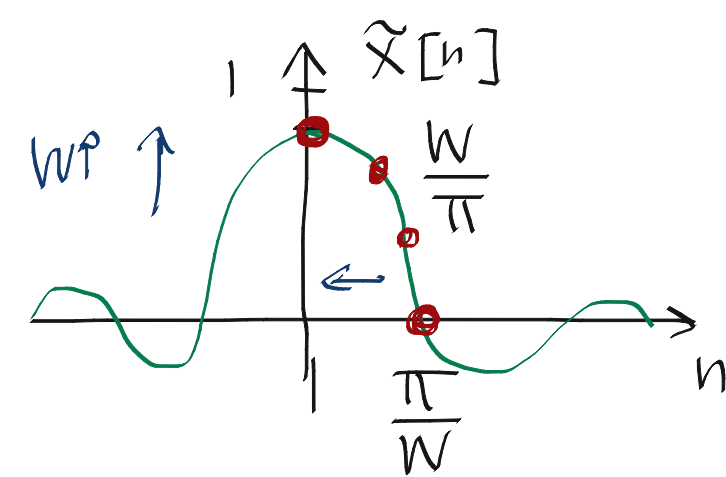
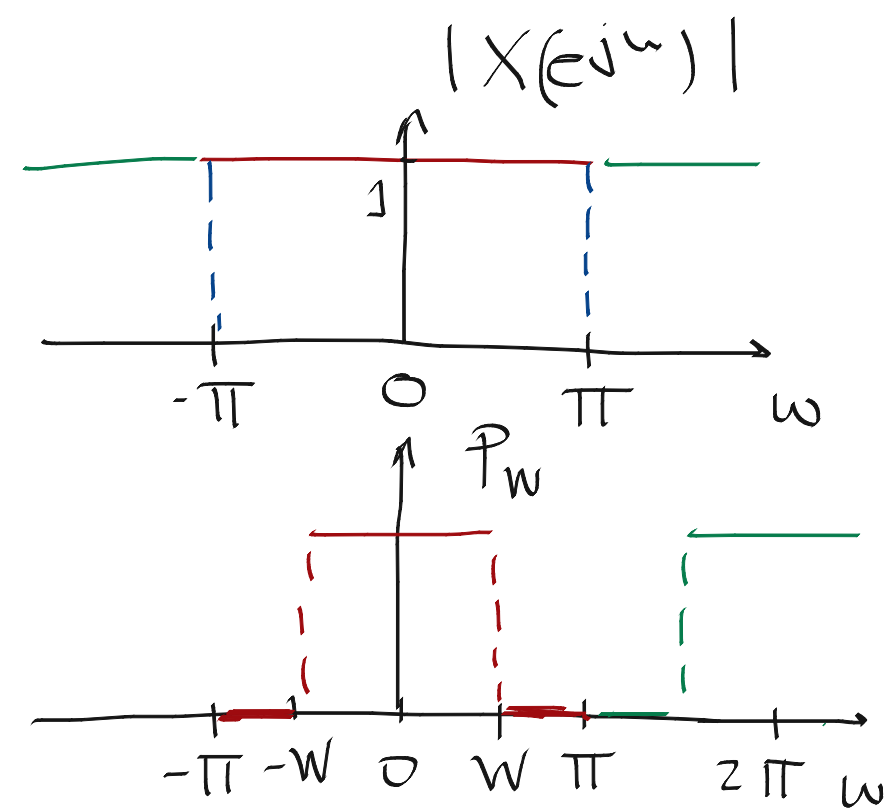
$$x[n] = \delta[n] \xleftrightarrow{\mathcal{F}} 1 = X(e^{j\omega})$$

$$\tilde{x}[n] = ? \xleftrightarrow{\mathcal{F}} P_W(e^{j\omega})$$

$$\tilde{x}[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_W(e^{j\omega}) \cdot e^{jn\omega} d\omega$$

$$\tilde{x}[n] = \frac{1}{2\pi} \int_{-W}^W 1 \cdot e^{jn\omega} d\omega \stackrel{n \neq 0}{=} \frac{1}{2\pi} \frac{e^{jWn} - e^{-jWn}}{jn} = \boxed{\frac{\text{sen}[Wn]}{\pi n}} \quad n \neq 0$$

$$\tilde{x}[0] = \frac{2W}{2\pi} = \boxed{\frac{W}{\pi}}$$



$$\stackrel{n \rightarrow 0}{\lim} \tilde{x}[n] = \lim_{n \rightarrow 0} \frac{\text{sen}[Wn]}{\pi n} = \frac{W}{\pi} \quad \boxed{\frac{W}{\pi}}$$

obs: ¿que pasa si $W \rightarrow \pi$? ¿será que $\tilde{x}[n] \rightarrow x[n]$?

$$x[n] = \lim_{W \rightarrow \pi} \tilde{x}[n] = \lim_{W \rightarrow \pi} \frac{\text{sen}[Wn]}{\pi n} = \frac{\text{sen}[\pi n]}{\pi n} = \begin{cases} 1 & n=0 \\ 0 & n \neq 0 \end{cases}$$

$$\boxed{x[n] = \begin{cases} 1 & n=0 \\ 0 & n \neq 0 \end{cases}}$$

$\Rightarrow$

$$\boxed{x[n] = \delta[n]}$$

PROPIEDADES

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega}), \quad y[n] \xleftrightarrow{\mathcal{F}} Y(e^{j\omega})$$

Periodicidad: $X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$

Linealidad: $a \cdot x[n] + b \cdot y[n] \xleftrightarrow{\mathcal{F}} a \cdot X(e^{j\omega}) + b \cdot Y(e^{j\omega})$

Desplazamiento en el tiempo: $x[n-n_0] \xleftrightarrow{\mathcal{F}} e^{-jn_0\omega} X(e^{j\omega})$

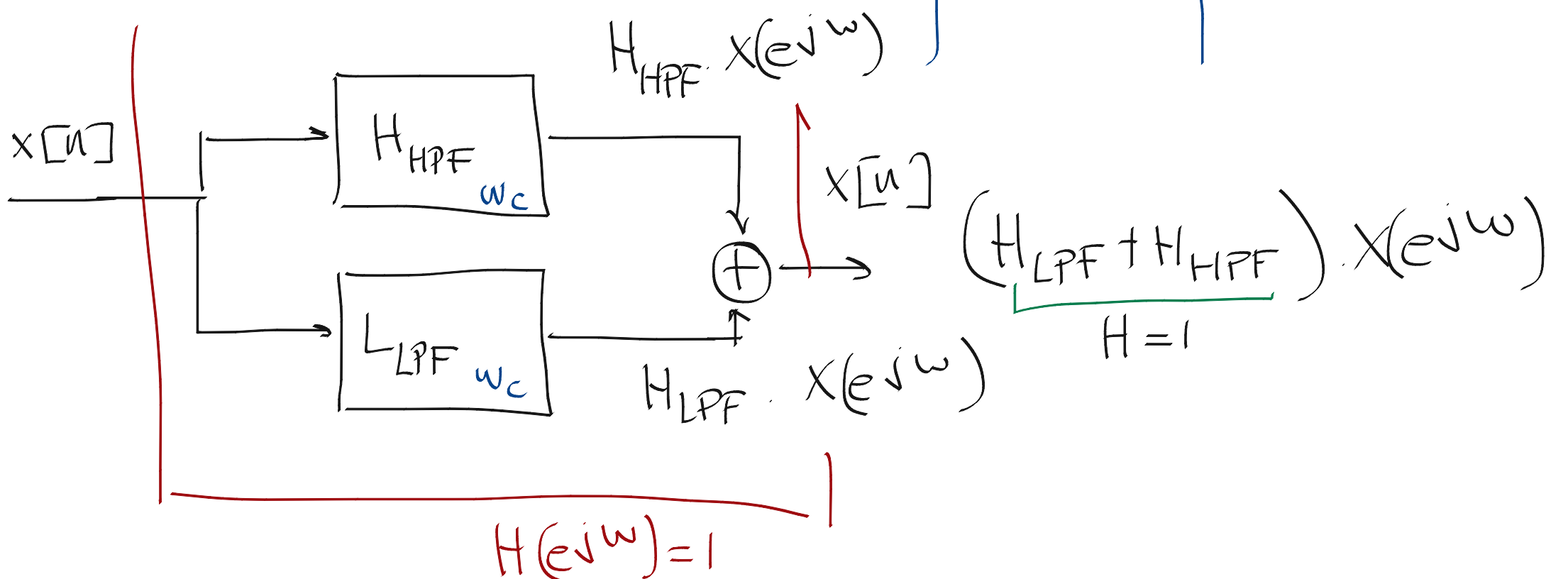
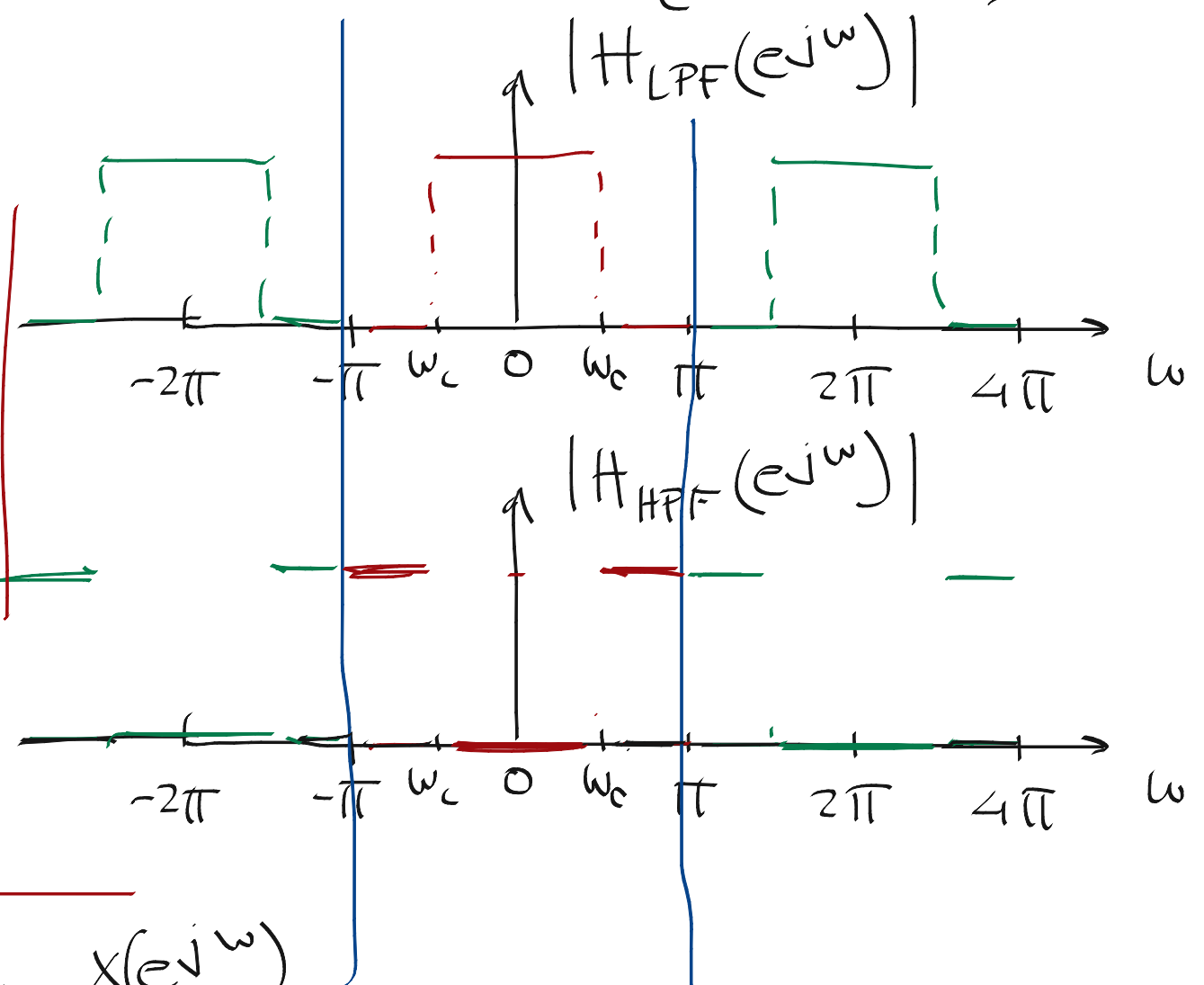
Desplazamiento en frecuencia: $e^{j\omega_0 n} x[n] \xleftrightarrow{\mathcal{F}} X(e^{j(\omega-\omega_0)})$

Ejemplo 5.7 "filtros ideales"

$$H_{HPF}(e^{j\omega}) = H_{LPF}(e^{j(\omega-\omega_c)})$$

$$h_{HPF}[n] = e^{j\omega_c n} h_{LPF}[n]$$

$$\omega_c = \frac{\pi}{2}$$



Conjugación y simetría conjugada

$$x^*[n] \xleftrightarrow{\mathcal{F}} X^*(e^{-j\omega})$$

$$* \quad x[n] \in \mathbb{R}, \quad X(e^{j\omega}) = X^*(e^{-j\omega})$$

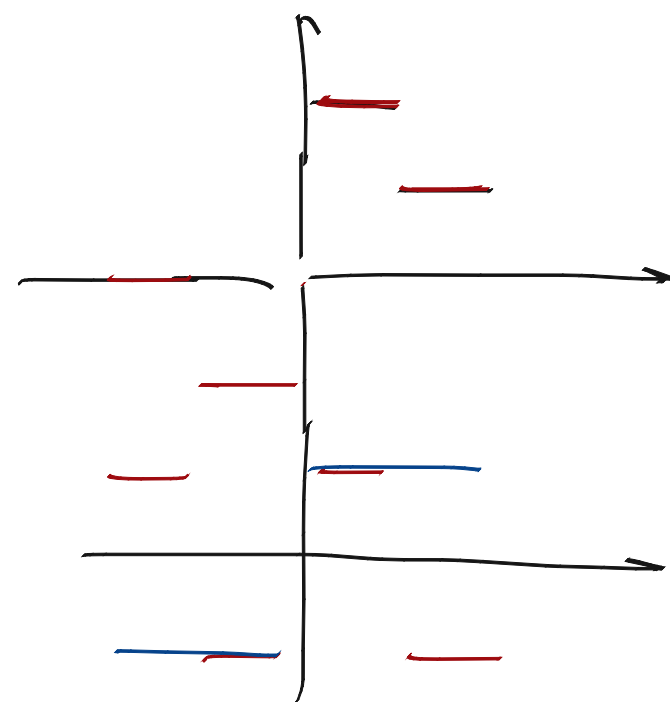
$$\rightarrow x[n] \text{ par} \xleftrightarrow{\mathcal{F}} X(e^{j\omega}) \text{ real y par}$$

$$\rightarrow x[n] \text{ impar} \xleftrightarrow{\mathcal{F}} X(e^{j\omega}) \text{ imaginaria pura e impar}$$

$$* \quad x[n] \in \mathbb{R} \rightarrow x[n] = x_{\text{par}}[n] + x_{\text{impar}}[n]$$

$$\rightarrow x_{\text{par}}[n] \xleftrightarrow{\mathcal{F}} \text{Re}\{X(e^{j\omega})\}$$

$$\rightarrow x_{\text{impar}}[n] \xleftrightarrow{\mathcal{F}} j \cdot \text{Im}\{X(e^{j\omega})\}$$



obs: descomposición par $f(x)$

$$* \quad f_{\text{par}}(x) = \frac{f(x) + f(-x)}{2}$$

$$f_{\text{par}}(-x) = \frac{f(-x) + f(x)}{2}$$

$$* \quad f_{\text{impar}}(x) = \frac{f(x) - f(-x)}{2}$$

$$* \quad \boxed{f_{\text{par}}(x) + f_{\text{impar}}(x) = f(x)}$$

Diferenciación

$$y[n] = x[n] - x[n-1] \xrightarrow{\mathcal{F}} X(e^{j\omega}) = (1 - e^{-j\omega}) \underbrace{X(e^{j\omega})}$$

$$Y(j\omega) = j\omega \cdot \underbrace{X(j\omega)} \quad \leftrightarrow \quad \underline{\underline{\text{DTF}}}$$

Acumulación

$$y[n] = \sum_{k=-\infty}^n x[k] \xleftrightarrow{\mathcal{F}} ?$$

C.V. $l = n - k, k = n - l \Rightarrow y[n] = \sum_{l=-\infty}^0 x[n-l]$

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n] \cdot e^{-jn\omega} = \sum_{n=-\infty}^{\infty} \sum_{l=0}^{\infty} \underbrace{x[n-l]}_m \cdot e^{-jn\omega} \quad \begin{array}{l} \text{C.V. } m = n-l \\ n = m+l \\ \Rightarrow \end{array}$$

$$Y(e^{j\omega}) = \sum_{m=-\infty}^{\infty} \sum_{l=0}^{\infty} x[m] \cdot e^{-j(m+l)\omega} = \sum_{l=0}^{\infty} e^{-jl\omega} \sum_{m=-\infty}^{\infty} x[m] \cdot e^{-jm\omega}$$

$$Y(e^{j\omega}) = X(e^{j\omega}) \cdot \sum_{l=0}^{\infty} e^{-jl\omega} \quad \underbrace{\sum_{m=-\infty}^{\infty} x[m] \cdot e^{-jm\omega}}_{X(e^{j\omega})}$$

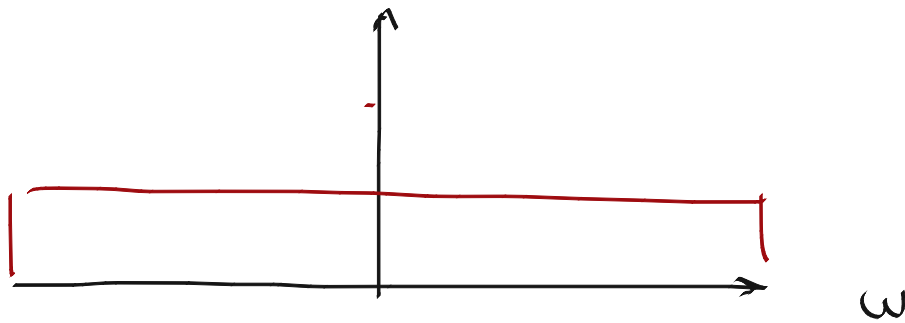
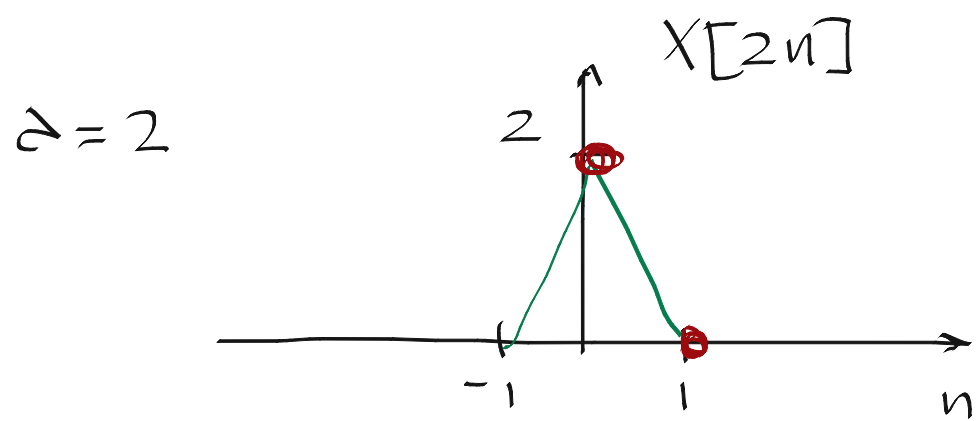
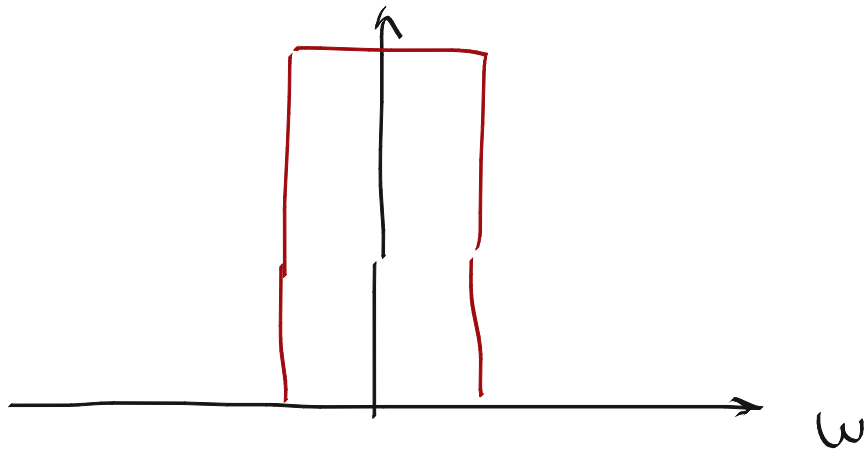
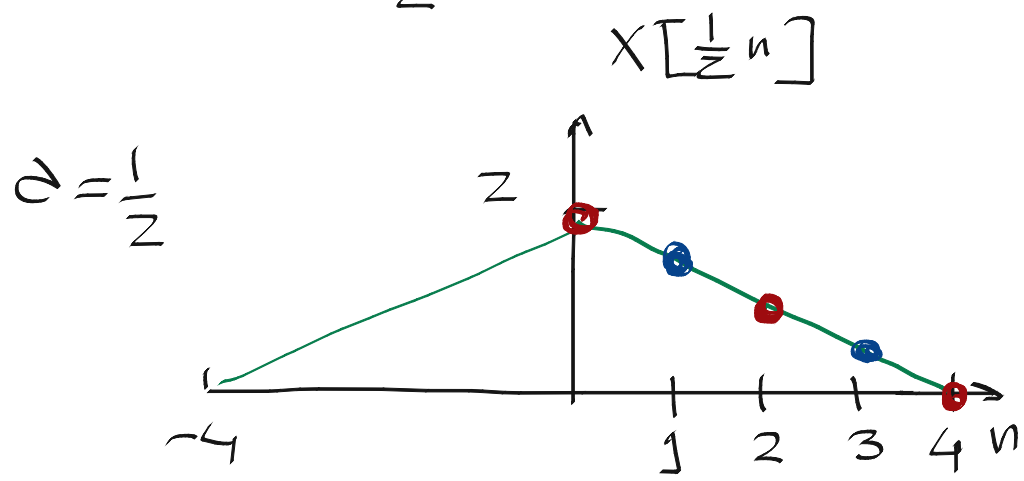
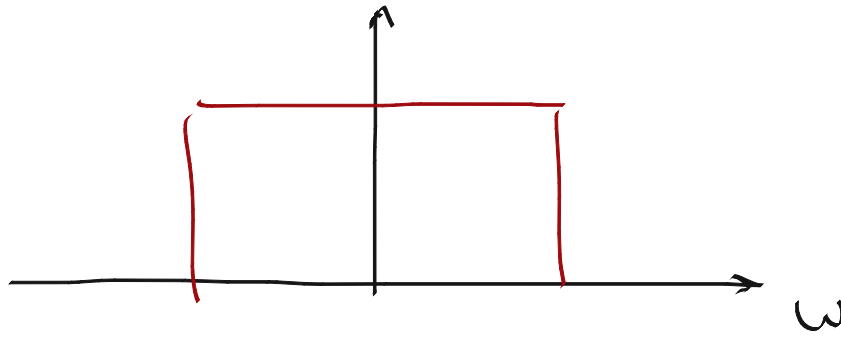
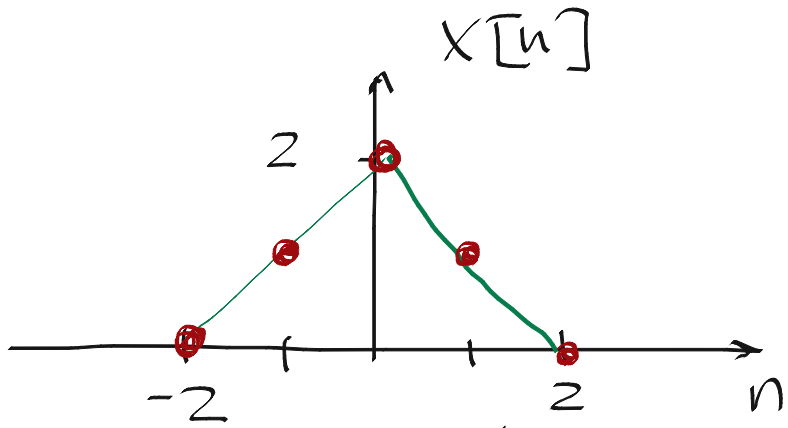
$$Y(e^{j\omega}) = X(e^{j\omega}) \sum_{l=0}^{\infty} \underbrace{(e^{-j\omega})^l}_r$$

$$Y(e^{j\omega}) = \frac{1}{1 - e^{-j\omega}} \cdot X(e^{j\omega})$$

$$y[n] = \sum_{k=-\infty}^n x[k] \xleftrightarrow{\mathcal{F}} Y(e^{j\omega}) = \frac{1}{1 - e^{-j\omega}} \cdot X(e^{j\omega}) + \pi \cdot X(e^{j0}) \cdot \delta(\omega)$$

Expansión temporal

$$x[an] \xleftrightarrow{\mathcal{F}} \frac{1}{|a|} X(e^{j\frac{\omega}{a}})$$



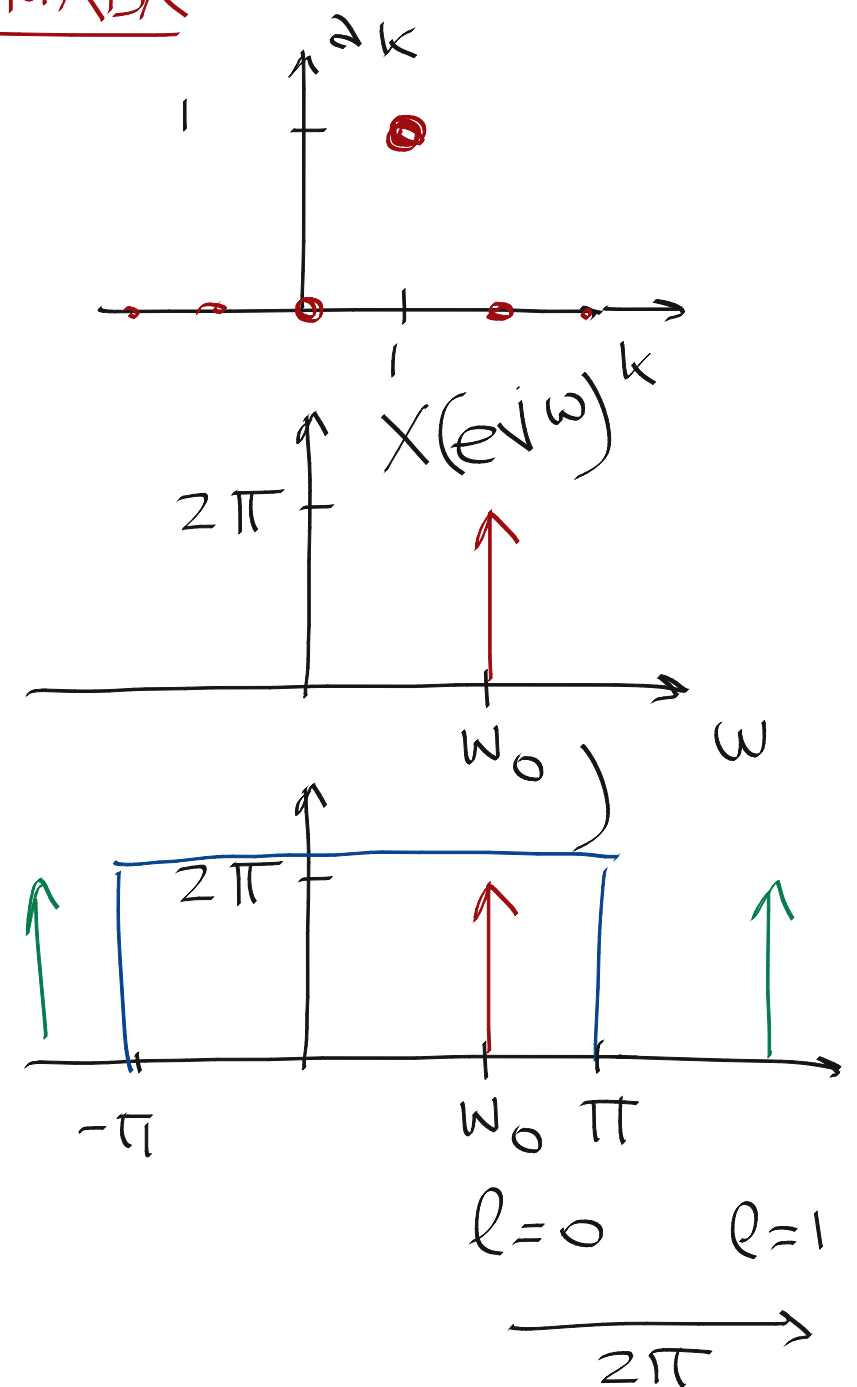
Inversión temporal

$$x[-n] \xleftrightarrow{\mathcal{F}} X(e^{-j\omega})$$

RELACIÓN SERIE-TRANSFORMADA

Topiandl]

a) $x[n] = e^{j\omega_0 n} \xleftrightarrow{\text{DTFS}} d_k = \begin{cases} 1 & k=1 \\ 0 & k \neq 1 \end{cases}$



DTFS $\longleftrightarrow$ DTFT

d_k

$X(e^{j\omega})$

$\delta[k]$

$2\pi \cdot \delta(\omega - \omega_0)$
periódica 2π

$$X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi \cdot \delta(\omega - \omega_0 - 2\pi l)$$

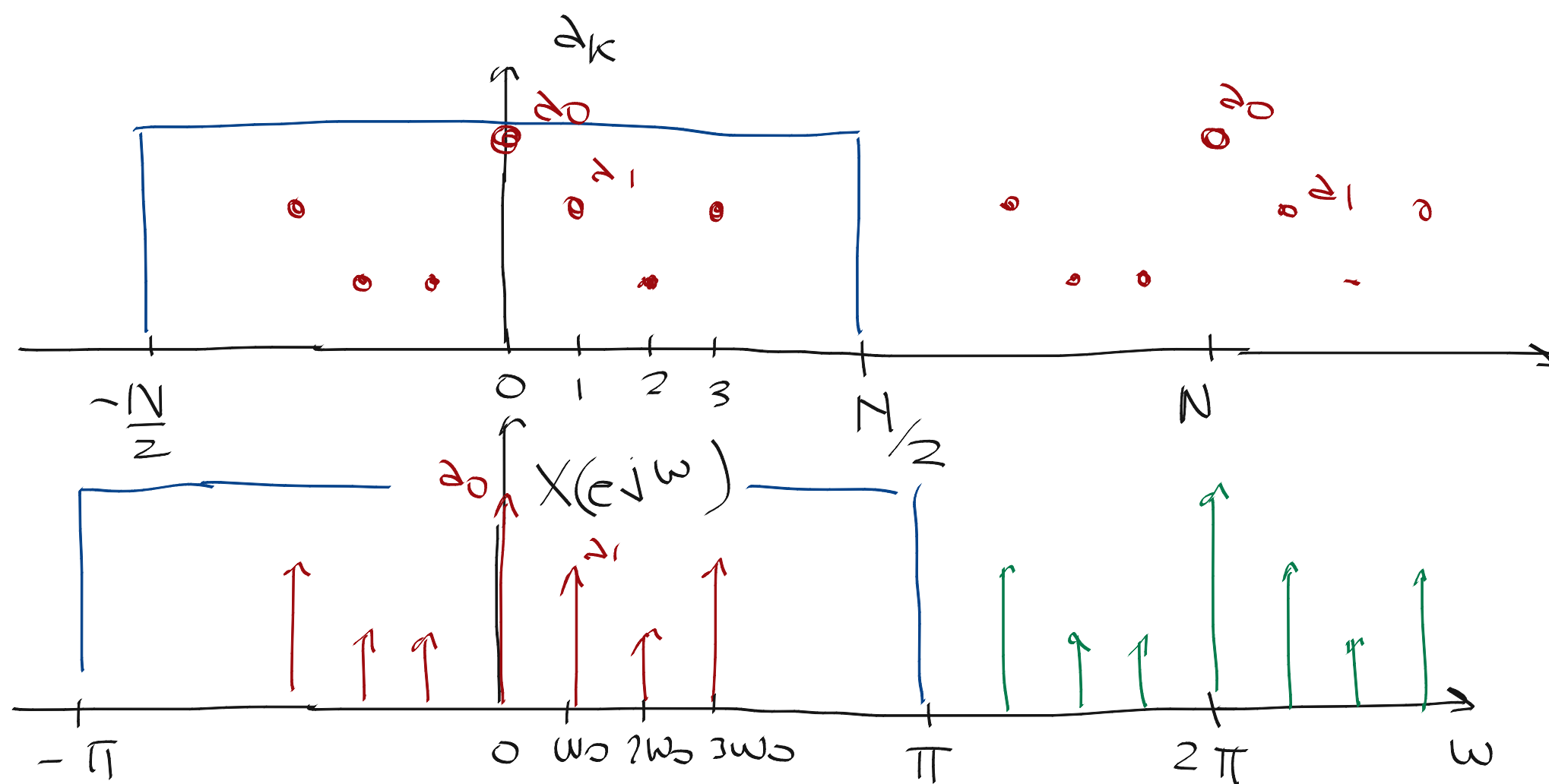
verif

$\tilde{x}[n] \xleftrightarrow{\text{DTFT}} X(e^{j\omega}) = 2\pi \cdot \delta(\omega - \omega_0)$

$$\tilde{x}[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} 2\pi \cdot \delta(\omega - \omega_0) \cdot e^{jn\omega} d\omega = e^{j\omega_0 n} \quad \checkmark$$

$$d_k = \frac{1}{2\pi} \cdot X(e^{j\omega}) \Big|_{\omega=k\omega_0} = \frac{1}{2\pi} \cdot 2\pi \cdot \delta(\underbrace{k\omega_0 - \omega_0}_0) = \begin{cases} 1 & k=1 \\ 0 & k \neq 1 \end{cases}$$

b)
$$X[n] = \sum_{k=\langle N \rangle} a_k \cdot e^{j k \omega_0 n}$$



$$X(e^{j\omega}) = \sum_{k=\langle N \rangle} 2\pi \cdot a_k \cdot \delta(\omega - k\omega_0) \quad \text{periódica } \pi$$

$$X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} \sum_{k=\langle N \rangle} 2\pi \cdot a_k \cdot \delta(\omega - k\omega_0 - 2\pi l)$$

escrito con repetición
explícita

Clase 24

T. de Fourier tiempo discreto

Propiedades

Ejemplo 5.9

$$z[n] = y_1[n] + y_2[n]$$

$$x[n] = p_2[n-2] *_{\text{I}}$$

$$y_1[n] = \begin{cases} x[\frac{n}{2}] & n \text{ par} \\ 0 & n \text{ impar} \end{cases} *_{\text{II}}$$

$$y_2[n] = \frac{1}{2} \cdot y_1[n-1] *_{\text{III}}$$

$$P_{N_1}(e^{j\omega}) = \frac{\text{sen}(\frac{N_1+1}{2}\omega)}{\text{sen}(\frac{\omega}{2})}$$

$$N_1 = 2$$

$$P_2(e^{j\omega}) = \frac{\text{sen}(\frac{3}{2}\omega)}{\text{sen}(\frac{\omega}{2})}$$

$$*_{\text{I}} \quad X(e^{j\omega}) = e^{-j2\omega} P_2(e^{j\omega})$$

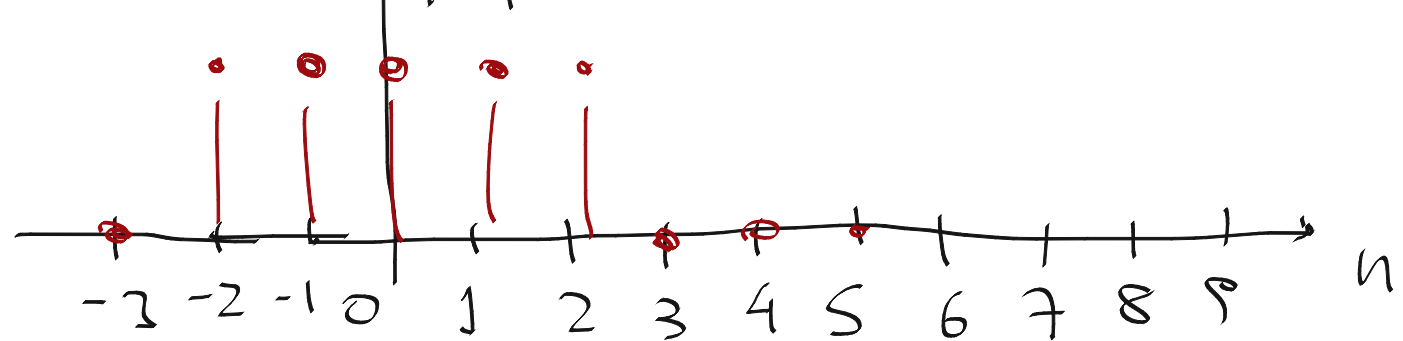
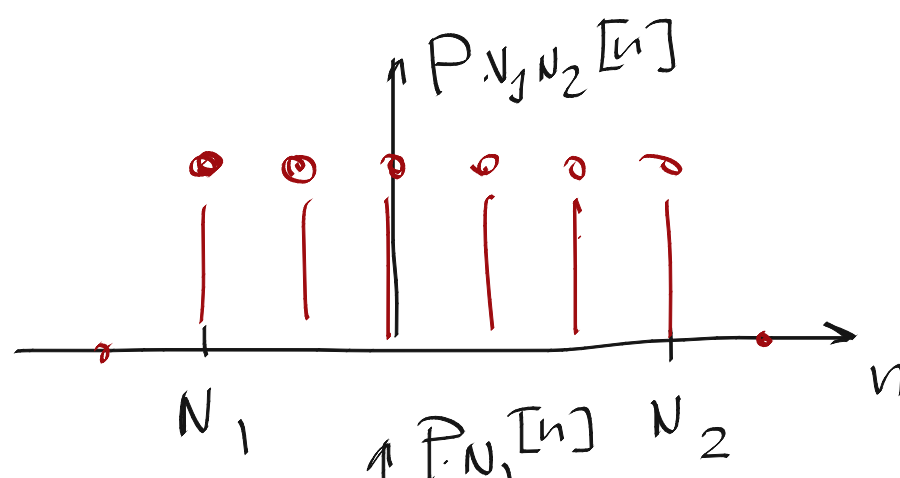
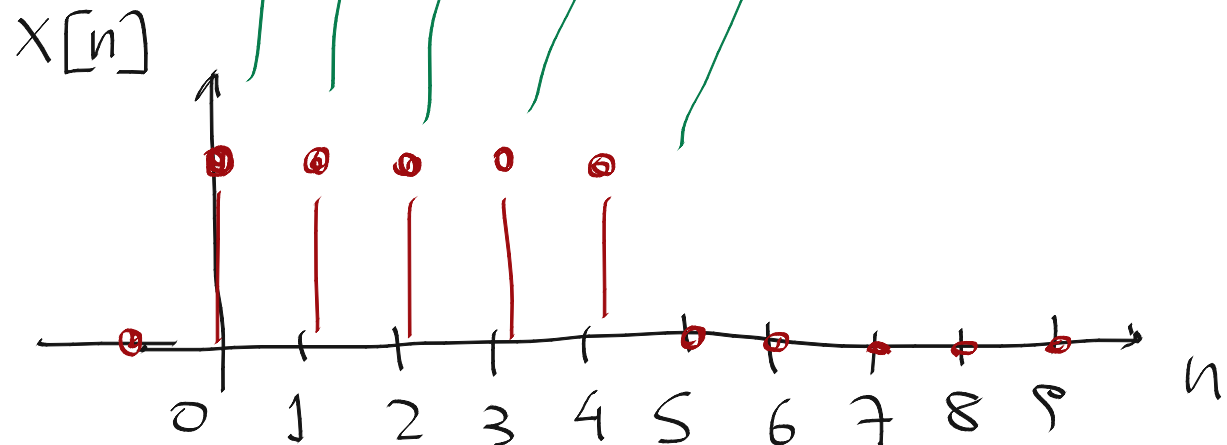
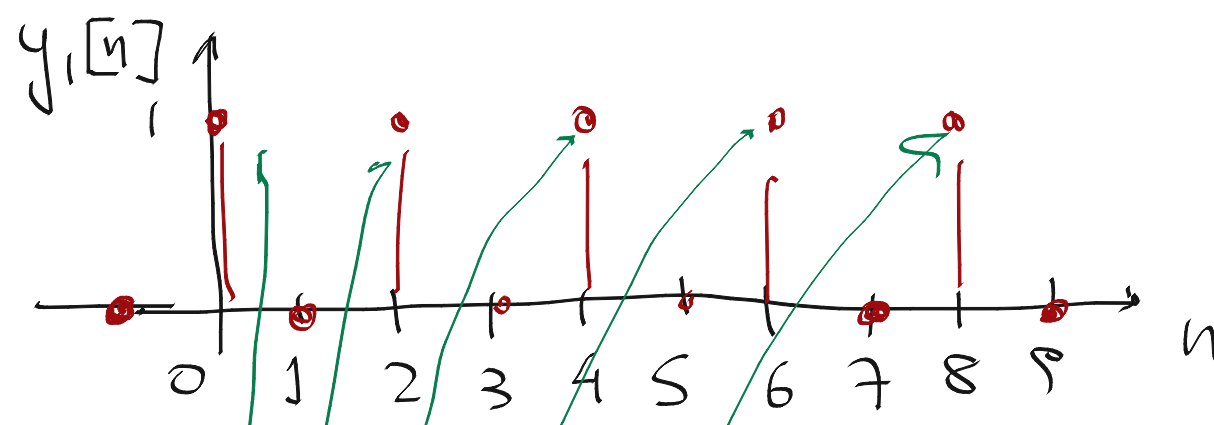
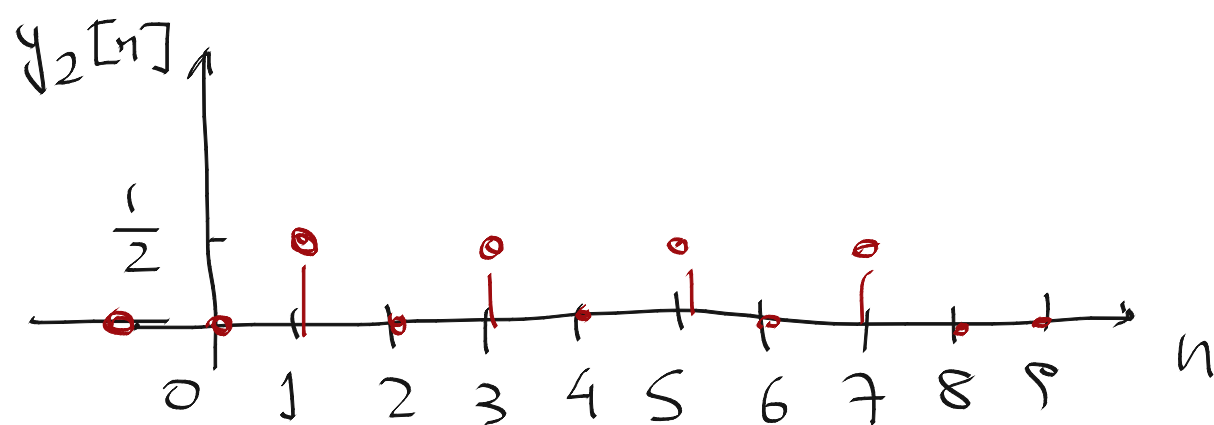
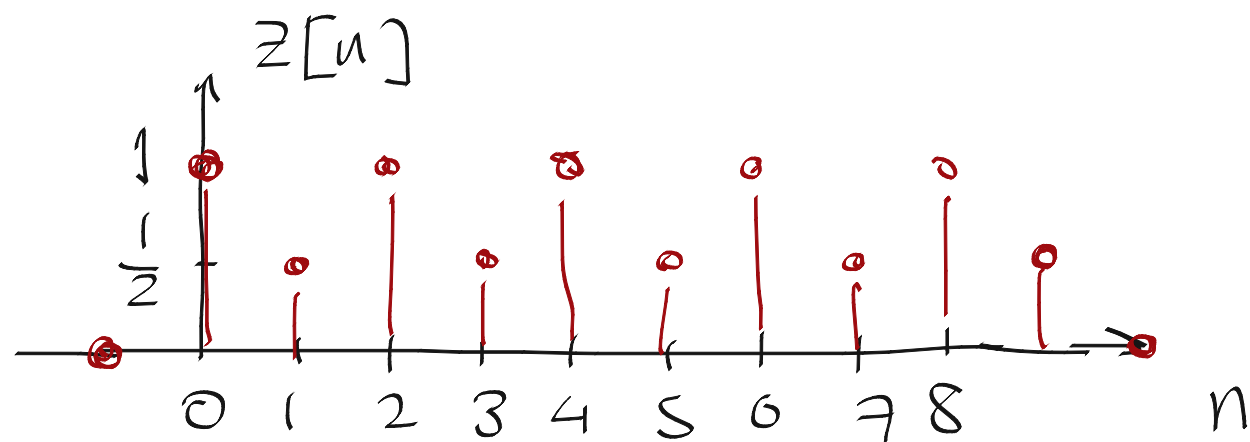
$$*_{\text{II}} \quad Y(e^{j\omega}) = 2 \cdot X(e^{j2\omega})$$

$$*_{\text{III}} \quad Y_2(e^{j\omega}) = \frac{1}{2} \cdot e^{-j\omega} Y(e^{j\omega})$$

$$Z(e^{j\omega}) = Y_1(e^{j\omega}) + Y_2(e^{j\omega})$$

$$Z(e^{j\omega}) = 2 \cdot X(e^{j2\omega}) + e^{-j\omega} X(e^{j2\omega})$$

$$Z(e^{j\omega}) = (2 + e^{-j\omega}) \cdot e^{-j2\omega} \frac{\text{sen}(3\omega)}{\text{sen}(\omega)}$$



Diferenciación en frecuencia

$$-jn \cdot x[n] \xleftrightarrow{\mathcal{F}} \frac{d}{d\omega} X(e^{j\omega})$$

Relación de Parseval

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} |X(e^{j\omega})|^2 d\omega$$

Convulsión:

$$y[n] = x[n] * h[n] \xleftrightarrow{\mathcal{F}} Y(e^{j\omega}) = X(e^{j\omega}) \cdot H(e^{j\omega})$$

Multiplicación (modulación)

$$x[n] \cdot y[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega}) * Y(e^{j\omega})$$
$$\frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(e^{j\theta}) \cdot Y(e^{j(\omega-\theta)}) d\theta$$

Conv. periódica

DUALIDAD

obs: * no hay dualidad en el sentido estricto

Dualidad de la DTFS

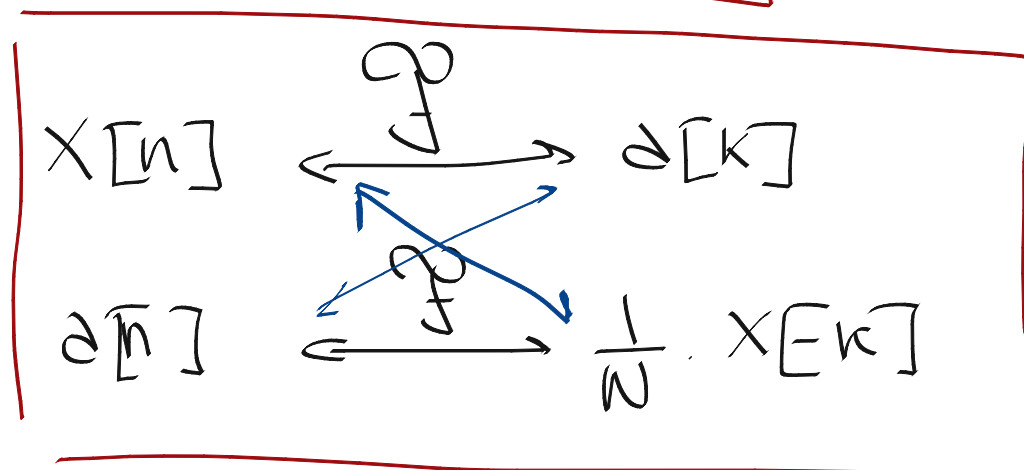


$$x[n] = \sum_{k=\langle N \rangle} a[k] e^{j k \omega_0 n} \Rightarrow x[n] = \sum_{k=\langle N \rangle} a[k] e^{-j k \omega_0 n}$$

$$\frac{1}{N} x[-n] = \frac{1}{N} \sum_{k=\langle N \rangle} \underbrace{a[k]}_{\substack{\text{f. periódica} \\ \text{f. conj.}}} \underbrace{e^{-j k \omega_0 n}}_{\substack{\text{exp. comp.} \\ \text{conj.}}}$$

→ es una ec. de análisis DTFS

$$a[k] \xleftrightarrow{\text{DTFS}} \frac{1}{N} x[-n]$$



obs: *

- cálculos
- propiedades (deducir)

Dualidad cruzada

* DTFT: $\rightarrow$ análisis: $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\omega n}$

$\rightarrow$ síntesis: $x[n] = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(e^{j\omega}) \cdot e^{j\omega n} d\omega$

* CTFS: $\rightarrow$ análisis: $a[k] = \frac{1}{T} \int_{\langle T \rangle} x(t) \cdot e^{-jk\omega_0 t} dt$

$\rightarrow$ síntesis: $x(t) = \sum_{k=-\infty}^{\infty} a[k] \cdot e^{jk\omega_0 t}$

obs: * ver resumen en slides

SIST. BASADOS EN EC. EN DIFERENCIAS

Ec. dif.

$\textcircled{N} \rightarrow$ orden: cant. retardos de la salida

$$\sum_{k=0}^N a_k \cdot y[n-k] = \sum_{l=0}^M b_l \cdot x[n-l]$$

$$y[n] = h[n] * x[n] \xleftrightarrow{\mathcal{F}} Y(e^{j\omega}) = X(e^{j\omega}) \cdot H(e^{j\omega}) \leftarrow \text{prop. conv.}$$
$$\sum_{k=0}^N a_k \cdot e^{-jk\omega} Y(e^{j\omega}) = \sum_{l=0}^M b_l \cdot e^{-jl\omega} X(e^{j\omega}) \leftarrow \text{prop. linealidad y resp. temporal.}$$
$$\sum_{k=0}^N a_k \cdot e^{-jk\omega} \underbrace{H(e^{j\omega}) X(e^{j\omega})}_{\text{no depende de } k} = \sum_{l=0}^M b_l \cdot e^{-jl\omega} X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{\sum_{l=0}^M b_l \cdot e^{-jl\omega}}{\sum_{k=0}^N a_k \cdot e^{-jk\omega}}$$

$\xrightarrow{\text{C.V. } z=e^{j\omega}}$

$$H(e^{j\omega}) = \frac{\sum_{l=0}^M b_l \cdot z^l}{\sum_{k=0}^N a_k \cdot z^k}$$

obs: * función racional en $e^{-j\omega}$

- * expresión general para cualquier sistema LTI discreto
- * Relación directa con la implementación: $\rightarrow$ software
 $\rightarrow$ electrónica
- * tema 2: sistemas en el tiempo (implementación)

La slides: ss-04_parte.2_slits.pdf
p. 65

Implementación en CPU

$$y[n] = \underbrace{\sum_{l=0}^M b_l x[n-l]}_{S_i} - \underbrace{\sum_{k=1}^N a_k y[n-k]}_{S_o}$$

for (int n=0; n<L; n++) {

```
float Si=0; Si  
  
for (int l=0; l<M; l++)  
    Si += b[l] * x[n-l];
```

n>l

↑

```
float So=0; So  
  
for (int k=0; k<N; k++)  
    So += a[k] * y[n-k];
```

n>k

↑

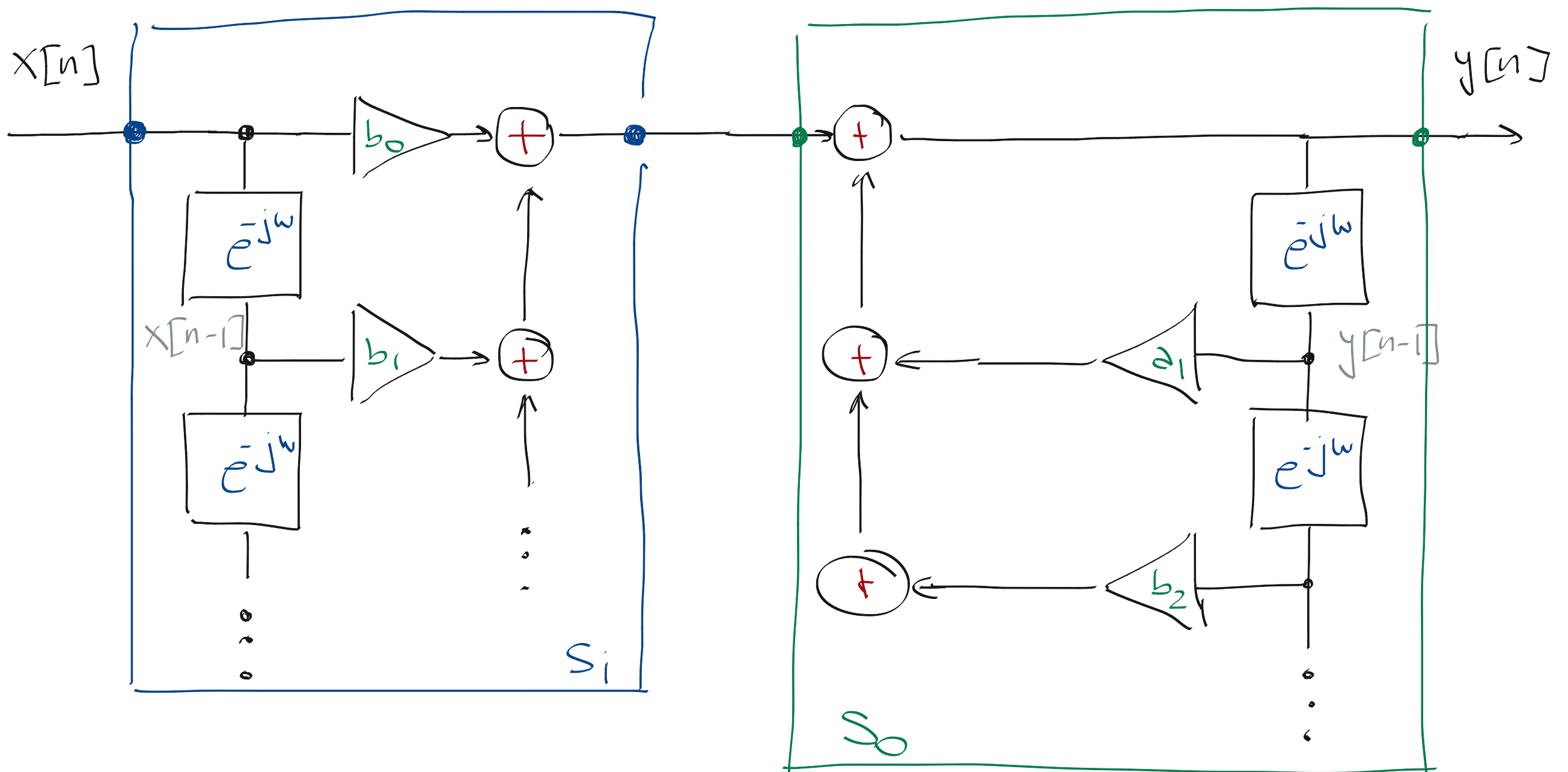
y[n] = S_i - S_o;

cond. de borde.

}

OBS: * es lo que está adentro del filter de Python

Implementación en electrónica



obs: * muy cerca de la implementación electrónica

* sólo 3 componentes: suma, multiplicador y memoria

Ejemplo 5.18 (p. 397)



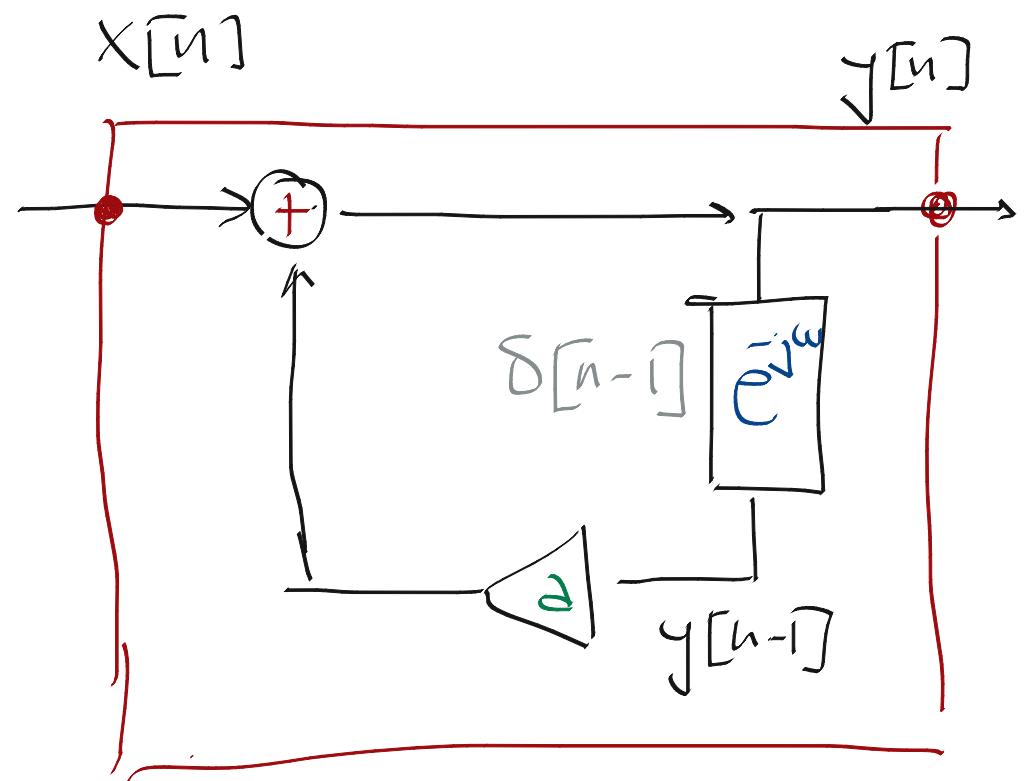
$$y[n] - a \cdot y[n-1] = x[n], |a| < 1 \leftarrow \text{ec. de diferencias}$$

a) Análisis en el tiempo (ejemplo IIR):

$$y[n] - a \cdot y[n-1] = x[n] \rightarrow h[n] = \delta[n] \cdot a^n$$

b) DTFS (ejemplo 3.17):

$$h[n] = \delta[n] \cdot a^n \xrightarrow{\text{indirecta}} H(e^{j\omega}) = \frac{1}{1 - a e^{-j\omega}}$$



c) DTFT (ej. 5.1):

$$x[n] = \delta[n] \cdot a^n \rightarrow X(e^{j\omega}) = \frac{1}{1 - a e^{-j\omega}}$$

$$y[n] - a y[n-1] = x[n]$$

$\uparrow$ $a_0 = 1$ $\uparrow$ $a_1 = -a$ $\uparrow$ $b_0 = 1$

$$H(e^{j\omega}) = \frac{1}{1 - a \cdot e^{-j\omega}}$$

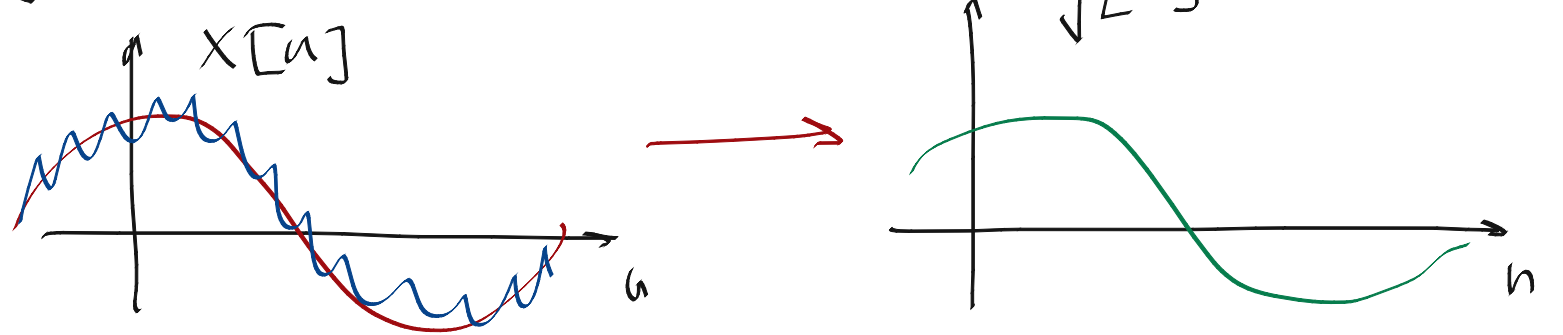
$$\xrightarrow{\mathcal{F}^{-1}} h[n] = \delta[n] \cdot a^n$$

obs.

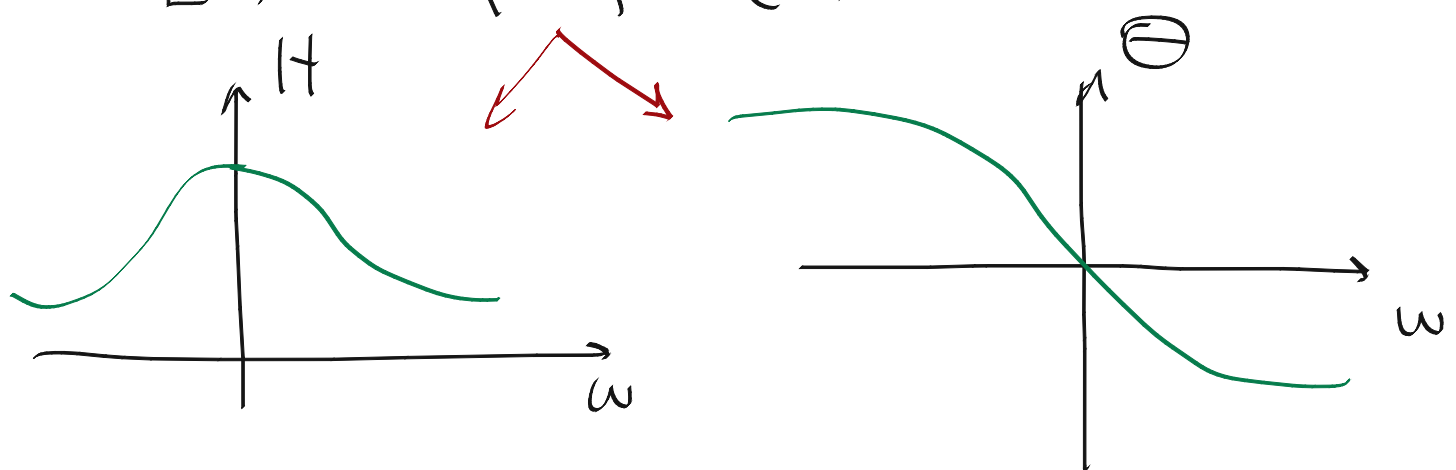
- * se implementa fácil en computadores
- * tanto en tiempo como en frecuencia.

Implementación (s.18)

e)
$$\left. \begin{array}{l} d = [1 \ -a] \\ b = [1] \end{array} \right\} \Rightarrow y = \text{signal.lfilter}(b, a, x)$$



f)
$$\left. \begin{array}{l} d = [1 \ -a] \\ b = [1] \end{array} \right\} \Rightarrow [H, \theta] = \text{freqz}(b, a)$$



abs: "formas de cálculo del espectro"

1) a) cálculo $H(e^{j\omega})$ 2) a) cálculo $H(e^{j\omega})$ 3) a) freqz

b) cálculo $|H|$
 $\angle H$

b) np.abs
 np.angle

c) plot

c) plot

Clase 25

Muestreo

Definición y representación en frecuencia

MUESTREO

Introducción: * hemos visto T.C. y T.D por separado
* cómo se juntan?

[slides hasta pág. 9]

Representación del muestreo en el dominio de la frecuencia

a) modulación

$$\left. \begin{aligned} X_S(t) &= X_C(t) \cdot s(t) \\ s(t) &= \sum_{n=-\infty}^{\infty} \delta(t - nT) \end{aligned} \right\} \text{modulación}$$

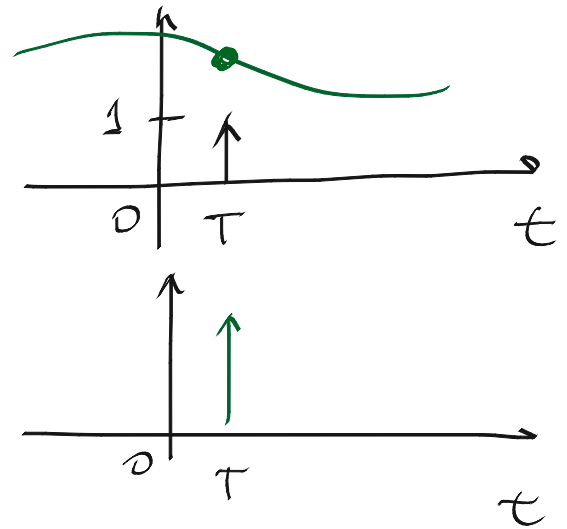
$$X_S(j\omega) = \frac{1}{2\pi} X_C(j\omega) * S(j\omega)$$

$$S(j\omega) = \left(\frac{2\pi}{T} \right) \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)$$

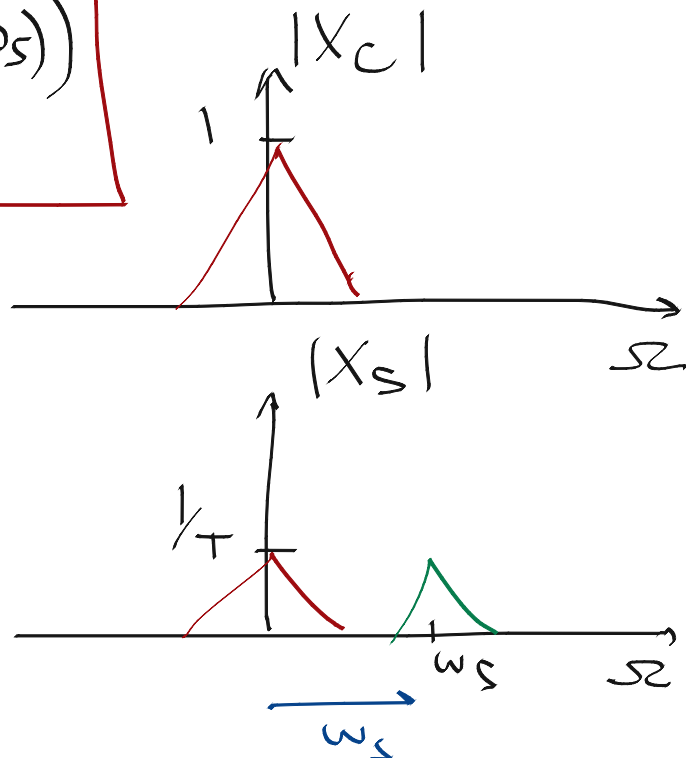
$$X_S(j\omega) = \frac{1}{2\pi} X_C(j\omega) * \left[\omega_s \cdot \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s) \right]$$

$$X_S(j\omega) = \left(\frac{\omega_s}{2\pi} \right) \sum_{n=-\infty}^{\infty} X_C(j\omega) * \delta(\omega - n\omega_s)$$

desplazo X_C en frecuencia



$$X_S(j\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X_C(j(\omega - n\omega_s))$$



Obs: * relación entrada-salida del modulador

* copias desplazadas y escaladas

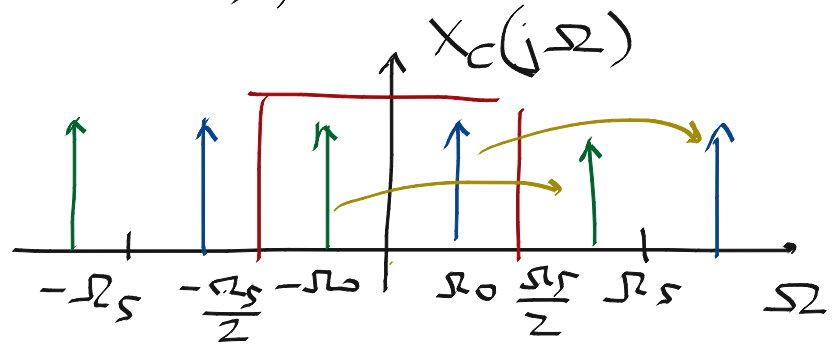
[volver a slide 13 a 17]

Ejemplo: $X_c(t) = \cos(\Omega_0 t)$, muestreo Ω_s

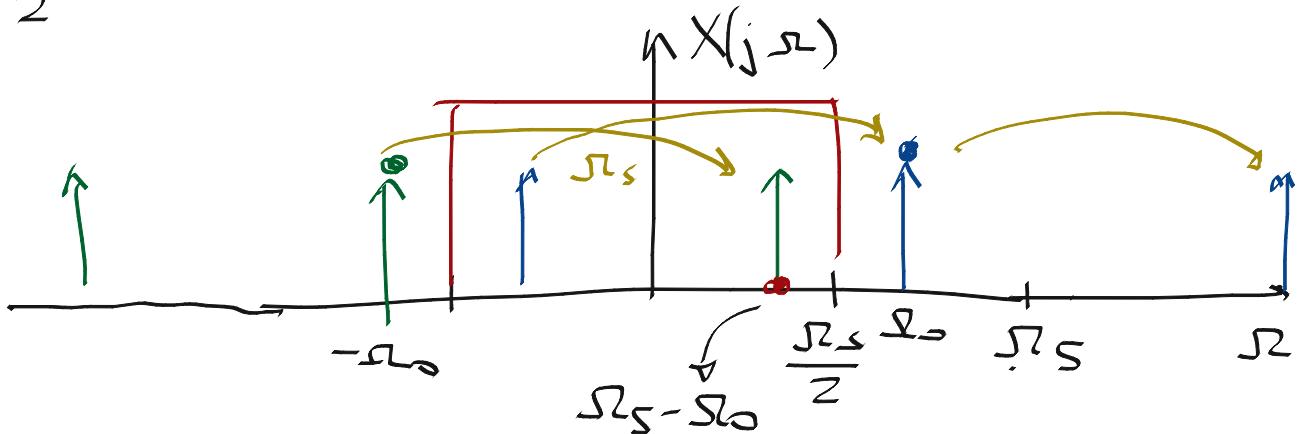
a) $\frac{\Omega_s}{2} > \Omega_0$

$$\Omega_0 < \Omega_c \leq \frac{\Omega_s}{2}$$

$$X_r(t) = \cos(\Omega_0 t)$$



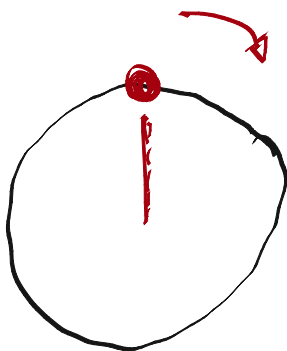
b) $\frac{\Omega_s}{2} < \Omega_0$



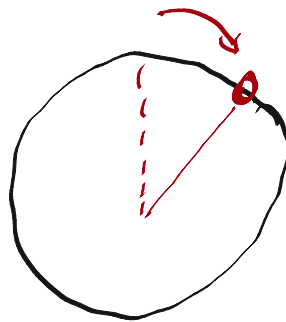
$$X_1(t) = \cos[(\Omega_s - \Omega_0)t]$$

[values of slide 18 y 20]

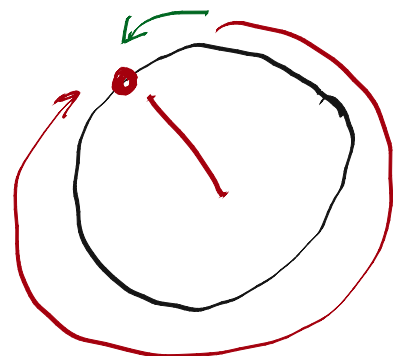
Obs:



$t=0s$



$t=1s$



$t=7s$

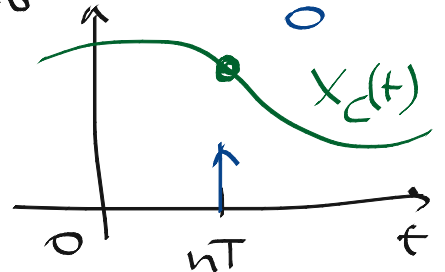
$$\Omega_0 = \frac{\pi}{4}$$

Representación en frecuencia

↳ expansión

$$X_s(t) = x_c(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} x_c(t) \cdot \delta(t - nT)$$

$$X_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT) \cdot \delta(t - nT) \quad *$$



$$X_s(j\omega) = \int_{-\infty}^{\infty} x_s(t) \cdot e^{-j\omega t} dt = \int_{-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} x_c(nT) \cdot \delta(t - nT) \right) e^{-j\omega t} dt$$

$$X_s(j\omega) = \sum_{n=-\infty}^{\infty} x_c(nT) \left[\int_{-\infty}^{\infty} \delta(t - nT) \cdot e^{-j\omega t} dt \right] \xrightarrow{t=nT} \sum_{n=-\infty}^{\infty} x_c(nT) \cdot e^{-jn\omega T}$$

$$X_s(j\omega) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-jn\omega T}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-jn\omega}$$

$$X(e^{j\omega}) = X_s(j\omega T)$$

$$\omega = \omega T$$

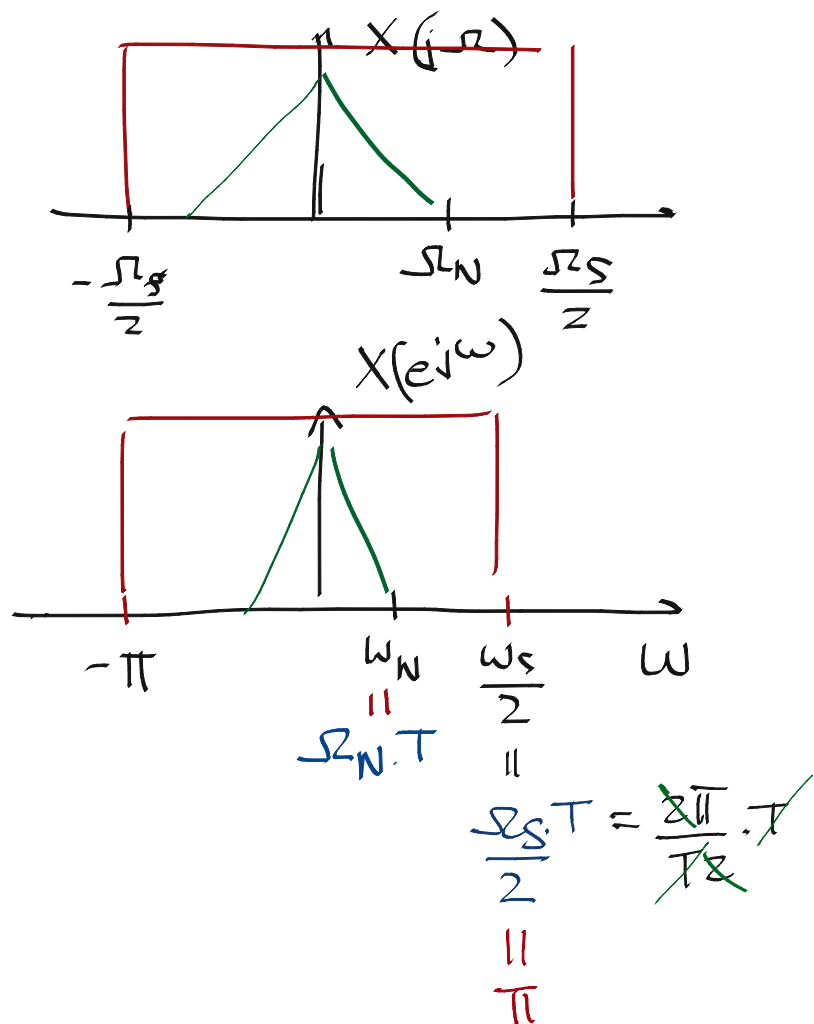
$$X(e^{j\omega}) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - k \cdot \omega_s\right)\right)$$

$$X(e^{j\omega}) = X_s\left(j\frac{\omega}{T}\right)$$

[cubre del slide 22 al 25]

obs. * normalización de
 $T \rightarrow 1 \left(\frac{t}{T}\right)$
 * equivale a expandir
 en frecuencia $\Omega \cdot T$

[mostrar junto con
 gráfico de slide 9]



Ejemplo: $f_0 = 2\text{kHz}$, $f_s = 6\text{kHz}$, $f_N = 4\text{kHz} \Rightarrow f_s > 2f_0$ ✓

$$T_s = \frac{1}{6000} \text{ s} \approx 166 \mu\text{s} \quad , \quad \Omega_s = 2\pi \cdot 6\text{k} \approx 36\text{k rad/s}$$

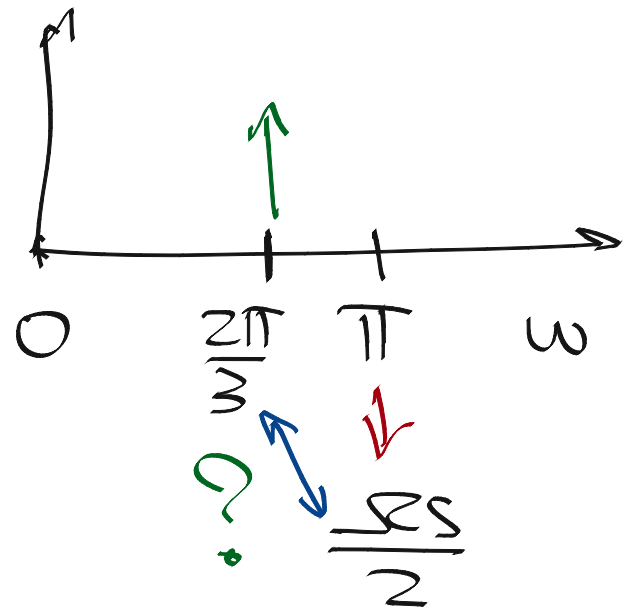
a) $x_c(t) = \cos(2\pi \cdot 2\text{k} \cdot t) = \cos(4000\pi t)$

$$x[n] = \cos(4000\pi \cdot nT_s) = \cos\left(\frac{4000\pi \cdot n}{6000}\right) = \cos\left[\frac{2}{3}\pi n\right]$$

b) $X_c(j\Omega) = \frac{1}{2} \left[\delta(\Omega - 4000\pi) + \delta(\Omega + 4000\pi) \right]$

$$X_s(j\Omega) = \frac{1}{T_s} \cdot X_c(j\Omega) \quad \text{en } \left[-\frac{\Omega_s}{2}, \frac{\Omega_s}{2}\right]$$

periódica Ω_s



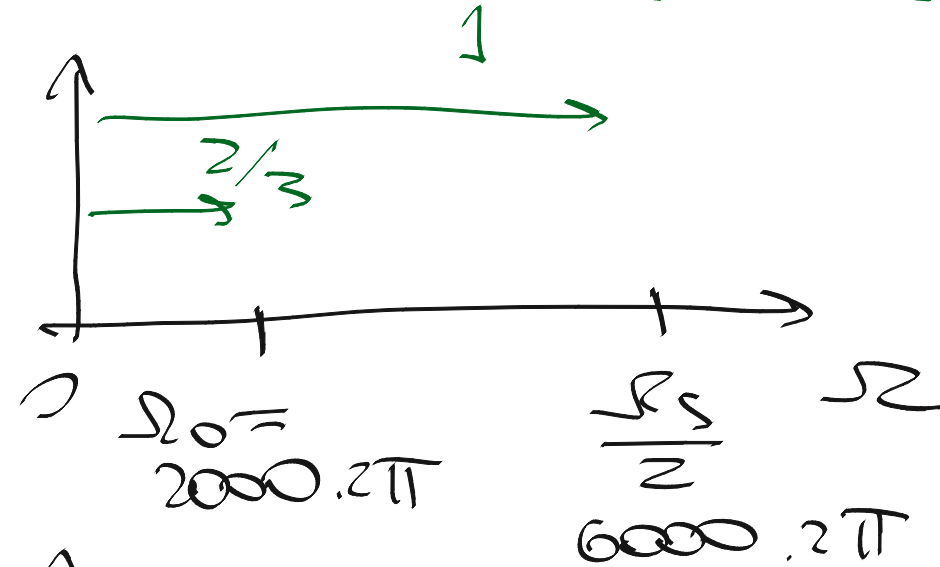
$$X(e^{j\omega}) = X_s\left(j\left(\frac{\omega}{T_s}\right)\right) = \frac{1}{T_s} X_c\left(j\frac{\omega}{T_s}\right)$$

$$X(e^{j\omega}) = \frac{1}{2T_s} \left[\delta\left(\frac{\omega}{T_s} - 4000\pi\right) + \delta\left(\frac{\omega}{T_s} + 4000\pi\right) \right]$$

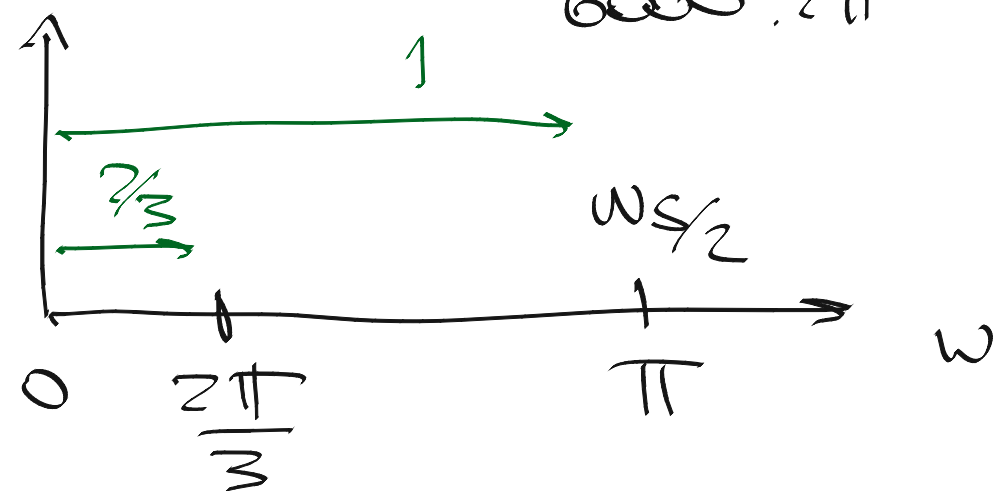
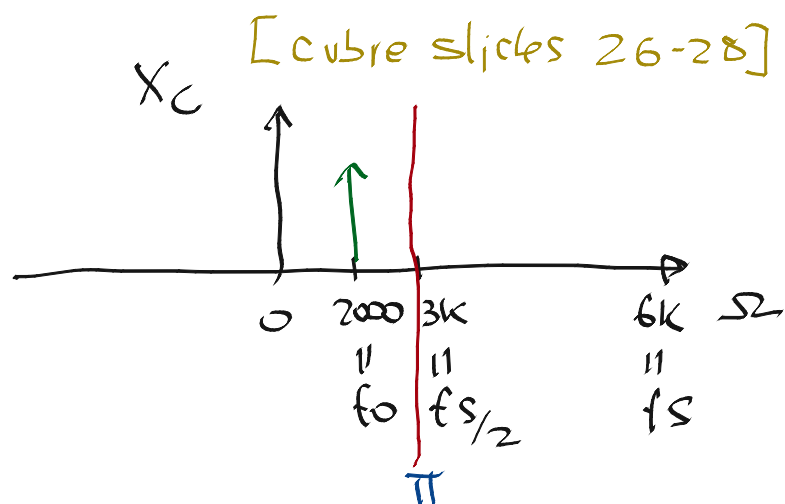
para saber la frec. del coseno.

$$\frac{\omega_0}{T_s} - 4000\pi = 0 \Rightarrow \omega_0 = 4000\pi \cdot T_s = \frac{4000\pi}{6000} = \boxed{\frac{2\pi}{3}}$$

$$x[n] = \cos\left[\frac{2\pi}{3}n\right]$$



$f_0 = 2000 \rightarrow \omega_0 = \frac{2\pi}{3}$



Clase 26

Muestreo

reconstrucción y procesamiento

RECONSTRUCCIÓN

[Empezar con slide 30, seguir con estas cuentas, son el slide 31]

$$x_r(t) = x_s(t) * h_r(t) \quad x_s(t) = \sum_{n=-\infty}^{\infty} x[n] \cdot \delta(t-nT)$$

$$x_r(t) = \left(\sum_{n=-\infty}^{\infty} x[n] \cdot \delta(t-nT) \right) * h_r(t)$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \cdot \left(\delta(t-nT) * h_r(t) \right) \quad \text{desplaza!!}$$

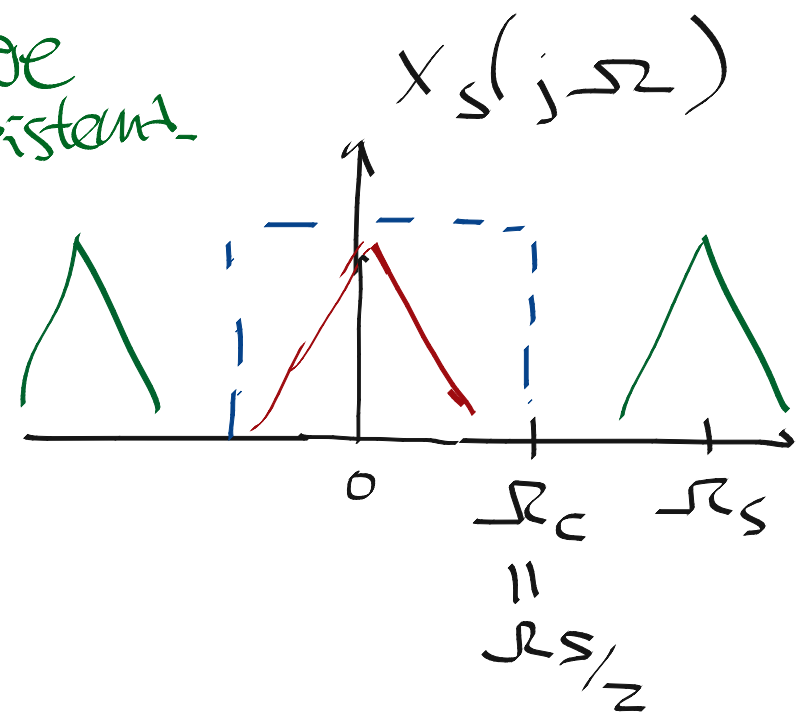
$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \cdot h_r(t-nT) \quad (s)$$

esto
sigue
en sistema

[reparar diagrama en slide 31]

[slide 32, empezar con figura]

$$H_r(j\omega) = \begin{cases} T & |\omega| < \omega_c = \frac{\omega_s}{2} = \frac{\pi}{T} \\ 0 & \text{e.o.c.} \end{cases}$$



$$h_r(t) = \frac{\text{sen}(\omega_c t)}{\omega_c t}$$

$$h_r(t) = \text{senc}(\omega_c t) \quad (6)$$

$$(s) \text{ y } (6) \Rightarrow x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \cdot \text{sen}(\omega_c (t-nT)) \quad *$$

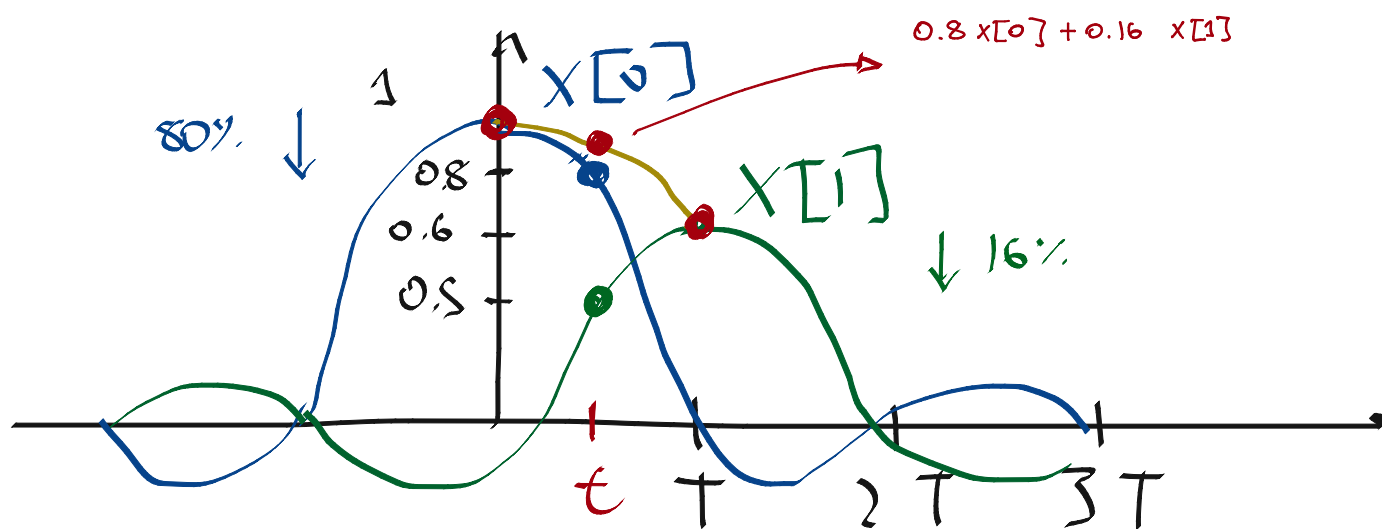
reconstrucción ideal.

en $t=nT$

[slide 33 - dibujo de $*$]

[slide 34 - conclusiones.]

$$x_i(t) = \sum_{n=-\infty}^{\infty} x[n] \cdot \frac{\sin(\pi(t-nT)/T)}{\pi(t-nT)/T}$$



obs: función interpolante

[slide 35 - retomando lo visto en la figura]

Conversor DC ideal

[slide 36]

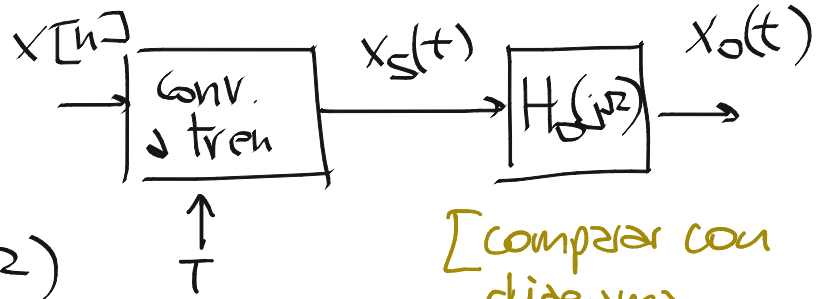
Procesamiento en tiempo discreto

slides 37 - 41

RECONSTRUCCIÓN REAL

Mantenedor de orden 0
(interpolador)

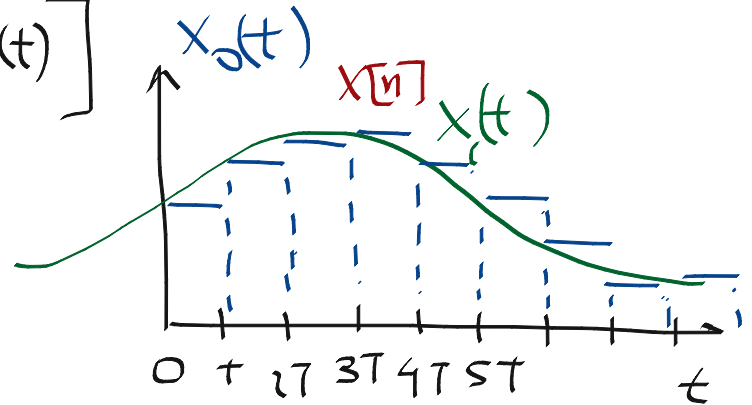
$$X_d(j\omega) = X_s(j\omega) \cdot H_d(j\omega)$$



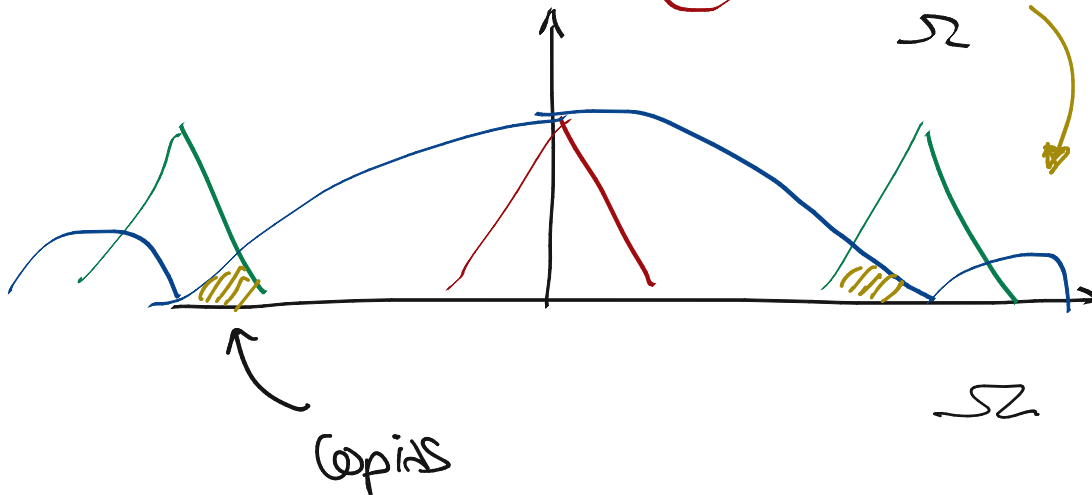
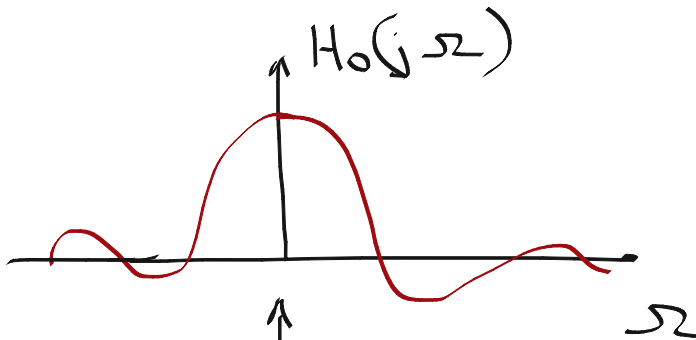
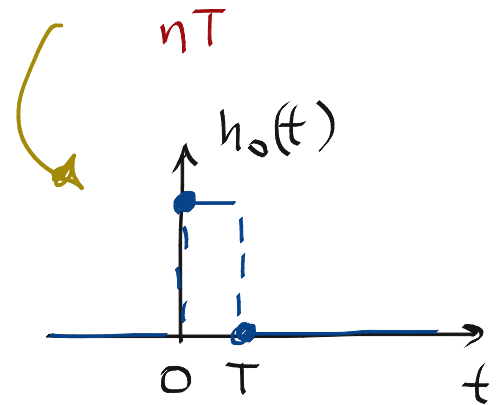
[comparar con diagrama en slide 36]

$$x_o(t) = \sum_{n=-\infty}^{\infty} x[n] \cdot [\delta(t-nT) * h_o(t)]$$

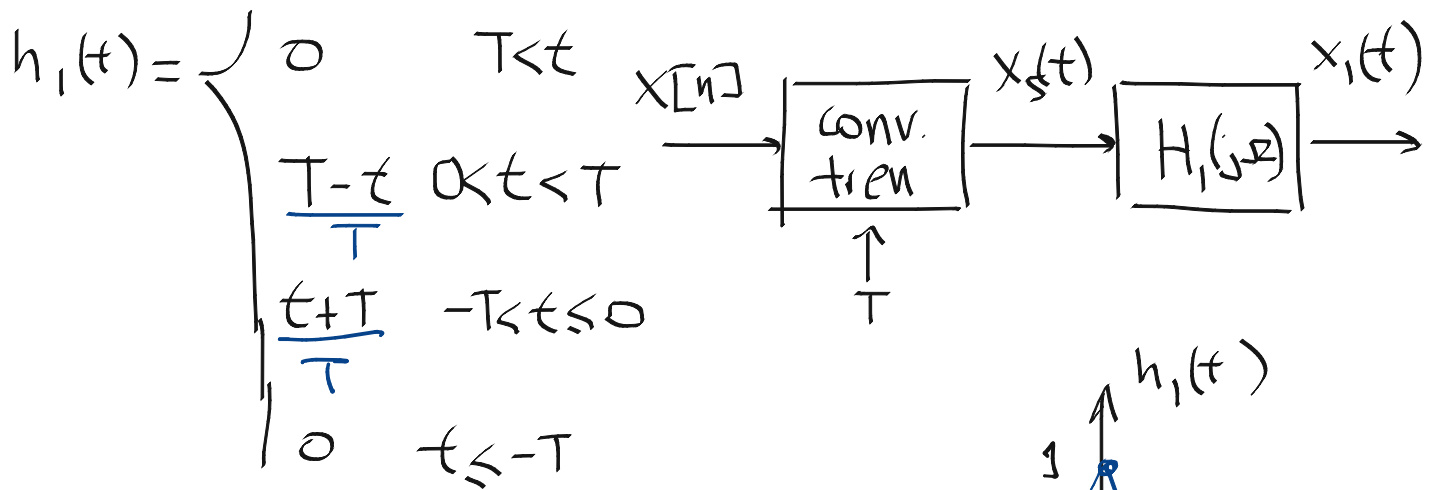
$$x_o(t) = \sum_{n=-\infty}^{\infty} x[n] \cdot h_o(t-nT)$$



$$|H_d(j\omega)| = T \cdot \text{sinc}\left(\frac{T}{2}\omega\right)$$



Interpolación lineal (orden 1)



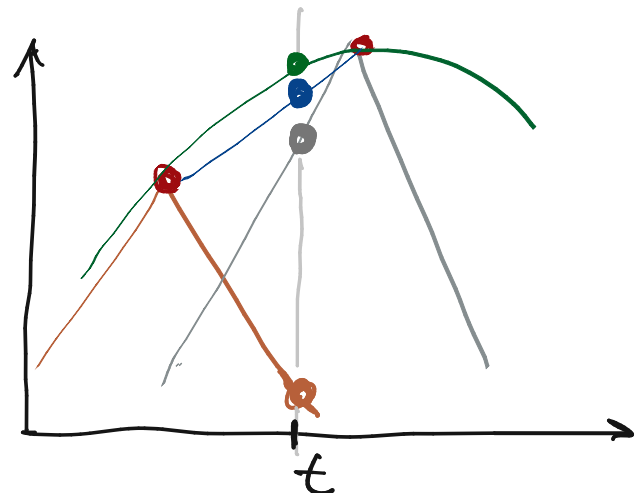
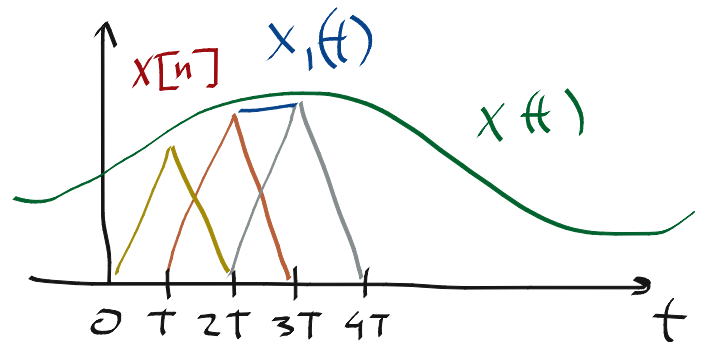
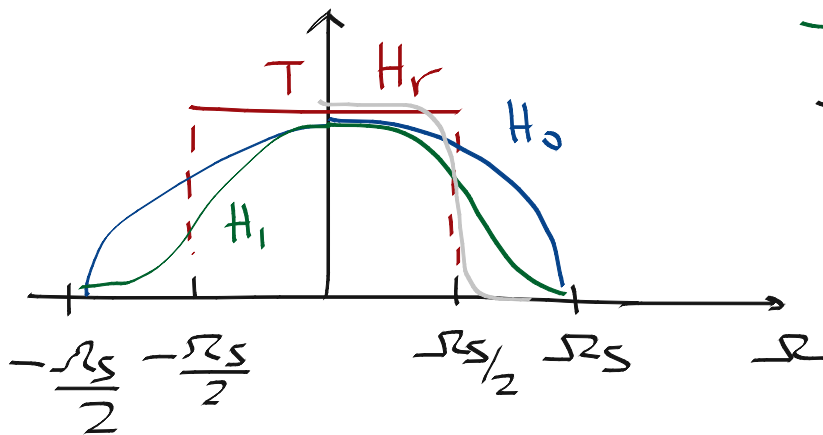
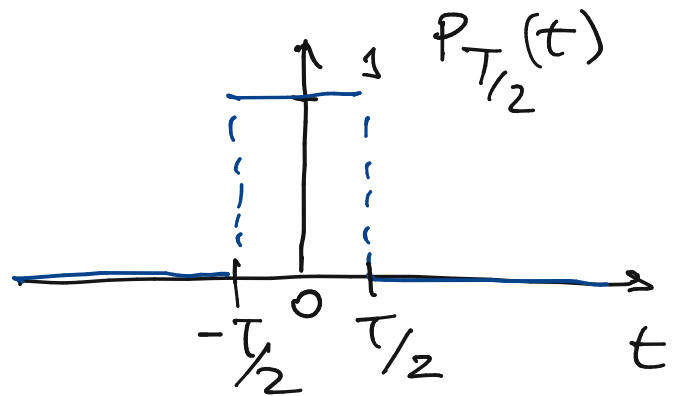
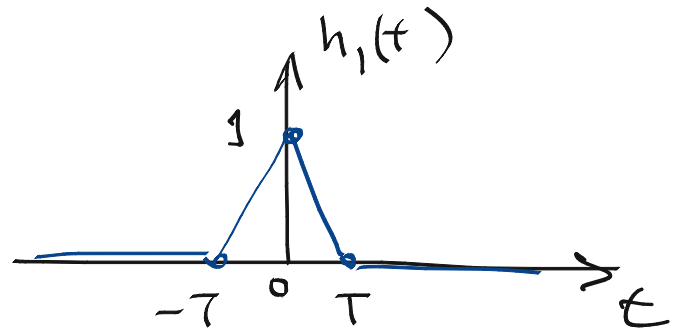
$$h_1(t) = p_{T/2}(t) * p_{T/2}(t)$$

$$\Rightarrow H_1(j\omega) = P_{T/2}^2(j\omega)$$

$$P_{T/2}(j\omega) = \text{sinc}\left(\frac{T}{2}\omega\right)$$

$$H_1(j\omega) = T \cdot \text{sinc}^2\left(\frac{T}{2}\omega\right)$$

↑
le agrego
después



Clase 27

Muestreo

Reconstrucción y cambio de frecuencia

Nota: Avanza en slides 42-45
Sigue en ppt de Aplicaciones

Nota: ver slides 46-47 junto con esta.

SUBMUESTREO (ANÁLISIS EN FRECUENCIA)

$$x_d[n] = x[Mn]$$

① $x_p[n] = x[n] \cdot p_M[n]$

$$p_M[n] = \sum_{k=-\infty}^{\infty} \delta[n - kM]$$

$$x_p[n] = \sum_{k=-\infty}^{\infty} x[kM] \cdot \delta[n - kM]$$

de ①

$$X_p(e^{j\omega}) = X(e^{j\omega}) * P_M(e^{j\omega})$$

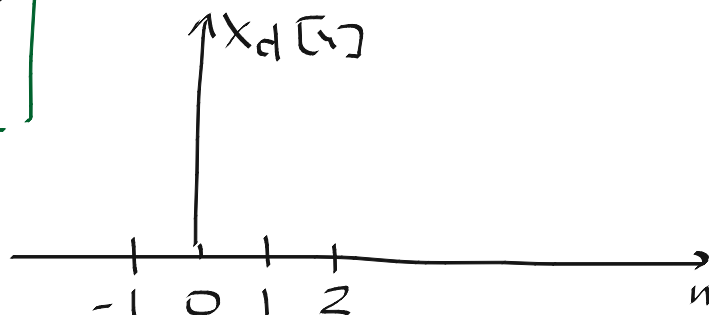
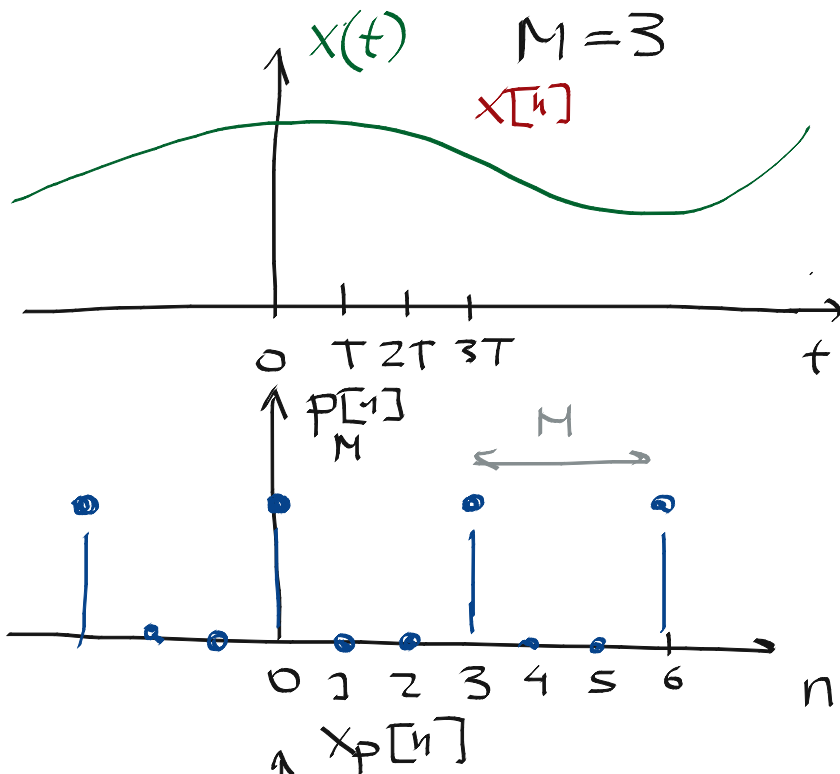
$$P_M(e^{j\omega}) = \frac{2\pi}{M} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$

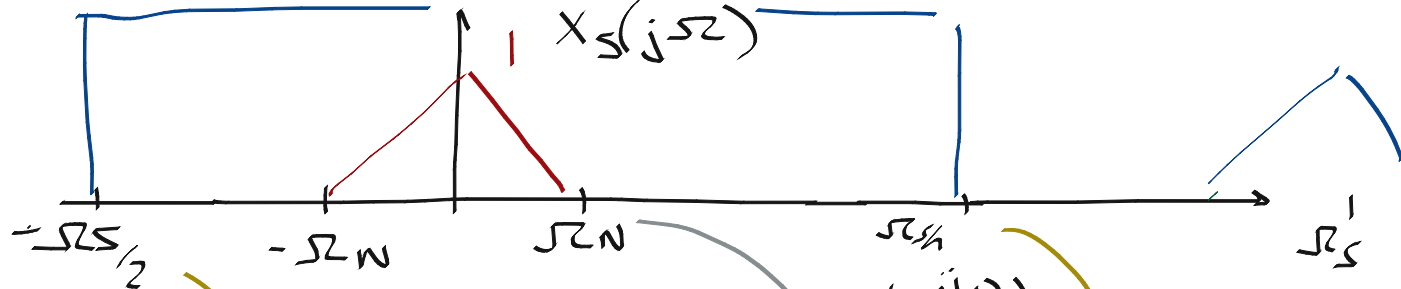
$\omega_s = \frac{2\pi}{M}$

②

$$X_p(e^{j\omega}) = \frac{1}{M} \sum_{k=-\infty}^{\infty} X(e^{j(\omega - k\omega_s)})$$

Obs: * 2 pareren copias!!



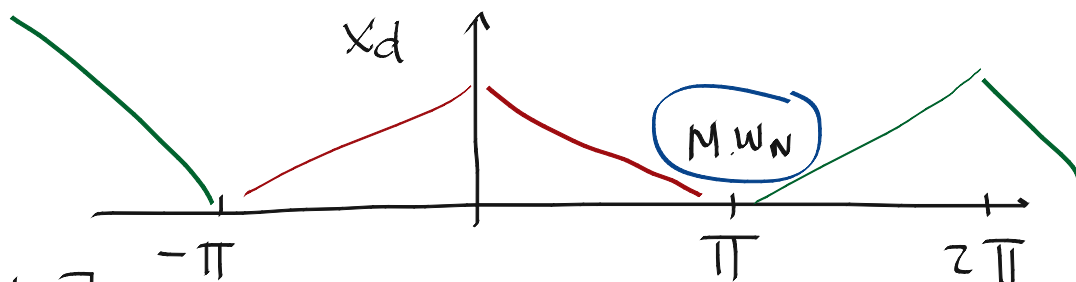
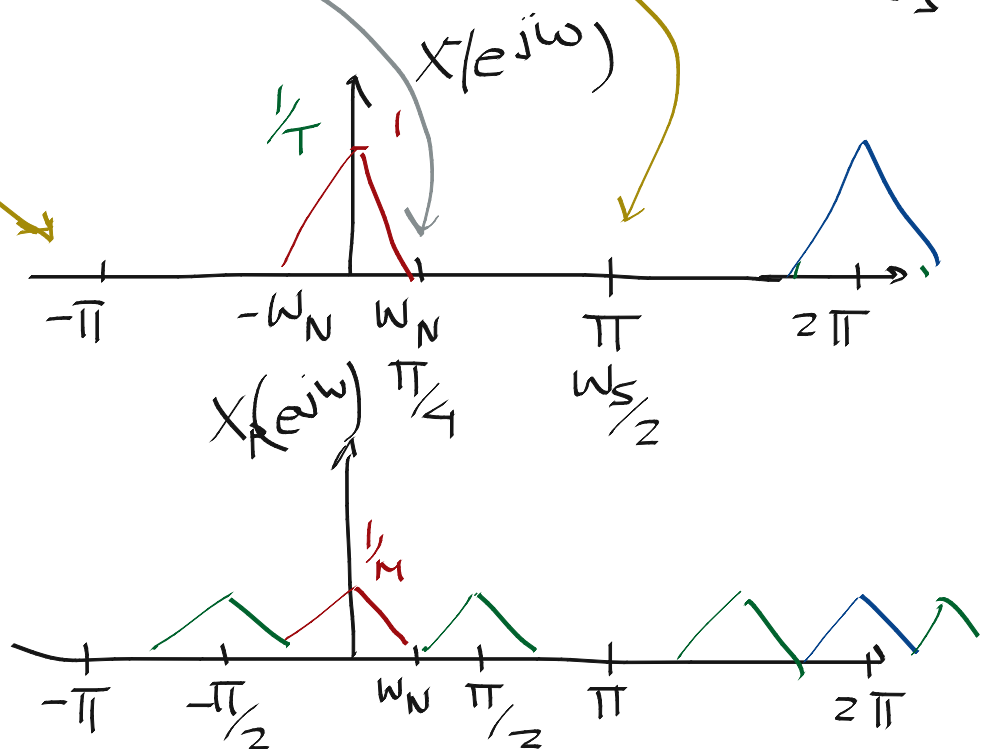


$$\omega_s = 8\omega_N$$

$$M=4$$

$$X(e^{j(\omega - k\omega'_s)})$$

$$\omega'_s = \frac{2\pi}{M} = \frac{\pi}{2}$$



$$(2) \quad X_d[n] = x[Mn]$$

$$X_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} X_d[n] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x[Mn] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x_p[\underline{Mn}] e^{-j\omega n} = \sum_{\substack{k=-\infty \\ \text{p}}}^{\infty} x_p[\underline{k}] e^{-j\omega \underline{\frac{k}{M}}}$$

$$\text{C.V.: } k = Mn \\ n = \frac{k}{M}$$

→ es una DTFT

$$X_d(e^{j\omega}) = X_p(e^{j\frac{\omega}{M}})$$

Note: sigue en slides 56-59

AUMENTO DE LA FRECUENCIA DE MUESTREO

[Slides 60-62]

$$x_e[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot \delta[n - kL]$$

$$X_e(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \underline{x_e[n]} \cdot e^{-j\omega n}$$

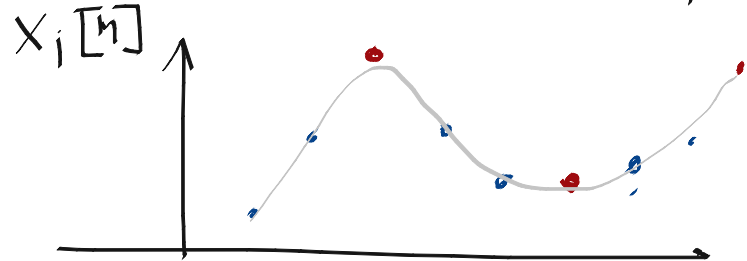
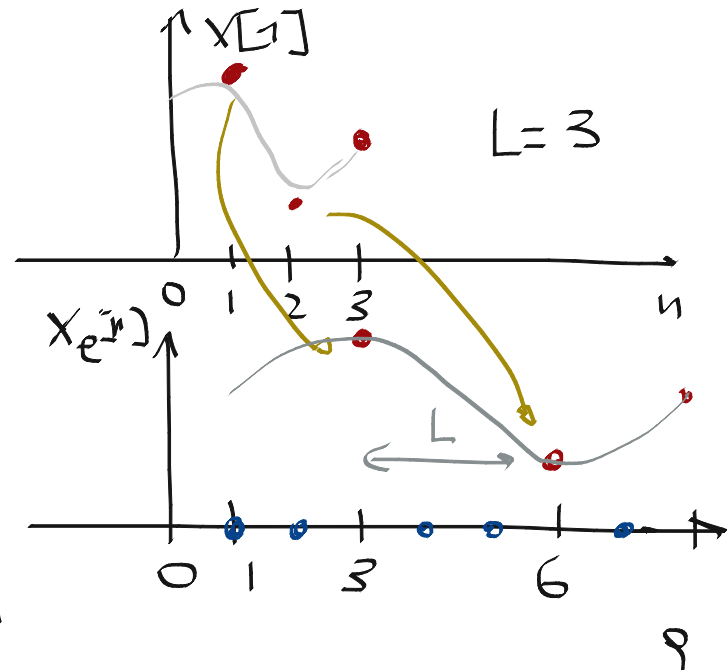
$$= \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x[k] \cdot \delta[n - kL] \right) e^{-j\omega n}$$

$$= \sum_{k=-\infty}^{\infty} x[k] \cdot \sum_{n=-\infty}^{\infty} \underline{\delta[n - kL]} \cdot e^{-j\omega n}$$

$n = kL$

$$= \sum_{k=-\infty}^{\infty} \underline{x[k]} \cdot e^{-j\omega kL}$$

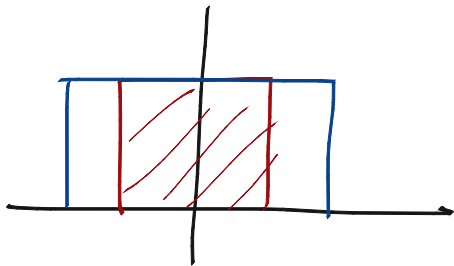
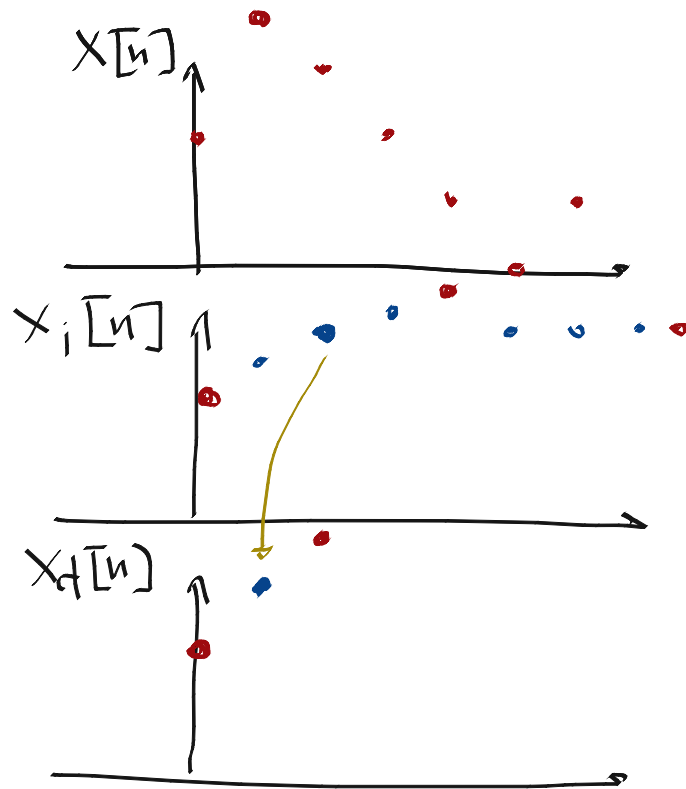
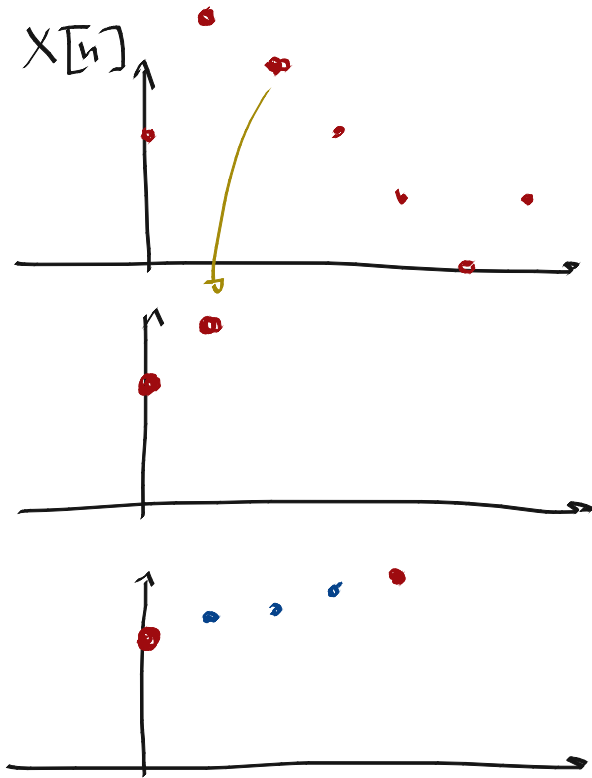
$$X_e(e^{j\omega}) = X(e^{j\omega L})$$



↓
después muestrear
6z

CAMBIO ARBITRARIO DE FS

$$L=4, M=2$$



Clase 28

Transformada de Laplace

Definición

RESUMEN DEL CURSO

1) Introducción y señales

1.1) Motivation

1.2) Resumen del curso

13) Señales: def., propiedades, clasificar $\rightarrow$ "lindas" ∞

T.C. $\Leftrightarrow$ T.P.

2) Systemas:

2.1) def., propiedades, clasifica

Sistemas LIT → "lindos" LIT

2.3) Análisis temporal:

- modelado
- resp. impulso
- convolución

3) Análisis espectral periódico:

→ frequências drônicas

→ CT FS, DT FS

4) Análisis espectral (generales):

→ ingredients: frequencies generales

→ CTFT, DTFT

5) Muestreo: $\rightarrow$ relación entre T.C. y T.D.

6) Analisis general :

→ T. Laplace y ~~T. Z.~~

→ Análisis "completo" de un sistema

→ "Design" de sistemas

L_{intro}

→ TDS

T. LAPLACE

Introducción

- * Fourier es útil para analizar SLITs:
 - base de funciones senoidales: exp. imaginarios puros
 - autovectores de los SLIT
 - descomposición y superposición
 - funciones de energía finita $\times$

$$s = j\omega$$

- * Si generalizo a exp. complejos generales e^{st}
 - muchas de estas propiedades se mantienen.
 - Análisis de sistemas "inestables" ✓
 - Herramienta de cálculo

Idea: $x(t) = e^{st}$, $s \in \mathbb{C}$ } $\Rightarrow y(t) = H(s) \cdot e^{st}$
sist. LIT
 $y(t) = \int h(\tau) x(t-\tau) d\tau$

$\downarrow$ vector propio
 $\downarrow$ valor propio

$$H(s) = \int_{-\infty}^{\infty} h(t) \cdot e^{-st} dt \xrightarrow{s=j\omega} H(j\omega) = \int_{-\infty}^{\infty} h(t) \cdot e^{-j\omega t} dt$$

$\downarrow$ es una T.L.

$\downarrow$ no es una CTFS $\times$
 $\downarrow$ si es una CTFT ✓

Def: $X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} dt$

$\left[\begin{array}{l} \text{ec. análisis} \\ \text{T. bilateral de Laplace} \\ \text{unilateral} \end{array} \right]$

Obs: * sigue siendo una proyección

$$* X(s) = \langle x(t), e^{st} \rangle$$

Notación:

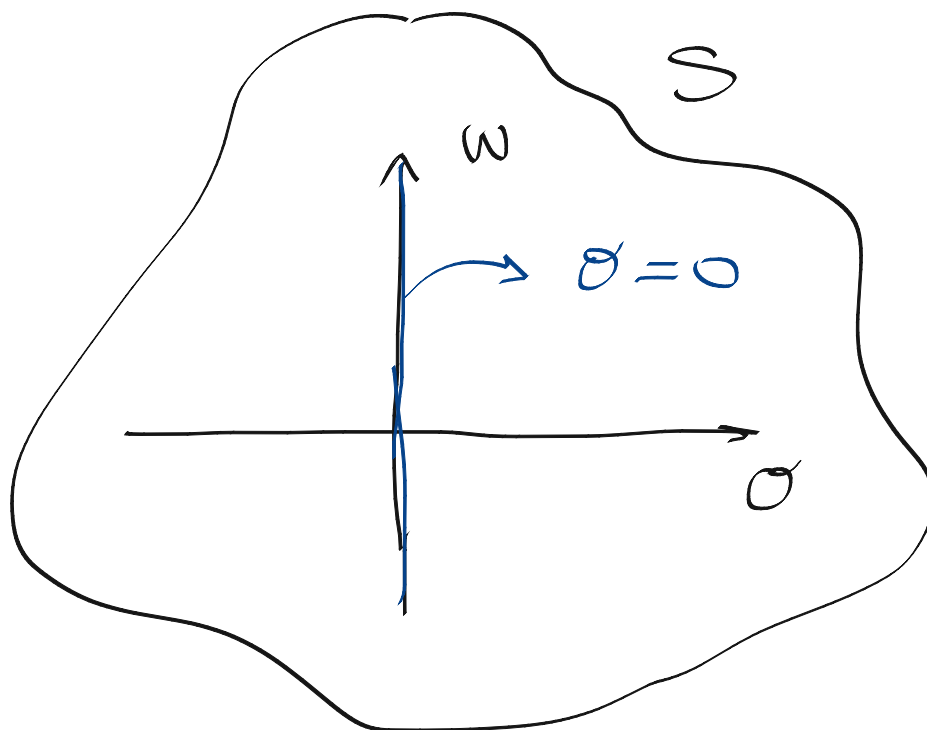
$$X(j\omega) : \mathbb{R} \rightarrow \mathcal{C}(\mathbb{R}^2)$$

$$* S = \sigma + j\omega$$

$$* x(t) \xleftrightarrow{\mathcal{L}} X(s)$$

$$* X(s) = \mathcal{L} \{ x(t) \}$$

$$* X(s) : \mathcal{C} \rightarrow \mathcal{C}$$



obs: $* X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt \Rightarrow \boxed{X(j\omega) = X(s) \Big|_{s=j\omega}}$

$$* X(s) = X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-(\sigma + j\omega)t} dt = \int_{-\infty}^{\infty} \underbrace{\left(x(t) \cdot e^{-\sigma t} \right)}_{f(t)} \cdot e^{-j\omega t} dt$$

CTFT

$$\boxed{X(s) = \mathcal{F} \{ x(t) \cdot e^{-\sigma t} \}}$$

* Hay una relación estrecha entre $\mathcal{F}$ y $\mathcal{L}$
↳ en particular en la convergencia.

Ejemplo 9.1: $x(t) = u(t) \cdot e^{-at}$, $a \in \mathbb{R}$

del ej. 4.1 $\rightarrow X(j\omega) = \int_{-\infty}^{\infty} u(t) \cdot e^{-at} \cdot e^{-j\omega t} dt = \int_0^{\infty} e^{-(a+j\omega)t} dt$

$$X(j\omega) = \frac{1}{a+j\omega}, \text{ converge si } a > 0$$

$$X(s) = \int_{-\infty}^{\infty} u(t) \cdot e^{-at} \cdot e^{-st} dt = \int_0^{\infty} e^{-(a+s)t} dt = \int_0^{\infty} e^{-(a+\sigma)t} e^{-j\omega t} dt$$

$$X(s) = \frac{1}{a+s}, \text{ converge si } \sigma > -a \text{ o } \operatorname{Re}\{s\} > -a$$

$$u(t) \cdot e^{-at} \xleftrightarrow{\mathcal{L}} \frac{1}{a+s}, \operatorname{Re}\{s\} > -a$$

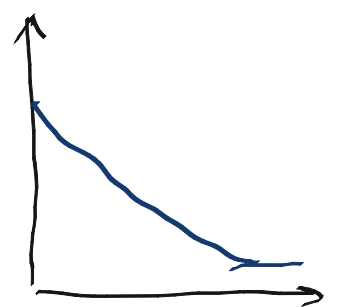
obs: * si $s = j\omega \Rightarrow$ convergen!!

* si $a = 0$, $u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s} \rightarrow$ integrador!!

obs:

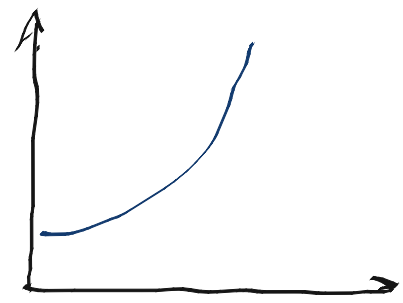
* si $a > 0$, converge en $\sigma = 0 \Rightarrow \exists \mathcal{L} \nexists \mathcal{F}$

* y se calcula como $X(0+j\omega) = \frac{1}{a+j\omega}$



* si $a < 0$, $\exists \mathcal{L}$ pero $\nexists \mathcal{F}$

$\rightarrow$ caso nuevo que antes no podíamos estudiar!!



Ejemplo 9.2: $x(t) = -u(-t) \cdot e^{-at}$

$$X(s) = \int_{-\infty}^0 -u(t) \cdot e^{-at} \cdot e^{-st} dt$$

$$X(s) = - \int_{-\infty}^0 e^{-(a+s)t} dt = \left[\frac{e^{-(a+s)t}}{-(a+s)} \right]_{-\infty}^0$$

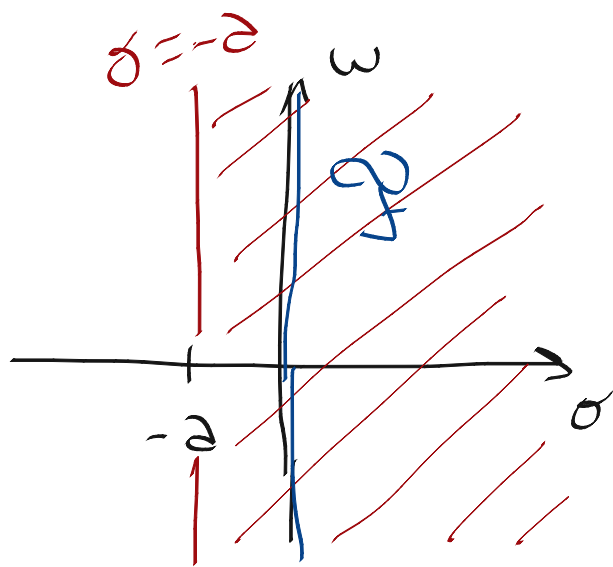
$$X(s) = \frac{1}{a+s}$$

$$-(a+s) > 0 \Rightarrow \sigma < -a \Rightarrow \text{Re}\{s\} < -a$$

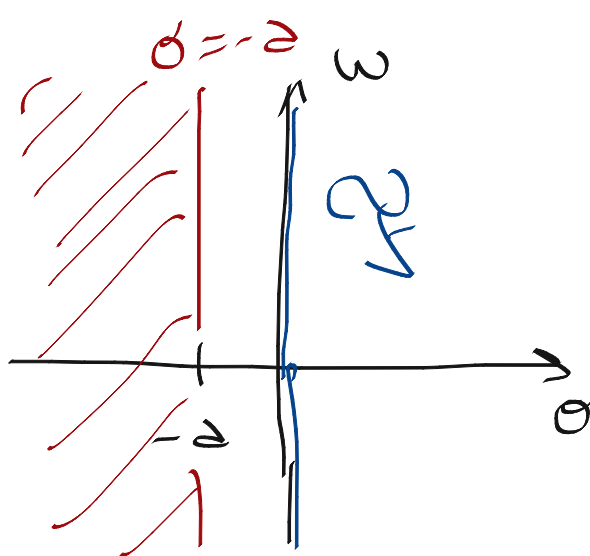
misma expresión algebraica
con diferente dominio.

Región de
Convergencia
(ROC)

$a > 0$



Ej. 9.1



Ej. 9.2

Def: "ROC"

$$R = \{s \in \mathbb{C} / \exists X(s)\}$$

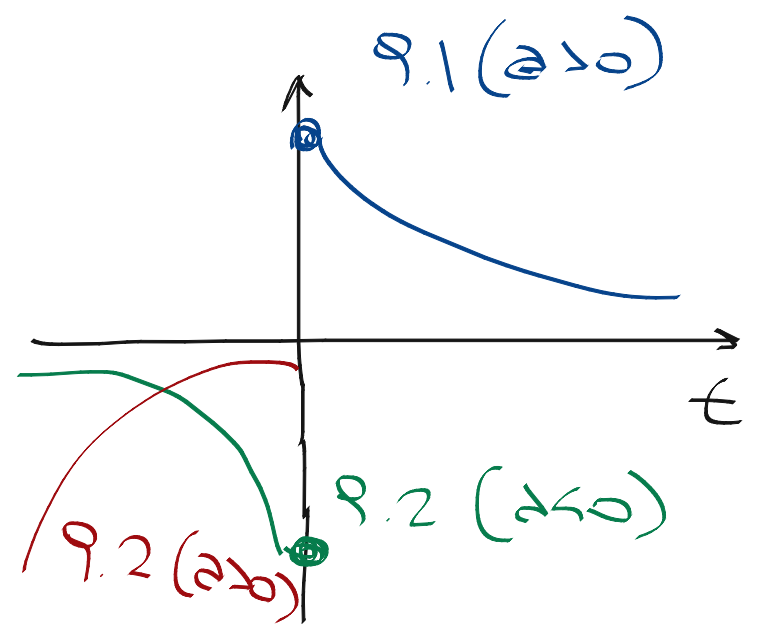
* Lugar geométrico

Obs:

* Expresión algebraica + ROC

* si el eje $\sigma=0$ (eje ω) está incluido en la ROC $\Rightarrow \exists \mathcal{F}$

* Dado $a \neq 0$, $\nexists \mathcal{F}$ en ambas a la vez



Ejemplo 9.3: $X(t) = 3 \cdot u(t) \cdot e^{-2t} - 2 \cdot u(t) \cdot e^{-t}$

$$X(s) = \int_{-\infty}^{\infty} \underbrace{(3u(t)e^{-2t} - 2u(t)e^{-t})}_{x(t)} e^{-st} dt$$

$$X(s) = 3 \int_{-\infty}^{\infty} u(t) e^{-2t-st} dt - 2 \int_{-\infty}^{\infty} u(t) e^{-t-st} dt$$

$\underbrace{\int_{-\infty}^{\infty} u(t) e^{-2t-st} dt}_{\mathcal{L}\{u(t) \cdot e^{-2t}\}}$
 $\underbrace{\int_{-\infty}^{\infty} u(t) e^{-t-st} dt}_{\mathcal{L}\{u(t) \cdot e^{-t}\}}$

linealidad del integral
 $\downarrow$
 pista: linealidad en props.

$$X(s) = 3 \cdot \underbrace{\mathcal{L}\{u(t) \cdot e^{-2t}\}}_A - 2 \cdot \underbrace{\mathcal{L}\{u(t) \cdot e^{-t}\}}_B$$

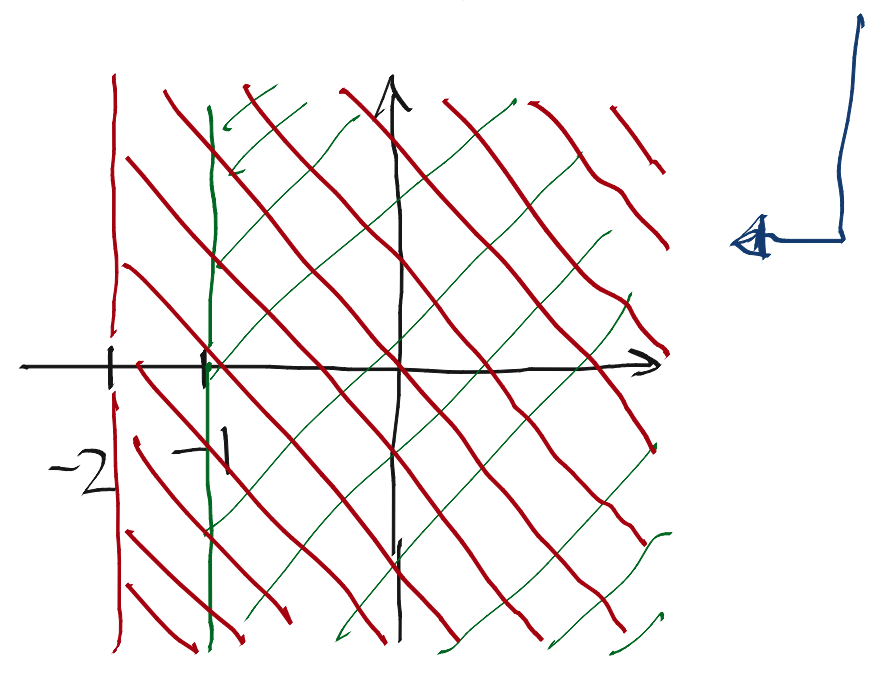
$$X(s) = 3 \cdot \frac{1}{s+2} \text{ (A)} - 2 \cdot \frac{1}{s+1} \text{ (B)}$$

obs: * aparecen funciones racionales

ROC_A: $\text{Re}\{s\} > -2$

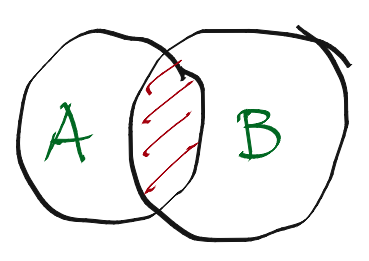
ROC_B: $\text{Re}\{s\} > -1$

$\Rightarrow \text{ROC} = \text{ROC}_A \cap \text{ROC}_B \Rightarrow \boxed{\text{ROC: } \text{Re}\{s\} > -1}$



obs:

* se combinan las ROCs



* La expresión final es una función racional de 2do grado (abajo)

$$X(s) = \frac{3s+3-2s-4}{(s+2)(s+1)} = \boxed{\frac{s-1}{(s+2)(s+1)}} \rightarrow \text{func. racional}$$

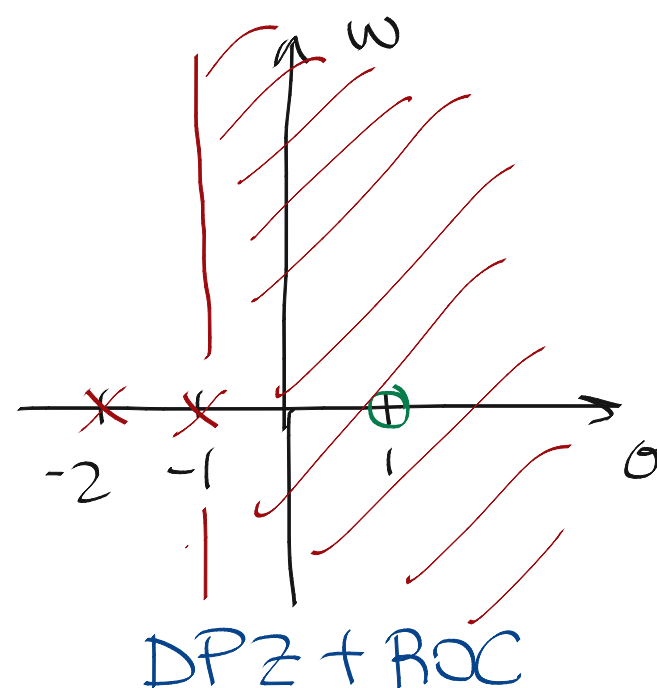
Diagrama de polos y ceros (DPZ)

* func. racional: $X(s) = \frac{N(s)}{D(s)} \rightarrow$ los polinomios quedan determinados por sus raíces.

Def: es una rep. gráfica de las raíces de la fun. racional

Notación

- * P_i : Polos $\times$ } singularidad
- * Z_i : Ceros $\circ$ }
- * Singularidades múltiples
 - $\rightarrow$ polos: $\times$
 - $\rightarrow$ ceros: $\odot$



obs:

* los polos quedan fuera de la ROC (porque $\nexists X(p_i)$)

* polos y ceros en infinito: $\lim_{s \rightarrow \infty} X(s) = \begin{cases} 0 & \text{cero} \\ \infty & \text{polo} \end{cases}$

* ejemplos: $\rightarrow \lim_{s \rightarrow \infty} X(s) = \infty \Rightarrow$ polo en infinito

$\rightarrow \lim_{s \rightarrow \infty} X(s) = 0 \Rightarrow$ cero en infinito

* si $\text{gr } N(s) > \text{gr } D(s) \Rightarrow$ polo en ∞ | cero en 0?

* si $\text{gr } N(s) < \text{gr } D(s) \Rightarrow$ cero en ∞ | polo en 0?

$$X(s) = \frac{b_N \cdot z^N + \dots + b_1 z + b_0}{a_M \cdot z^M + \dots + a_1 z + a_0}$$

Ejemplo 9.5 $x(t) = \underbrace{\delta(t)}_{(A)} - \frac{4}{3} \underbrace{u(t)}_{(B)} e^{-t} + \frac{1}{3} \underbrace{u(t)}_{(C)} e^{2t}$

9.1 9.1
↳ b.s. tengo.

(A) $\mathcal{L}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) \cdot e^{-st} dt = e^{-st} \Big|_{t=0} = \boxed{1}$ ROC: $\mathbb{C}$

$X(s) = 1 - \frac{4}{3} \cdot \frac{1}{s+1} + \frac{1}{3} \cdot \frac{1}{s-2}$

$X(s) = \frac{3(s+1)(s-2) - 4(s-2) + (s+1)}{3(s+1)(s-2)} = \frac{3s^2 - 3s - 6 - 4s + 8 + s + 1}{3(s+1)(s-2)}$

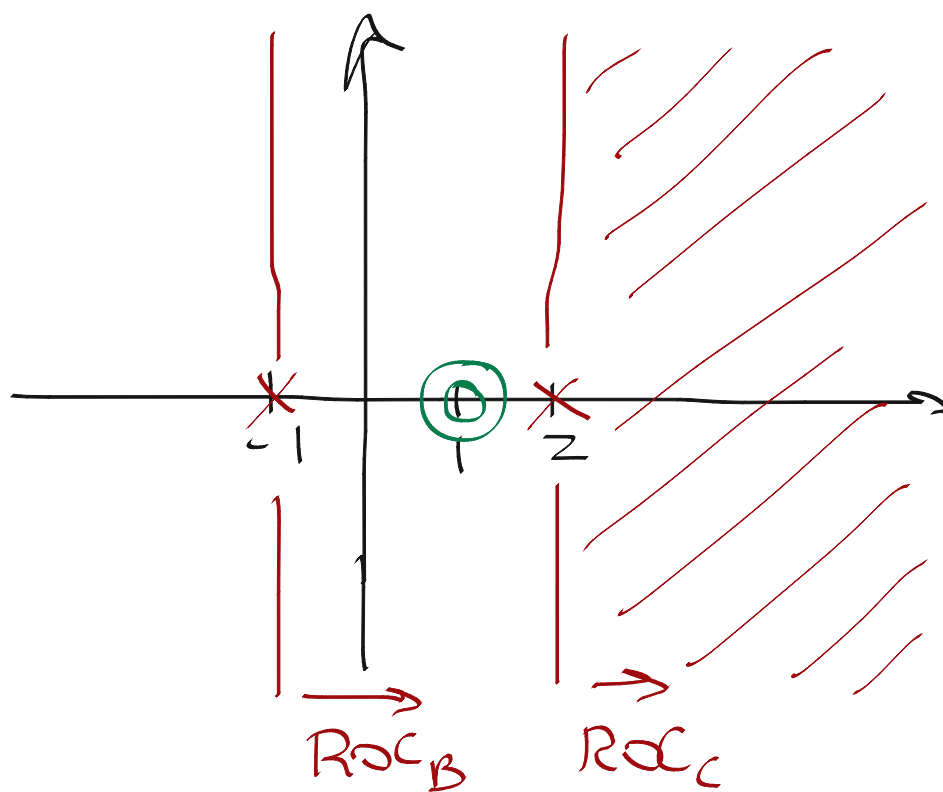
$X(s) = \frac{\cancel{3}s^2 - \cancel{6}s + \cancel{3}}{\cancel{3}(s+1)(s-2)} = \frac{s^2 - 2s + 1}{(s+1)(s-2)} = \boxed{\frac{(s-1)^2}{(s+1)(s-2)}}$

ROC_A: $\mathbb{C}$

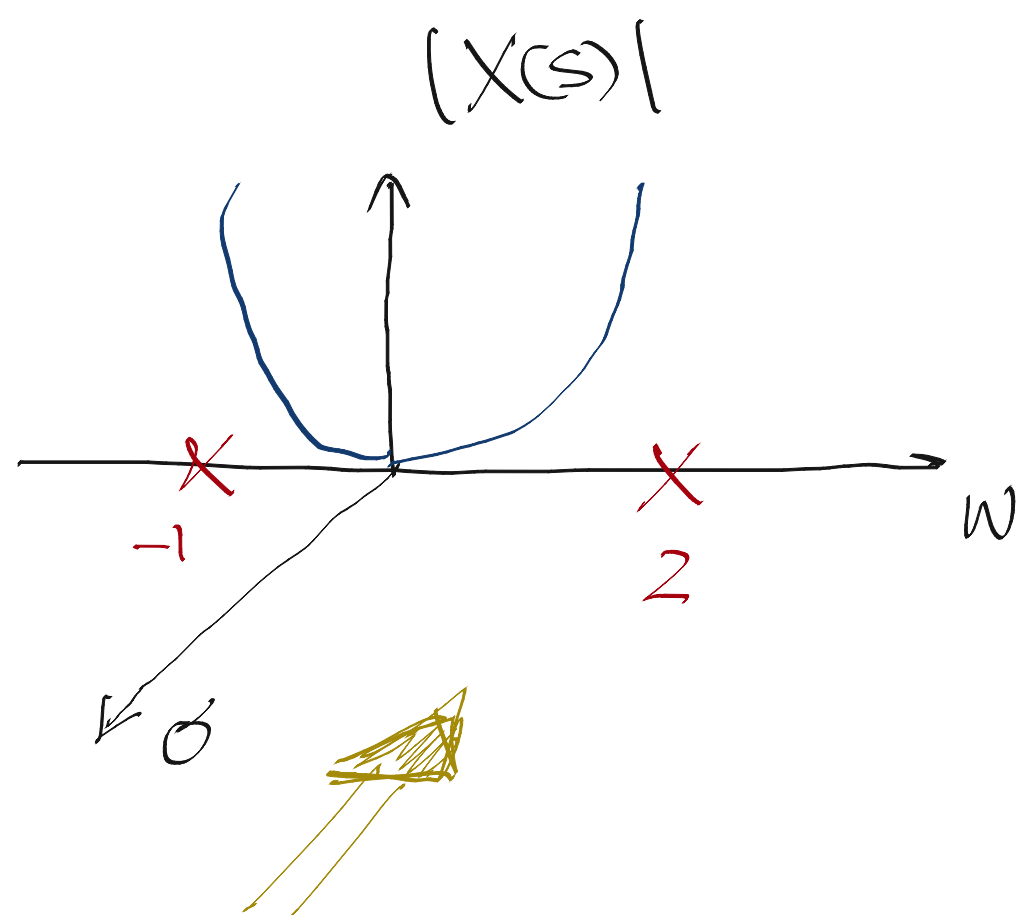
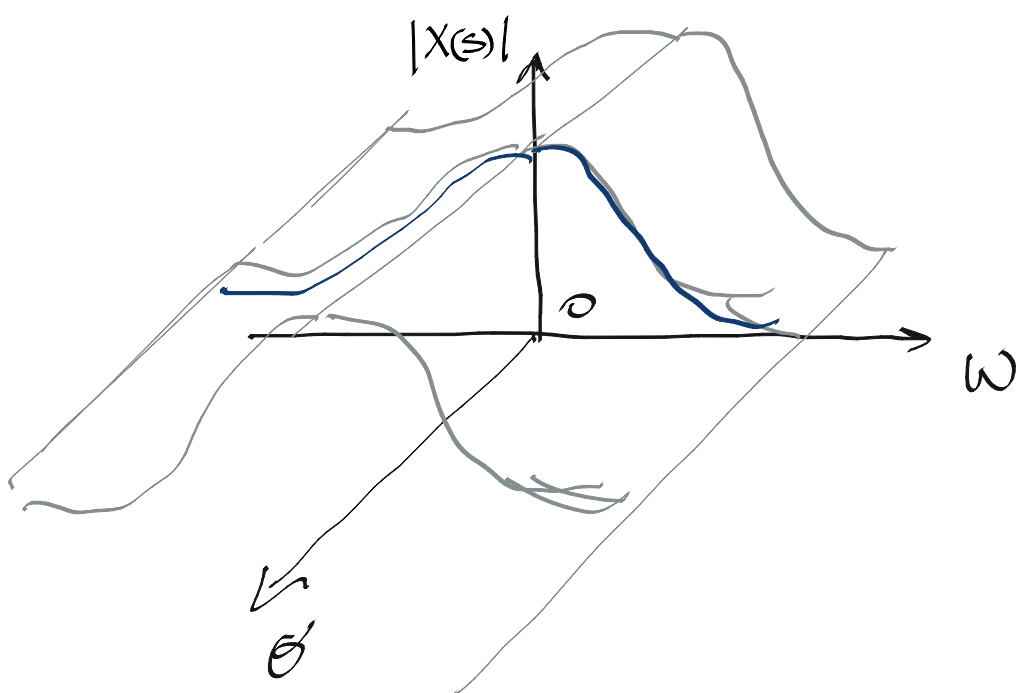
ROC_B: $\text{Re}\{s\} > -1$

ROC_C: $\text{Re}\{s\} > 2$

ROC: $\text{Re}\{s\} > 2$



obs: $X(s): \mathbb{C} \rightarrow \mathbb{C}$



Clase 29

Transformada de Laplace

Región de convergencia

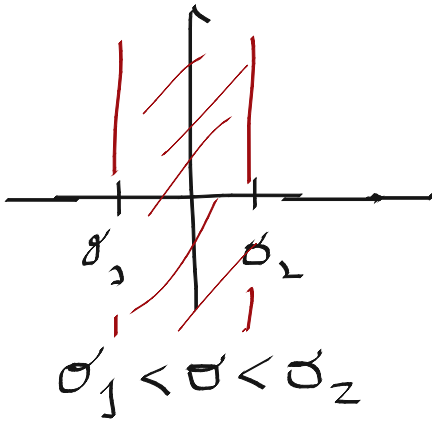
REGIÓN DE CONVERGENCIA

Propiedad 1: La ROC consiste en bandas paralelas

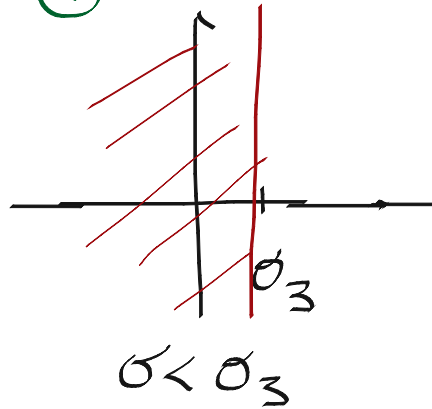
al eje $\underline{w}$

$\mathcal{L}\{x(t)\} = \int \underbrace{x(t) \cdot e^{-\sigma t}}_{\text{tamaño}} dt \Rightarrow$ convergencia depende de $\|x(t) \cdot e^{-\sigma t}\|$ solo σ

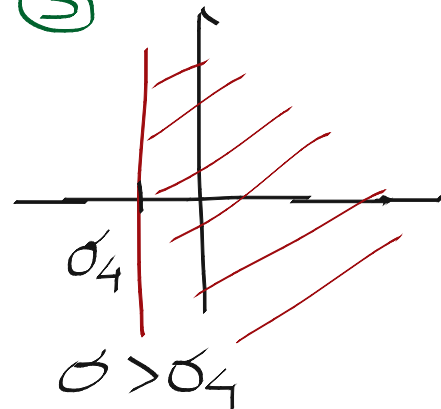
①



②



③

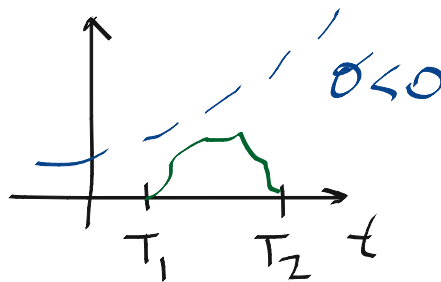
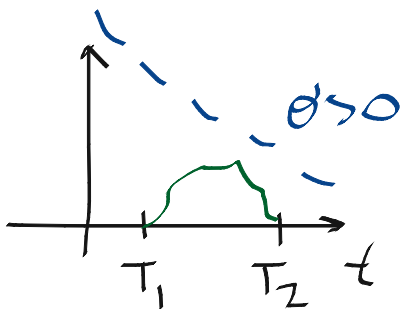


Propiedad 2 Para $X(s)$ racional, no contiene polos

$$\text{a polo } X(s) \Rightarrow \boxed{\lim_{s \rightarrow \infty} X(s) = \infty}$$

Propiedad 3 $x(t)$ soporte finito y absolutamente integrable

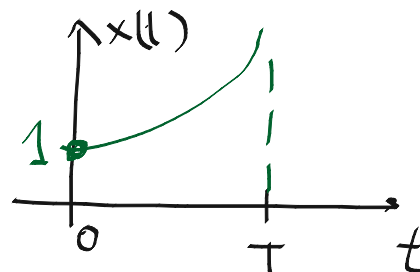
$$\Rightarrow ROC = \mathbb{C}$$



$$\| \underline{x(t)} \cdot \underline{e^{-\sigma t}} \|$$

Ejemplo 9.6

$$x(t) = \begin{cases} e^{-\lambda t} & 0 < t < T \\ 0 & \text{e.o.c.} \end{cases}$$

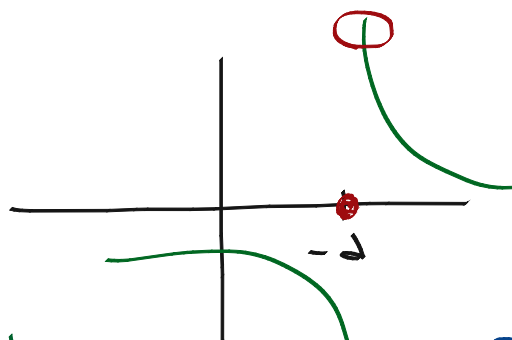


$$X(s) = \int_0^T e^{-\lambda t} e^{-st} dt = \frac{e^{-(\lambda+s)T} - 1}{-(\lambda+s)} = \boxed{\frac{1 - e^{-(\lambda+s)T}}{\lambda+s}}$$

$\uparrow$
 $\forall s \neq -\lambda$

$s = -\lambda$

$$X(-\lambda) = \int_0^T \underbrace{e^{-\lambda t}}_1 \cdot e^{\lambda t} dt = \boxed{T}$$



$$\lim_{s \rightarrow -\lambda} X(s) = \lim_{s \rightarrow -\lambda}$$

$$\frac{1 - e^{-(\lambda+s)T}}{\lambda+s}$$

$\downarrow$
 $\rightarrow 0$

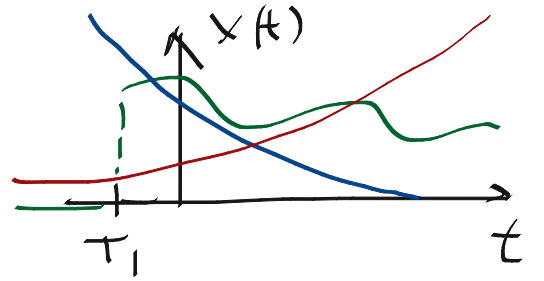
$$= \lim_{s \rightarrow -\lambda} \frac{T \cdot e^{-(\lambda+s)T}}{1} = \boxed{T}$$

$\uparrow$
Indeterminación $\frac{0}{0}$
L'Hopital

Roc: $\mathbb{C}$

Propiedad 4:

$x(t)$ decrece y converge para un cierto σ_0 , entonces converge $\forall \sigma > \sigma_0$



$$|X(s)| \leq \int_{-\infty}^{\infty} |x(t)| \cdot e^{-\sigma t} dt = \int_{T_1}^{\infty} |x(t)| \cdot e^{-\sigma_0 t} \cdot e^{-(\sigma - \sigma_0)t} dt \leq \int_{T_1}^{\infty} |x(t)| \cdot e^{-\sigma_0 t} dt \cdot \underbrace{e^{-(\sigma - \sigma_0)T_1}}_{\text{finito}}$$

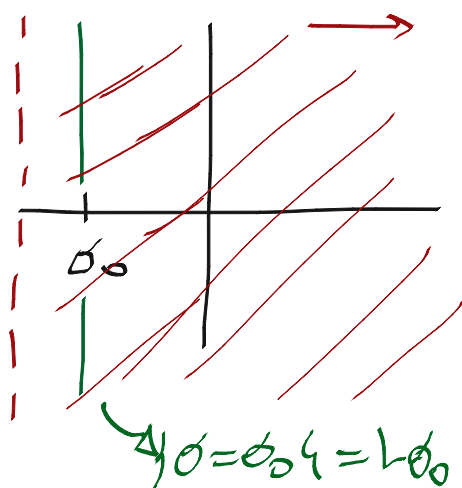
$$\Rightarrow \boxed{|X(s)| < \infty}$$

obs: * no puede crecer infinitamente en la dirección negativa

↳ que es donde puede diverger.

* $\text{ROC} \supset \{\text{Re}\{s\} > \sigma_0\}$

* ROC es un Π^+



Propiedad 5:

$x(t)$ izquierda

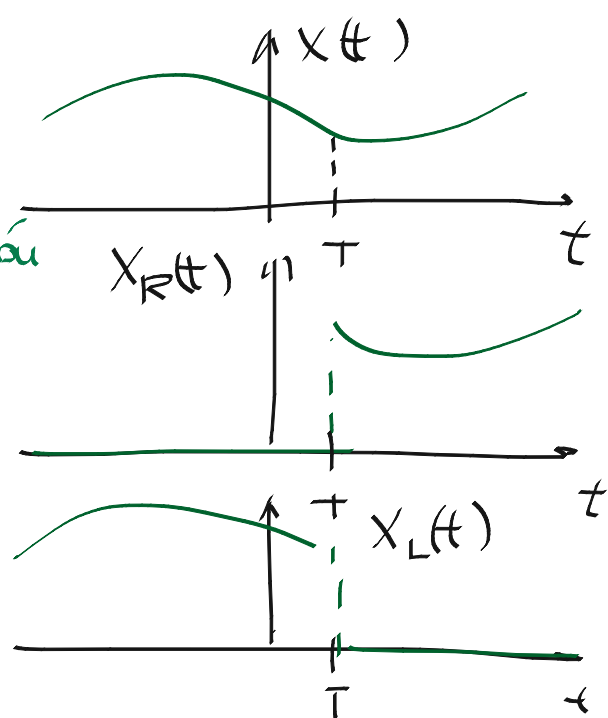
$\text{ROC} \supset \{\sigma = \sigma_0\}$

$\Rightarrow \text{ROC} \supset \{\text{Re}\{s\} < \sigma_0\}$
(ROC es un Π^-)

Propiedad 6 $x(t)$ bilateral

$$\exists x_R(t) \text{ y } x_L(t) / x(t) = x_R(t) + x_L(t)$$

↑ descomposición



(H) $\sigma = \sigma_0$

$$L = \{ \operatorname{Re}\{s\} = \sigma_0 \} \subset \text{ROC}$$

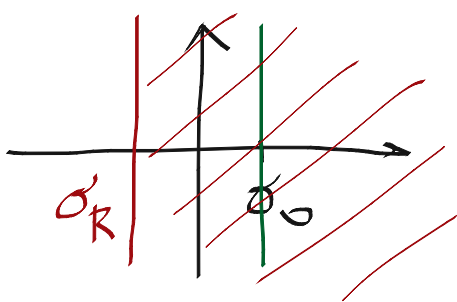
prop. 4

$$* \text{ROC}_R \supset \Pi_{\sigma_0}^+ = \{ \operatorname{Re}\{s\} > \sigma_0 \}$$

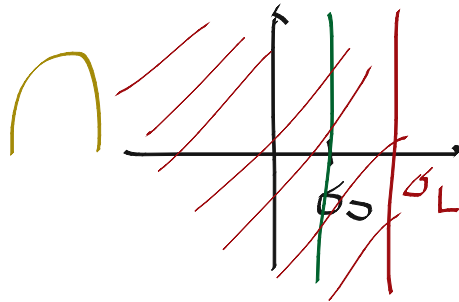
$$* \text{ROC}_L \supset \Pi_{\sigma_0}^- = \{ \operatorname{Re}\{s\} < \sigma_0 \}$$

(I) $\text{ROC} = \text{ROC}_R \cap \text{ROC}_L$

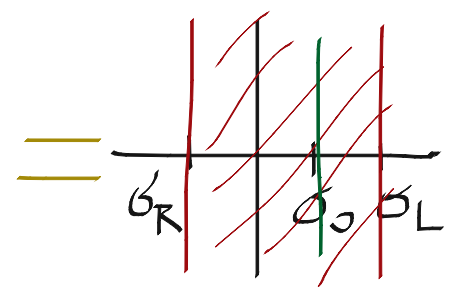
$$\text{ROC} = \{ \sigma_R < \operatorname{Re}\{s\} < \sigma_L \}$$



ROC_R



ROC_L



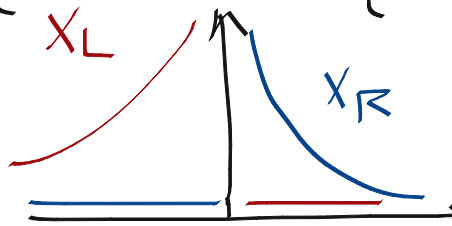
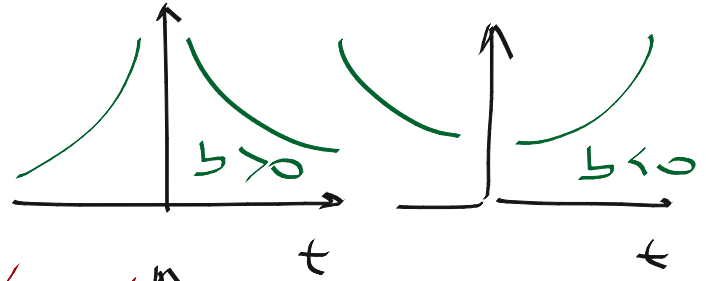
ROC

Ejemplo 9.7

$$x(t) = e^{-b|t|}$$

descomposición:

$$x(t) = \underbrace{u(t) \cdot e^{-bt}}_{x_R(t)} + \underbrace{u(-t) \cdot e^{+bt}}_{x_L(t)}$$



$$X_R(s) = \frac{1}{s+b} \quad 9.1$$

$$\text{ROC}_R: \text{Re}\{s\} > -b$$

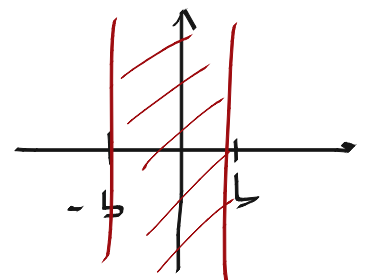
$$x_L(t) = -\left(-u(t-\tau) \cdot e^{\underbrace{(-b)}_{d} \tau} \right) \quad 9.2$$

$$X_L(s) = -\frac{1}{s-b} \quad d = -b$$

$$\text{ROC}: \{ \text{Re}\{s\} < b \}$$

$$X(s) = \frac{1}{s+b} - \frac{1}{s-b} = -\frac{2b}{(s+b)(s-b)}$$

$$\text{ROC}: \{ -b < \text{Re}\{s\} < b \}$$



obs: 5 casos de ROC

- * semiplano derecho
- * semiplano izquierdo
- * banda
- * todo el plano
- * conjunto vacío

Propiedad 7:

→ se va infinito

si $X(s)$ racional, su ROC: → limitada por los polos.

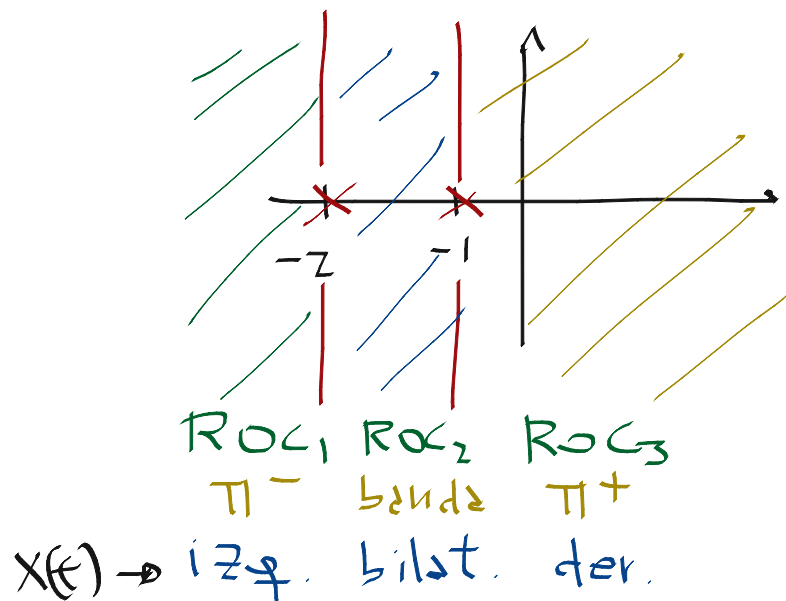
Propiedad 8:

① $X(s)$ racional
 $x(t)$ derecha } ⇒ ROC es el semiplano derecho
a la derecha de polo más
a la derecha.

② $X(s)$ racional
 $x(t)$ izquierda } ⇒ ROC es el semiplano izquierdo
a la izquierda del polo de
más a la izquierda.

Ejemplo 9.8

$$X(s) = \frac{1}{(s+1)(s+2)}$$



TRANSFORMADA INVERSA DE LAPLACE

$$X(s) = X(\sigma + j\omega) = \mathcal{F}\{x(t) \cdot e^{-\sigma t}\} \Rightarrow$$

$$\mathcal{F}^{-1}\{X(s)\} = x(t) \cdot e^{-\sigma t} \Rightarrow \boxed{x(t) = e^{\sigma t} \mathcal{F}^{-1}\{X(s)\}}$$

$$\boxed{x(t) = e^{\sigma t} \frac{1}{2\pi} \int_{-\infty}^{\infty} \underline{X(\sigma + j\omega)} \cdot e^{j\omega t} \cdot d\omega}$$

$$X(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underline{X(\sigma + j\omega)} \cdot e^{\frac{(\sigma + j\omega)t}{s}} \cdot d\omega$$

C.V.: $s = \sigma + j\omega$

$$X(t) = \frac{1}{2\pi} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) \cdot e^{st} \cdot ds$$

$$\boxed{X(t) = \frac{1}{2\pi} \oint_{L_{\sigma_0}} X(s) \cdot e^{st} \cdot ds}$$

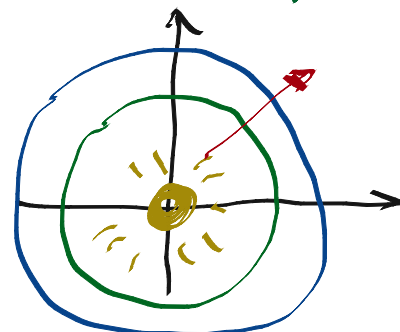
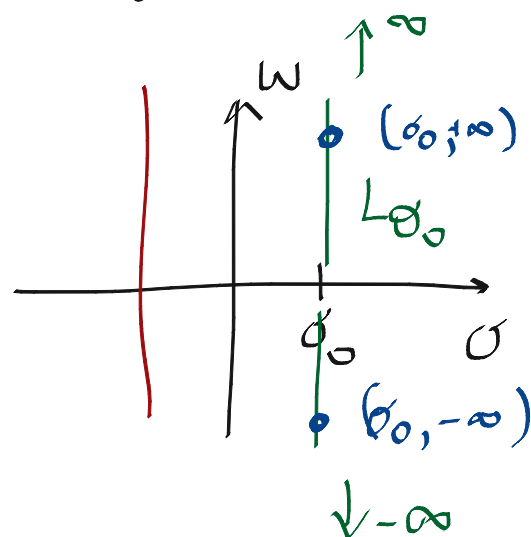
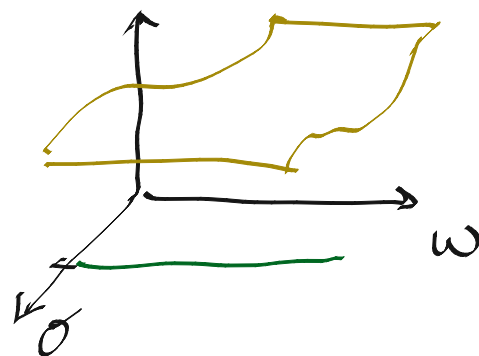
Integral de línea

obs:

* los integrales de línea son difíciles de resolver

* se puede calcular para cualquier $\underline{\sigma}$

* para $X(s)$ racional se pueden resolver por fracciones simples



Ejemplo genérico

* no hay polos múltiples

* $\text{gr}(N) < \text{gr}(D) \rightarrow$ sino, hay truco!!

puedo usar fracciones simples.

$$X(s) = \sum_{i=1}^N \frac{A_i}{s + \lambda_i} \rightarrow N \text{ polos } \lambda_i, \text{ donde cada uno se antitransforma como en 9.1}$$

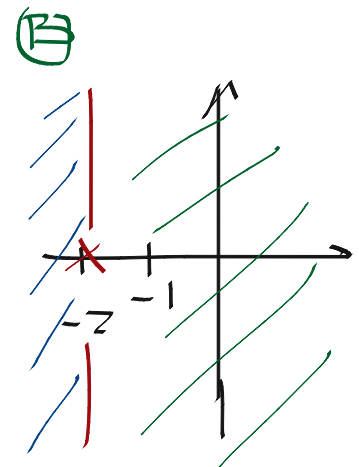
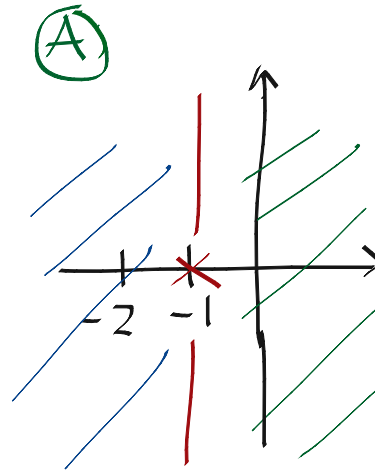
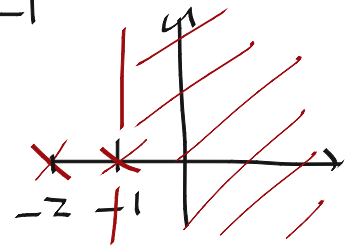
$$\frac{1}{s + \lambda_i} \xrightarrow{\mathcal{L}^{-1}} x_i(t) = \begin{cases} u(t) \cdot e^{-\lambda_i t} & \text{si ROC es } \Pi^+ \\ -u(t) \cdot e^{-\lambda_i t} & \text{si ROC es } \Pi^- \end{cases}$$
$$X(t) = \sum_{i=1}^N x_i(t) = \sum_{i=1}^N A_i \cdot u(t) \cdot e^{-\lambda_i t}$$

Asumí x_i derecha $\forall i$

Ejemplo 9.9 $X(s) = \frac{1}{(s+1)(s+2)}$, $\text{Re}(s) > -1$

$$X(s) = \frac{A}{s+1} + \frac{B}{s+2}$$

$$X(s) = \frac{1}{\underbrace{s+1}_{\textcircled{A}}} - \frac{1}{\underbrace{s+2}_{\textcircled{B}}}$$

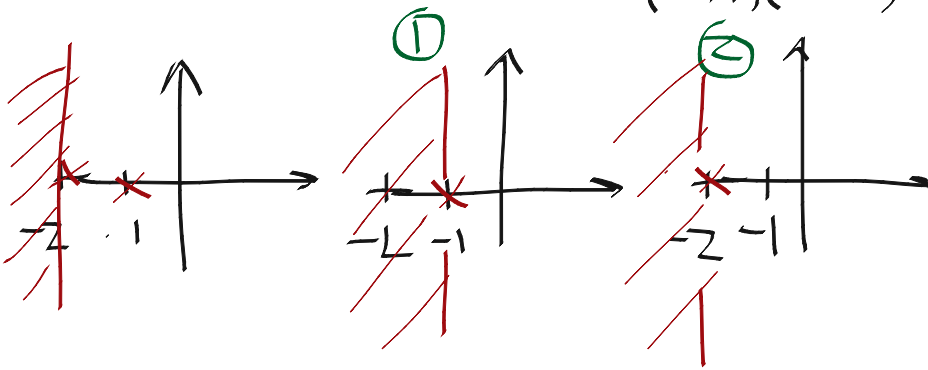


① $\frac{1}{s+1}$, $\text{Re}(s) > -1 \xrightarrow{\mathcal{L}^{-1}} u(t) \cdot e^{-t}$

② $\frac{1}{s+2}$, $\text{Re}(s) > -2 \xrightarrow{\mathcal{L}^{-1}} u(t) \cdot e^{-2t}$

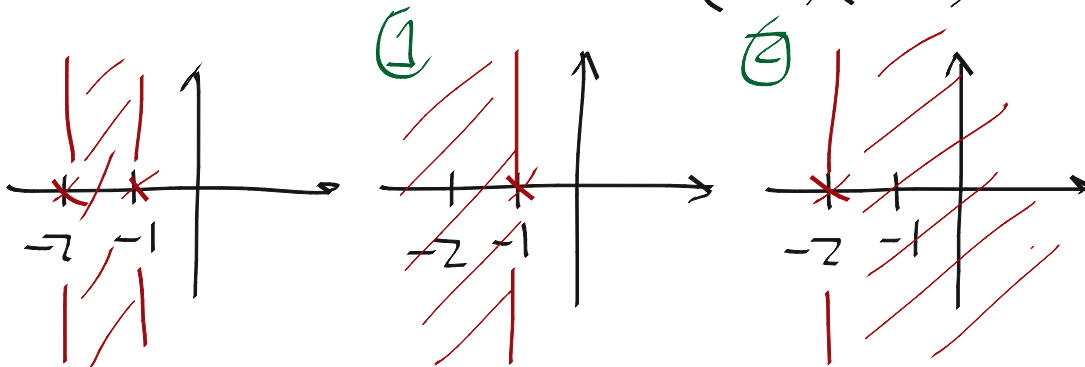
$$x(t) = u(t) \cdot \begin{bmatrix} e^{-t} & -e^{-2t} \end{bmatrix}$$

Ejemplo 9.10 $X(s) = \frac{1}{(s+1)(s+2)}$, $\text{Re}(s) < -2$



$$X(t) = -U(-t) \cdot \begin{bmatrix} e^{-t} & e^{-2t} \end{bmatrix}$$

Ejemplo 9.11 $X(s) = \frac{1}{(s+1)(s+2)}$ $-2 < \text{Re}(s) < -1$



$$X(t) = -U(t) \cdot e^{-t} - U(t) \cdot e^{-2t}$$

Clase 30

Análisis de sistemas

En tiempo y frecuencia

ANÁLISIS DE SISTEMAS EN T y F

Representación magnitud-fase

Análisis: * módulo: distribución de energía (frec)
* fase: comportamiento en el tiempo (ret)

Ejemplos: * barco: interferencia constructiva (super) (das)
* audio: → fase lineal (retardo constante)
→ reverberación (especialización)
→ flanging
→ phase shifting
* imágenes: → fase: geometría y bordes
→ módulo: direcciones y colores

fase lineal: $\phi(\omega) = \tau_g \cdot \omega$

retardo de grupo: $\tau_g(\omega) = -\frac{d\phi}{d\omega}$

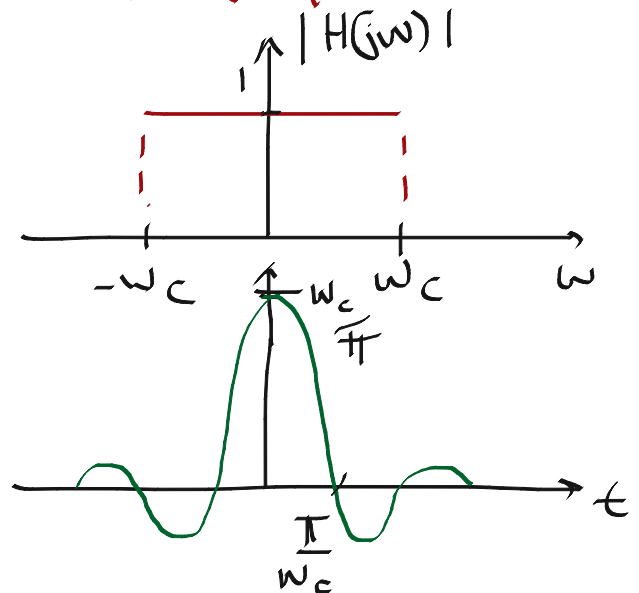
Bode: → escala logarítmica y asintótica
→ producto en logaritmo es aditivo.

Propiedades de filtros en tiempo y frecuencia

Filtro ideal: (pasabajas)

$$H(j\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & |\omega| \geq \omega_c \end{cases}$$

$$h(t) = \text{senc}(\omega_c t)$$



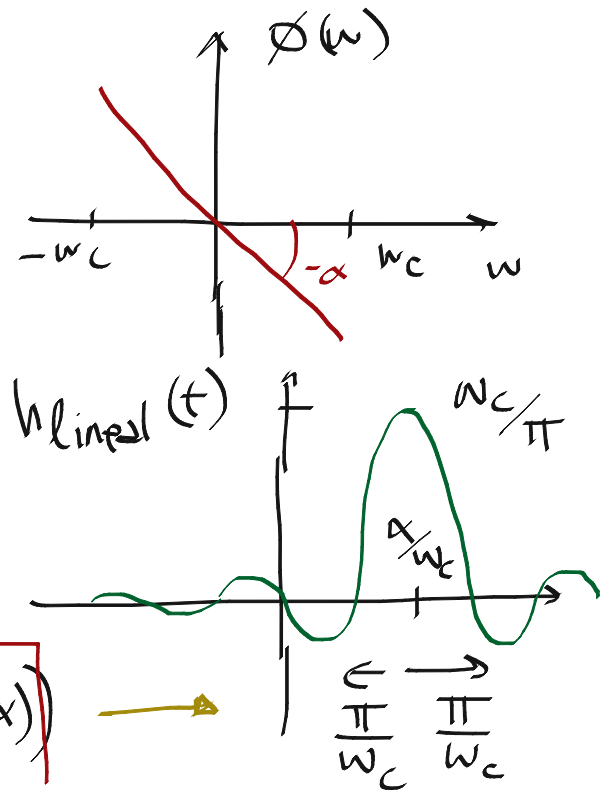
Fase lineal:

$$\phi(\omega) = -\alpha \cdot \omega$$

$$H_2(j\omega) = e^{-j\alpha\omega}, \quad h_2(t) = \delta(t - \alpha)$$

$$H_{\text{lineal}}(j\omega) = H(j\omega) \cdot H_2(j\omega)$$

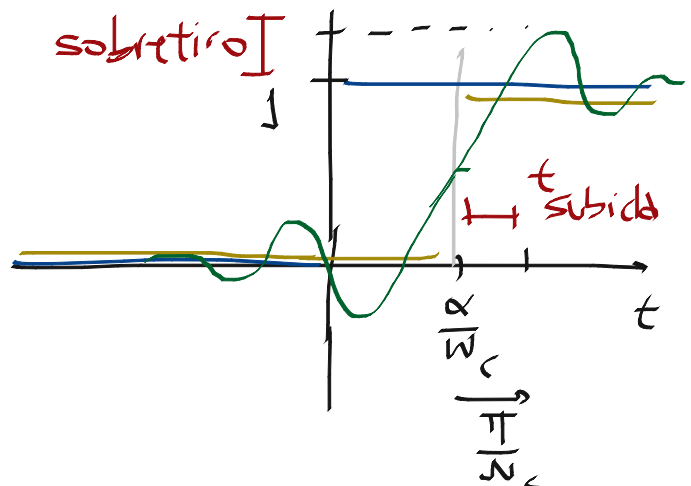
$$H_{\text{lineal}} = \begin{cases} e^{-j\alpha\omega} & |\omega| < \omega_c \\ 0 & |\omega| \geq \omega_c \end{cases}$$



$$h_{\text{lineal}}(t) = h(t) * h_2(t) = \boxed{\text{senc}(\omega_c(t - \alpha))}$$

Respuesta al escalón:

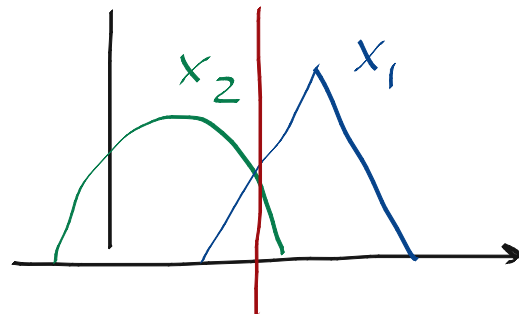
$$s(t) = \int_{-\infty}^t h(\tau) d\tau$$



Propiedades en tiempo y frecuencia de filtros reales

obs: propiedades no deseadas de filtros reales

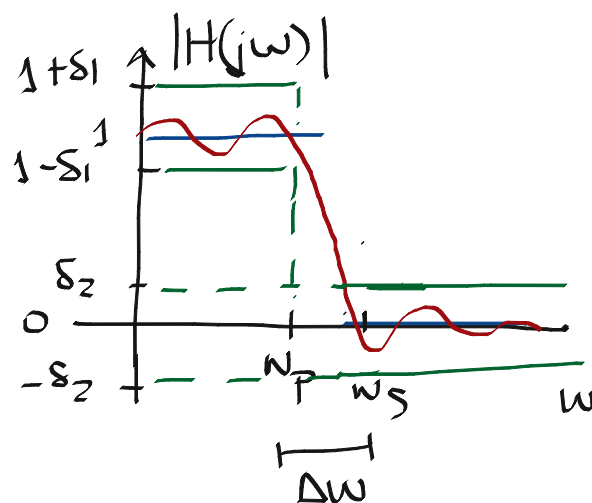
- * cuando hay solapamiento: mejor transiciones
- * oscilaciones: discontinuidad. suaves
- * facilidad de construcción
- * causalidad.



obs: * necesitamos flexibilizar los requerimientos
* no puedo fijar todos parámetros a la vez

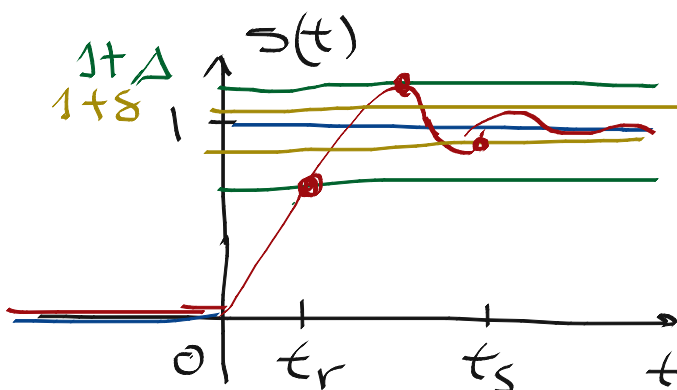
Frecuencia

- * δ_1 : oscilación permitida B.P.
- * δ_2 : oscilación permitida B.S.
- * ω_p : ancho de la B.P.
- * ω_s : ancho de la B.S.
- * ω_c : frecuencia de corte
- * $\Delta\omega$: tolerancia en la f.c.



Tiempo

- * Δ : sobretiro
- * δ : precisión
- * t_r : t. de subida
- * t_s : t. de asentamiento.



Ejemplos de filtros

Ejemplo 6.3

$f_c = 500 \text{ Hz}$, tolerancia 0.05

$N = 5$ (orden)

Comparación Butterworth vs elíptico

Obs. * respuesta de 5^{to} orden [slides 2 y 3]

ANÁLISIS DE SISTEMAS DE 1^{er} y 2^{do} ORDEN en TyF

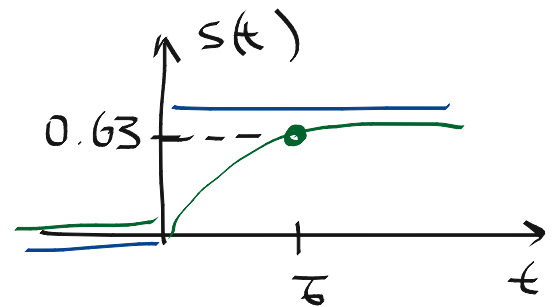
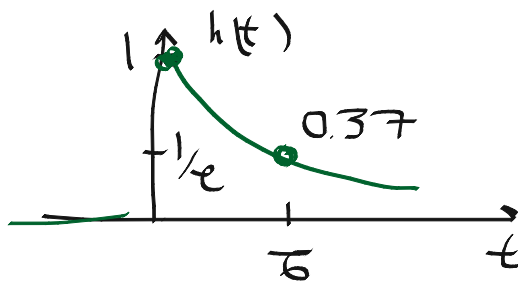
- * muchos sistemas reales de interés que se pueden modelar así
- * en general son fáciles de construir
- * sistemas de orden mayor se pueden descomponer en combinaciones de 1^{er} y 2^{do} orden.

Primer orden

* modelo: $\tau \cdot y'(t) + y(t) = x(t)$

* tiempo:

$$h(t) = \frac{1}{\tau} \cdot e^{-\frac{t}{\tau}} \quad s(t) = \left(1 - e^{-\frac{t}{\tau}}\right) u(t)$$



* frecuencia:

$$H(j\omega) = \frac{1}{1 + j\omega\tau}$$

$$|H(j\omega)|^2 = \frac{1}{\omega^2\tau^2 + 1}$$

$$\Theta(\omega) = -\arctan \omega\tau$$

Bode:

$$B(\omega) = 20 \cdot \log_{10} |H(\omega)| = -10 \cdot \log_{10} (\omega \tau)^2 + 1$$

a) si $\omega \tau \ll 1$ ($\omega \ll \frac{1}{\tau}$) $\Rightarrow B(\omega) = 0 \text{ dB}$

b) si $\omega \tau \gg 1$ ($\omega \gg \frac{1}{\tau}$) $\Rightarrow$

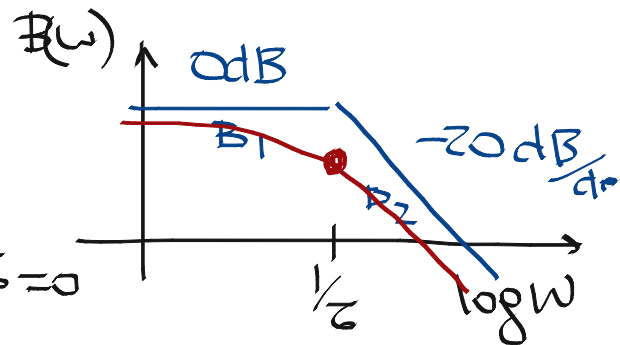
$$B(\omega) = \underbrace{-20 \log_{10} \omega}_{\text{a}} \underbrace{-20 \log_{10} \tau}_{\text{b}}$$

* verif. lecta:

$$B_1\left(\frac{1}{\tau}\right) = 0$$

$$B_2\left(\frac{1}{\tau}\right) = -20 \cdot \log_{10}\left(\frac{1}{\tau}\right) - 20 \log_{10} \tau = 0$$

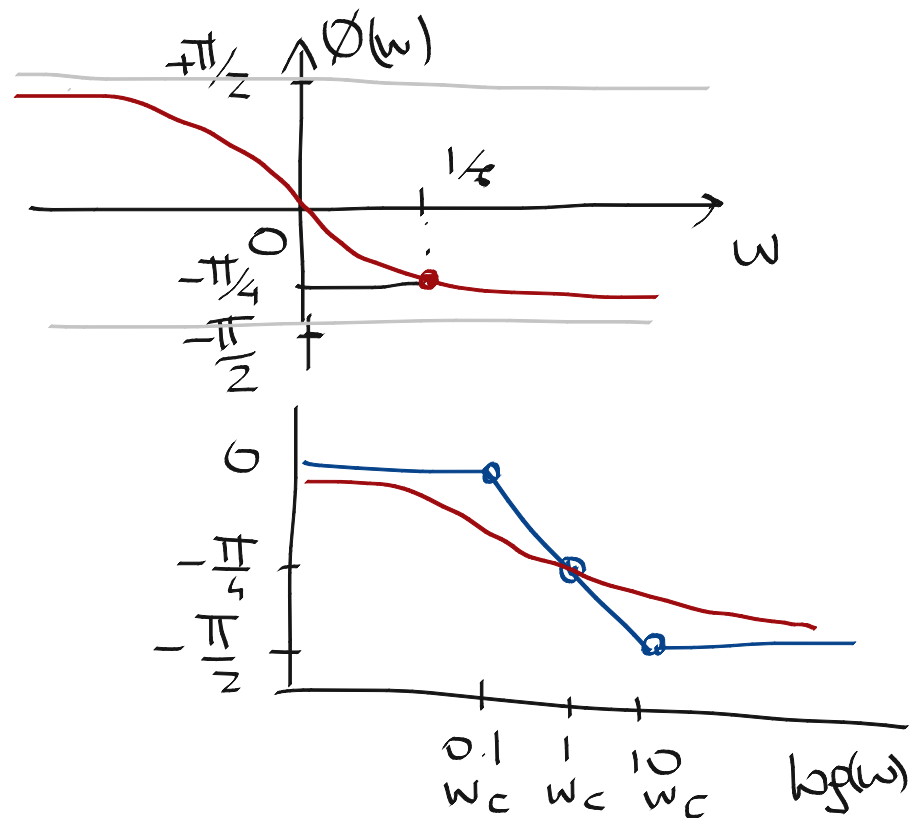
$$B\left(\frac{1}{\tau}\right) = -10 \cdot \log_{10} \left(\left(\frac{1}{\tau} \tau\right)^2 + 1 \right) = -10 \cdot \log_{10}(2) = \boxed{-3.02 \text{ dB}}$$



Case:

$$\phi(\omega) = -\arctan(\omega\tau)$$

$$\Theta(\omega) \approx \begin{cases} 0 & \omega \ll \omega_c & \omega < 0.1 \frac{1}{\tau} \\ -\frac{\pi}{4} \left[\log(\omega) + \log(\tau) + 1 \right] & 0.1 \frac{1}{\tau} < \omega < 10 \frac{1}{\tau} \\ -\frac{\pi}{2} & \omega \gg \omega_c & \omega > 10 \frac{1}{\tau} \end{cases}$$



Clase 31

Análisis de sistemas

En tiempo y frecuencia

Sistemas de 2º orden en TyF

modelo gen: $y''(t) + 2\epsilon\omega_n y'(t) + \omega_n^2 y(t) = \omega_n^2 x(t)$

obs: * sistemas amortiguados o viscosos

* ejemplos: $\rightarrow$ circuitos RLC
 $\rightarrow$ sistemas mecánicos

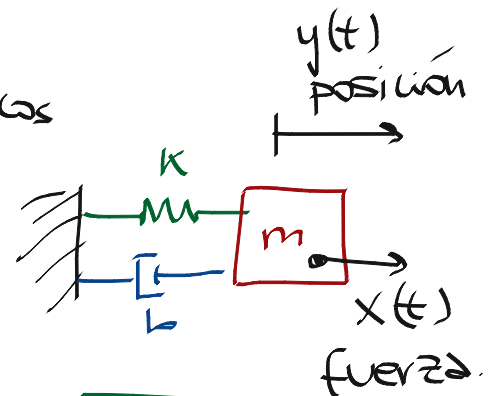
ejemplo:

$$m y''(t) = x(t) - k \cdot y(t) - b \cdot y'(t)$$

$$y''(t) + \frac{b}{m} y'(t) + \frac{k}{m} y(t) = \frac{1}{m} x(t)$$

$$y''(t) + 2\epsilon\omega_n y'(t) + \omega_n^2 y(t) = \frac{\omega_n^2}{k} x(t)$$

$\rightarrow$ sólo
pendencia



$$\omega_n = \sqrt{\frac{k}{m}}$$

$$\epsilon = \frac{b}{\sqrt{2km}}$$

Análisis en frecuencia #1

$$x(t) = e^{j\omega t} \rightarrow y(t) = H(j\omega) \cdot e^{j\omega t}$$

$$H(j\omega) \cdot (j\omega)^2 \cdot e^{j\omega t} + 2\epsilon\omega_n \cdot j\omega \cdot e^{j\omega t} \cdot H(j\omega) + \omega_n^2 \cdot H(j\omega) \cdot e^{j\omega t} = \omega_n^2 e^{j\omega t}$$

$$H(j\omega) \cdot [(j\omega)^2 + 2\epsilon\omega_n j\omega + \omega_n^2] = \omega_n^2$$

$$H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\epsilon\omega_n j\omega + \omega_n^2}$$

$$H(s) = \frac{\omega_n^2}{s^2 + 2\epsilon\omega_n s + \omega_n^2}$$

$$s = j\omega$$

$$H(j\omega) = \frac{1}{\left(j \frac{\omega}{\omega_n}\right)^2 + 2\varepsilon j \frac{\omega}{\omega_n} + 1}$$

Obs: * ω_n : frecuencia natural del sistema (escala la frecuencia)

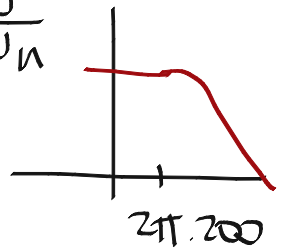
* ε : razón de amortiguamiento

* $H(j\omega)$: depende solo de la razón $\frac{\omega}{\omega_n}$

→ $\omega_n = 2\pi \cdot 100$ $\omega = 2\pi \cdot 200$

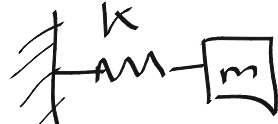
→ $\omega_n = 2\pi \cdot 1$ $\omega = 2\pi \cdot 2$

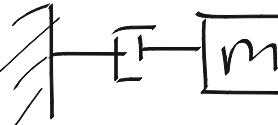
→ $\tilde{\omega} = \frac{\omega}{\omega_n} = 2$



Obs: "bajando al ejemplo"

* $\varepsilon = \sqrt{\frac{b}{2km}}$ y $\omega_n = \sqrt{\frac{k}{m}}$

* $\varepsilon = 0 \Rightarrow \boxed{b = 0} \Rightarrow$  no freno $\Rightarrow$ oscila

* $\omega_n = 0 \Rightarrow \boxed{k = 0} \Rightarrow$  no oscila, solo freno

Análisis en el tiempo

$$H(s) = \frac{\omega_n^2}{(s-c_1)(s-c_2)}$$

$$C_i = -\varepsilon \omega_n \pm \sqrt{\varepsilon^2 - 1}$$

obs: * dos casos: $\rightarrow \varepsilon = 1$, raíz doble
 $\rightarrow \varepsilon \neq 1$, raíces distintas.

$\varepsilon \neq 1$ $C_1 \neq C_2$ (reales o complejas)

$$H(s) = \frac{M}{s-c_1} - \frac{M}{s-c_2}$$

$\uparrow$ coef. iguales

$$M = \frac{\omega_n^2}{2\sqrt{\varepsilon^2 - 1}}$$

$$h(t) = M \cdot u(t) \cdot (e^{c_1 t} - e^{c_2 t})$$

$\varepsilon = 1$ $C_1 = C_2 = -\omega_n$

$$H(s) = \frac{\omega_n^2}{(s+\omega_n)^2}$$

Cuadrado de $\frac{1}{s}$ es su derivada

$$h(t) = u(t) \cdot \omega_n^2 \cdot t \cdot e^{-\omega_n t}$$

tabla 4.2

caso $\xi \neq 1$:

a) $0 < \xi < 1 \rightarrow$ subamortiguado

$$h(t) = \frac{\omega_n}{2\sqrt{\xi^2 - 1}} e^{-\xi \omega_n t} \left(e^{\frac{\sqrt{\xi^2 - 1}}{\omega_n} \omega_n t} - e^{-\frac{\sqrt{\xi^2 - 1}}{\omega_n} \omega_n t} \right)$$

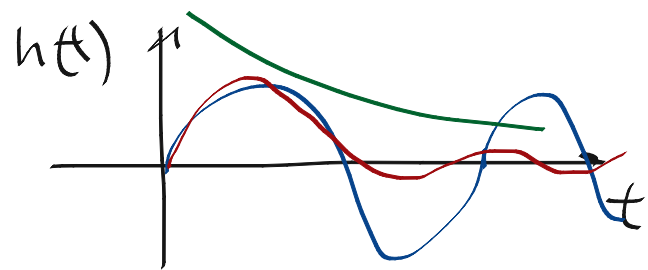
$\xi < 1$

$$h(t) = \frac{\omega_n}{2j\sqrt{1 - \xi^2}} e^{-\xi \omega_n t} \left(e^{j\sqrt{1 - \xi^2} \omega_n t} - e^{-j\sqrt{1 - \xi^2} \omega_n t} \right)$$

seno

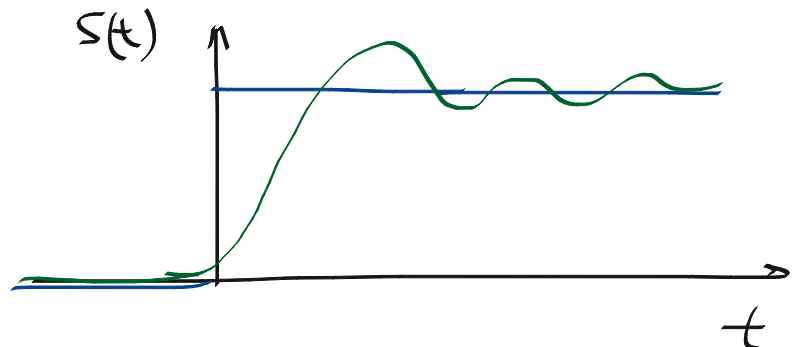
$$h(t) = \frac{\omega_n}{\sqrt{1 - \xi^2}} \underbrace{e^{-\xi \omega_n t}}_{\text{decrecimiento}} \underbrace{\text{sen}(\sqrt{1 - \xi^2} \omega_n t)}_{\text{oscilatorio}}$$

ds: * raíces complejas conjugadas
* comportamiento oscilatorio
* "no frené lo suficiente"



$$s(t) = 1 - \frac{e^{-\xi \omega_n t}}{\sqrt{1 - \xi^2}} \text{sen}(\underbrace{\sqrt{1 - \xi^2} \omega_n t}_{\omega_{eff}} + 2 \cos(\xi))$$

$$\omega_{eff} = \sqrt{1 - \xi^2} \omega_n$$



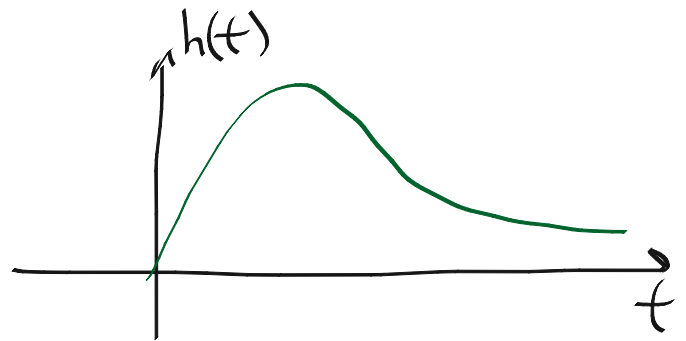
CASO $\xi \neq 1$:

b) $\xi > 1 \rightarrow$ sobre amortiguado

$$h(t) = \frac{w_n}{2\sqrt{\xi^2 - 1}} e^{-\xi w_n t} \left(e^{\frac{\sqrt{\xi^2 - 1}}{w_n} w_n t} - e^{-\frac{\sqrt{\xi^2 - 1}}{w_n} w_n t} \right)$$

↑ crece
↑ decrece
 $\in \mathbb{R}$

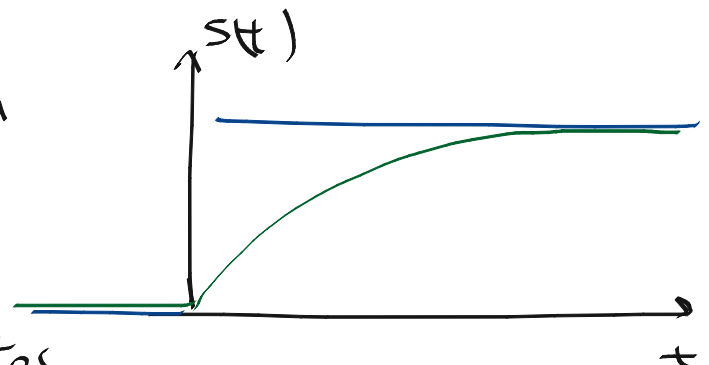
este término
siempre good
 $\xi > \sqrt{\xi^2 - 1}$
 $\xi^2 > \xi^2 - 1 \checkmark$



$$s(t) = v(t) + \frac{w_n}{2\sqrt{\xi^2 - 1}} \left(\frac{e^{\frac{\sqrt{\xi^2 - 1}}{w_n} w_n t}}{c_1} - \frac{e^{-\frac{\sqrt{\xi^2 - 1}}{w_n} w_n t}}{c_2} \right)$$

$$c_1 = -\xi w_n + \sqrt{\xi^2 - 1} \cdot w_n$$

$$c_2 = -\xi w_n - \sqrt{\xi^2 - 1} \cdot w_n$$

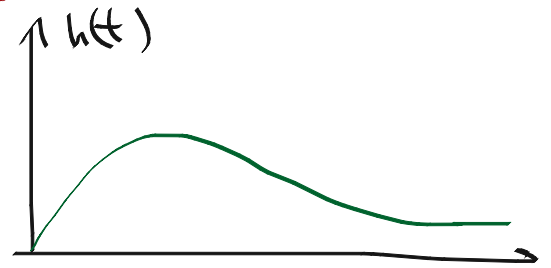


- obs:
- * raíces reales
 - * no hay oscilaciones
 - * exponenciales decrecientes
 - * "lo frené demasiado"

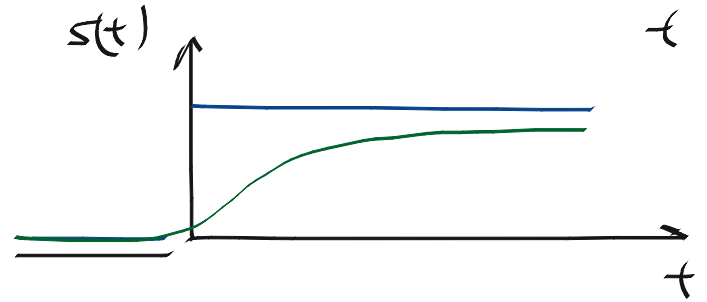
Caso $\zeta=1$ amortiguamiento crítico.

$$h(t) = u(t) \cdot \omega_n^2 \cdot t \cdot e^{-\omega_n t}$$

$$s(t) = u(t) \cdot \left(1 - e^{-\omega_n t} - \omega_n t \cdot e^{-\omega_n t} \right)$$



- Obs:
- * raíz doble
 - * no hay oscilaciones
 - * más rápido que el subamortiguado



Análisis en frecuencia #2

Bode:

$$H(j\omega) = \frac{1}{\left(j\frac{\omega}{\omega_n}\right)^2 + 2\varepsilon j\frac{\omega}{\omega_n} + 1}$$

1) Módulo:

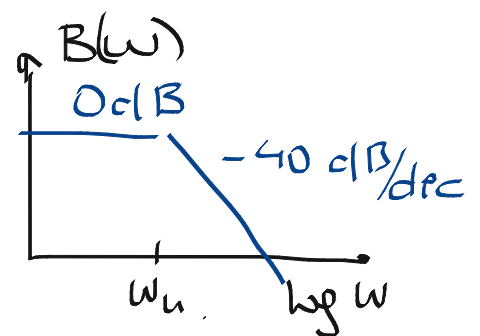
$$|H(j\omega)|^2 = \frac{1}{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + 4\varepsilon^2\left(\frac{\omega}{\omega_n}\right)^2}$$

$$B(\omega) = -10 \log_{10} \left[\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + 4\varepsilon^2\left(\frac{\omega}{\omega_n}\right)^2 \right]$$

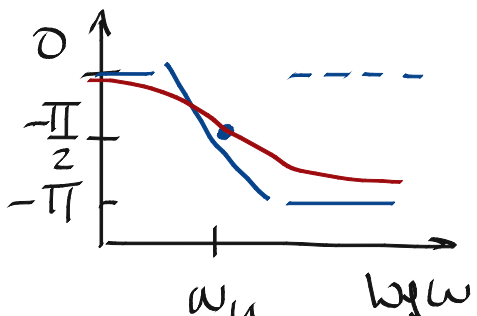
a) $\omega < \omega_n$, $B_1(\omega) = -10 \log_{10} 1 = \boxed{0 \text{ dB}}$

b) $\omega \gg \omega_n$, $B_2(\omega) = -10 \log_{10} \left[\left(\frac{\omega}{\omega_n}\right)^4 + 4\varepsilon^2\left(\frac{\omega}{\omega_n}\right)^2 \right]$

$$B_2(\omega) = -40 \log_{10} \left(\frac{\omega}{\omega_n} \right) \Rightarrow \boxed{B_2(\omega) = \underbrace{-40 \log \omega}_A + \underbrace{40 \log \omega_n}_B}$$



2) fase: $\phi(\omega) = -\arctan\left(\frac{2\varepsilon\omega/\omega_n}{1 - \frac{\omega^2}{\omega_n^2}}\right)$



$$\phi(\omega) = \begin{cases} a) & 0 & \omega < 0.1\omega_n \\ b) & -\frac{\pi}{2} \left[\log_{10}\left(\frac{\omega}{\omega_n}\right) + 1 \right] & 0.1 < \frac{\omega}{\omega_n} < 10 \\ c) & -\pi & \omega > 10\omega_n \end{cases}$$

b) $\phi(\omega) = A \cdot \log_{10}\left(\frac{\omega}{\omega_n}\right) + B = A \cdot \log_{10}(1) + B$

$$\phi(\omega_n) = B = -\frac{\pi}{2}$$

$$\phi(10\omega_n) = A \cdot \log_{10}(10) + B = -\pi \Rightarrow A = -\frac{\pi}{2}$$

$$\phi(\omega) = -\frac{\pi}{2} \left[\log_{10}\left(\frac{\omega}{\omega_n}\right) + 1 \right]$$

c) $\phi(\omega) \approx -\arctan\left(\frac{2\varepsilon\omega/\omega_n}{\frac{\omega^2}{\omega_n^2}}\right) = -\arctan\left(2\varepsilon \frac{\omega_n}{\omega}\right)$

$\neq -\pi$

Clase 32

Análisis de sistemas

En tiempo y frecuencia

Respuesta en frecuencia real

si derivamos $|H(j\omega)|^2$, para $\xi < \frac{\sqrt{2}}{2} \approx 0.7$

tiene máximo !!

$$\boxed{\omega_{max} = \sqrt{1-2\xi^2} \cdot \omega_n}$$

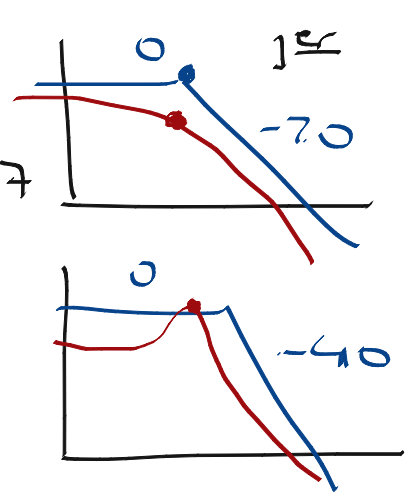
$$|H(j\omega_{max})| = \frac{1}{2\xi\sqrt{1-\xi^2}}$$

$$\text{si } \xi = 0.5 \rightarrow \omega_{max} = 0.7\omega_n \Rightarrow |H(j\omega_{max})|^2 = 2.3$$

obs: * cambio cualitativo

* pico: $\rightarrow$ amplifique una frecuencia o banda
 $\rightarrow$ calidad: $\boxed{Q = \frac{1}{2\xi}}$

obs: * $\xi < 0$: sist. inestable



Bate para respuestas racionales

caso: * sabemos analizar polos de 1er y 2do orden

* los ceros se deducen fácilmente.
1er y 2do orden

$$H_1^c(j\omega) = 1 + j\omega\tau$$

$$H_2^c(j\omega) = 1 + 2\epsilon j\frac{\omega}{\omega_n} + \left(\frac{\omega}{\omega_n}\right)^2$$

} $\Rightarrow$ son los inversos de $H_1(j\omega)$ y $H_2(j\omega)$

* módulo:

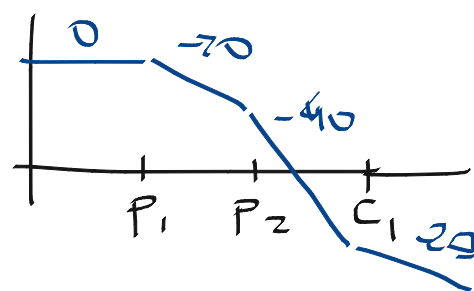
$$20 \log_{10} |H_1^c(j\omega)| = 20 \log_{10} \frac{1}{|H_1(j\omega)|} = -20 \log_{10} |H_1(j\omega)|$$

$$* \text{ fase: } \angle H_1^c(j\omega) = -\angle H_1(j\omega)$$

Caso constante

$$H(j\omega) = K \Rightarrow 20 \log_{10} |K| = \text{cte}$$

0 dB/dec



polo orden 2to

$$* \left(\frac{1}{s+a}\right)^2 = -\frac{d}{ds} \left(\frac{1}{s+a}\right) \rightarrow \text{derivadas en frecuencia}$$

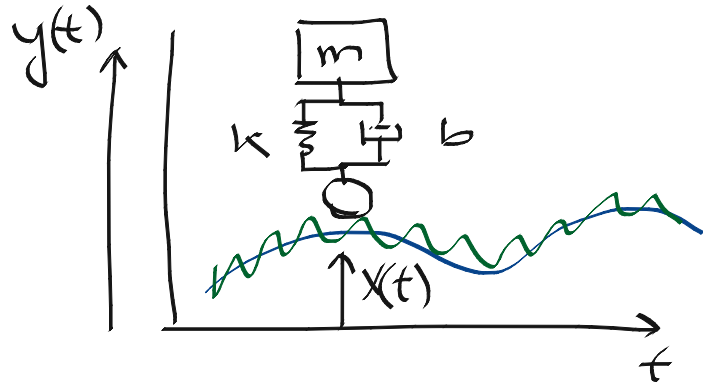
$$* \left(\frac{1}{s+a}\right)^n \approx -\frac{d^n}{ds^n} \left(\frac{1}{s+a}\right) \rightarrow \text{factor } t^n \text{ en el tiempo.}$$

$$* \text{ caída frecuencia } -20n \text{ dB/dec}$$

Ejemplo de análisis en el dominio del tiempo

suspensión auto

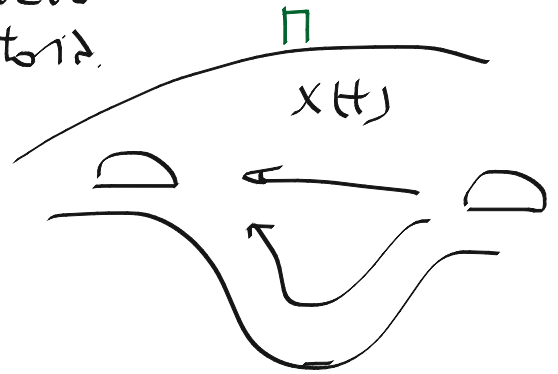
- * $x(t)$ posición del piso
 - ↳ topografía
 - ↳ rugosidad



- * $y(t)$ posición del auto

requerimientos

- * objetivo: filtrar la rugosidad del suelo.
 - ↳ transición suave
- * no queremos respuesta oscilatoria.



- * transición suave entre las frecuencias. (sin corte abrupto)
- * sistema usado en la práctica: bajo costo
 - ↳ prueba: apretar y soltar

Modelado

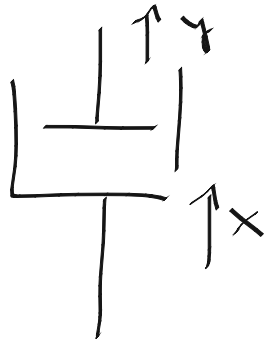
$$m y''(t) = -k \underbrace{(y(t) - x(t))}_{\text{importa la diferencia}} - b \underbrace{(y'(t) - x'(t))}_{\text{velocidad relativa}}$$

$$\underbrace{m y''(t) + b y'(t) + k y(t)}_{\text{C.L. Salidas}} = \underbrace{b x'(t) + k x(t)}_{\text{Entradas}}$$

$$H(j\omega) = \frac{b j\omega + k}{m(j\omega)^2 + b j\omega + k}$$

cero

$$\omega_n = \sqrt{\frac{k}{m}}$$



$$H(j\omega) = \frac{1 + 2\zeta j \frac{\omega}{\omega_n}}{\left(j \frac{\omega}{\omega_n}\right)^2 + 2\zeta j \frac{\omega}{\omega_n} + 1}$$

* ceros: $2\zeta s + \omega_n = 0 \Rightarrow$

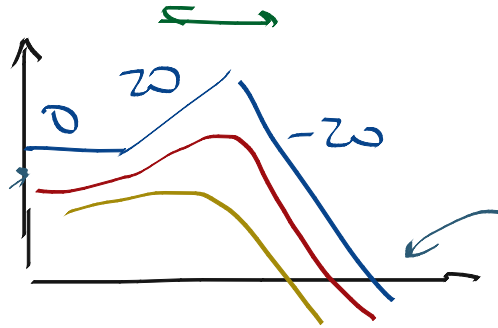
$$C = -\frac{\omega_n}{2\zeta}$$

* polos: $\rightarrow p_1 = -\zeta\omega_n + \sqrt{\zeta^2 - 1} \cdot \omega_n$

$$\rightarrow p_2 = -\zeta\omega_n - \sqrt{\zeta^2 - 1} \cdot \omega_n$$

$$\zeta = 0.5 \Rightarrow C = -\omega_n$$

$$p_1 = -0.5\omega_n + j\sqrt{0.75} \cdot \omega_n$$

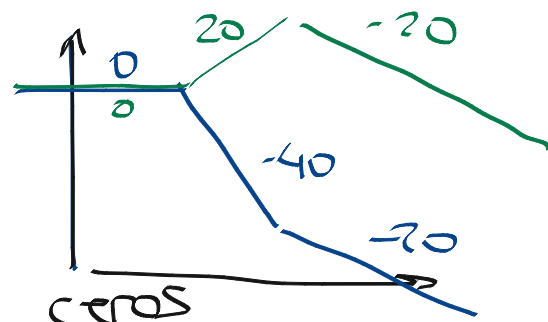


DS:

* asintótico no funciona
(están rca)

* depende del orden de p' y ceros

* asintótico funciona a los ω DS.



RESUMEN

- * Análisis en frecuencia:
 - módulo
 - fase:
 - retardo grupo
 - lineal
- * Análisis temporal:
 - transitorios
 - tiempos
- * Específica en tiempo y frecuencia: compromiso
- * probamos las herramientas en sistemas de segundo orden.
- * Factores:
 - restricciones Tyf
 - causalidad
 - estabilidad
 - construcción (real)

Clase 33

Transformada de Laplace

Propiedades

PROPIEDADES (tabla 9.1)

Linealidad $a \cdot x(t) + b \cdot y(t) \xrightarrow{\mathcal{L}} a \cdot X(s) + b \cdot Y(s), \text{ ROC} = R_1 \cap R_2$

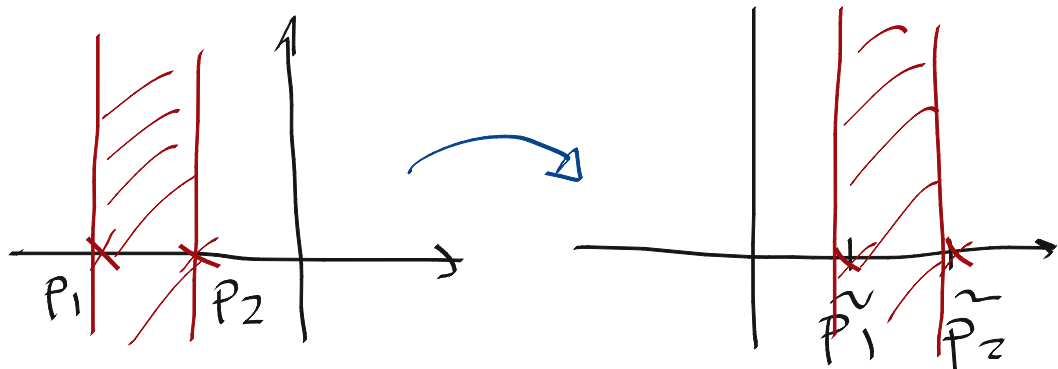
$\downarrow$ $\downarrow$
 R_1 R_2

Desplazamiento en s

$e^{s_0 t} x(t) \xrightarrow{\mathcal{L}} X(s - s_0), \text{ ROC} = \{ s + \text{Re}\{s_0\} / s \in R \}$

$X(s) = \frac{N(s)}{D(s)} = 0$

$= \underline{R + \text{Re}\{s_0\}}$



ej:

$e^{-3t} \cdot e^{-t} \cdot e^{-\sigma t}$

$e^{-3t} \cdot e^{-t} \cdot e^{-\sigma t}$

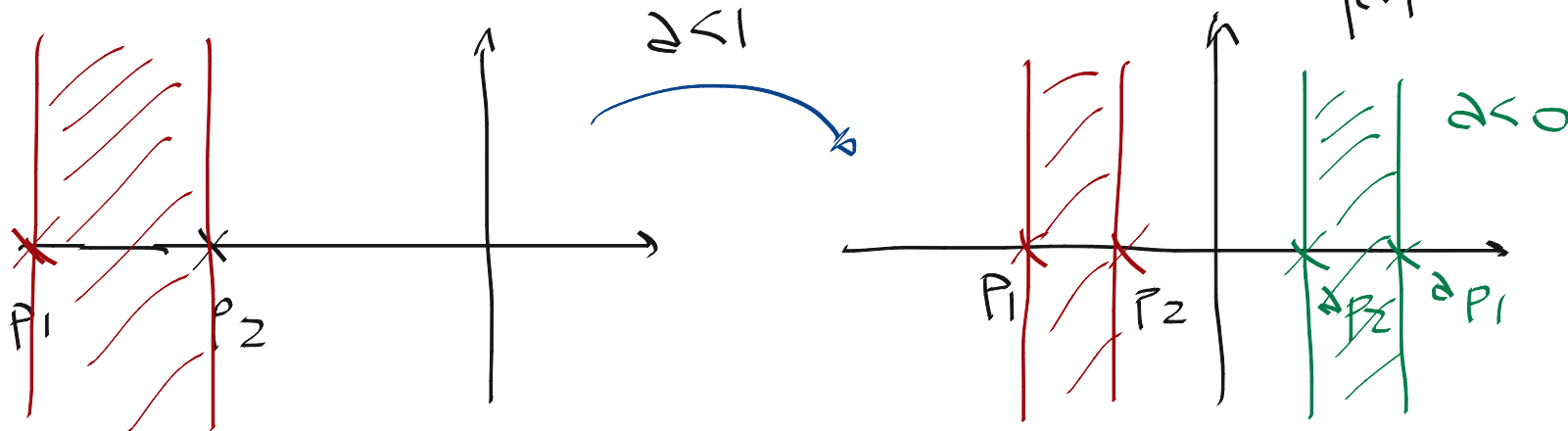
$\sigma > -1$

$\sigma > -4$

$D(p_1) = 0 \quad \tilde{D}(\tilde{p}_1 + s_0) = 0$
 $\tilde{p}_1 = p_1 + \text{Re}\{s_0\}$

Escalaamiento temporal

$x(at) \xrightarrow{\mathcal{L}} \frac{1}{|a|} X\left(\frac{s}{a}\right)$



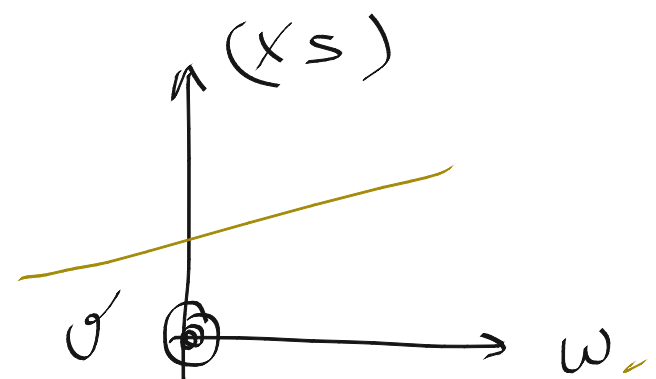
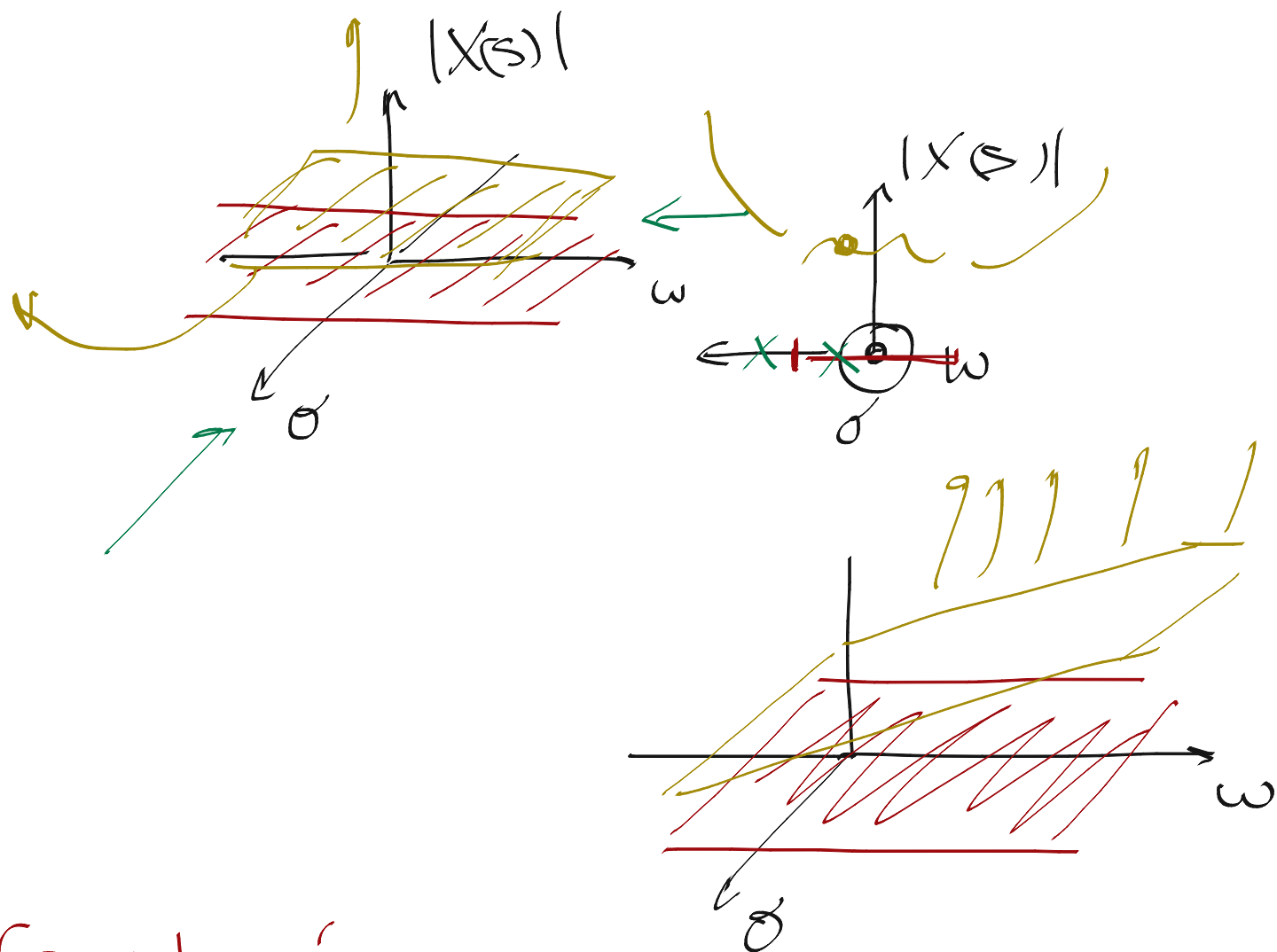
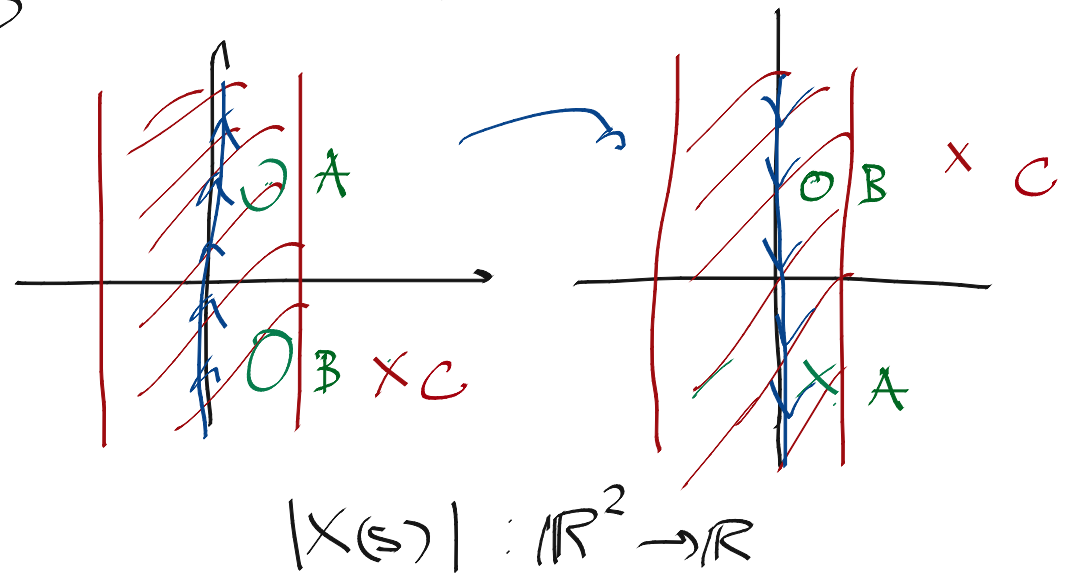
ej:

$e^{-t} \cdot e^{-\sigma t}, \exists \text{ si } \sigma > -1$

$e^{-2t} \cdot e^{-\sigma t}, \exists \text{ si } \sigma > -2$

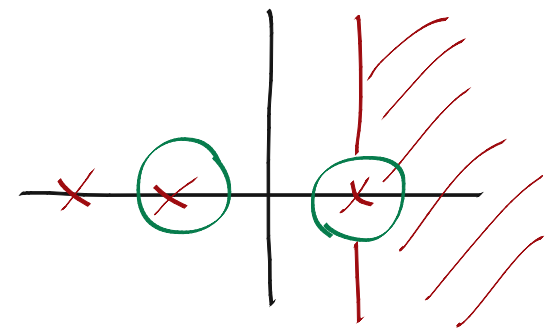
Conjugación $x^*(t) \xleftrightarrow{\mathcal{L}} X^*(s^*)$, $\text{ROC} = R$

* $x(t) \in \mathbb{R} \rightarrow X(s^*) = X(s)$



Convolution

$x_1(t) * x_2(t) \xleftrightarrow{\mathcal{L}} X_1(s) \cdot X_2(s)$



$\text{ROC} \supset (R_1 \cap R_2)$

- Obs:
- * no se agregan polos
 - * pueden eliminarse polos
 - * ROC es igual o mayor.

Diferenciación en t

$$x'(t) \xleftrightarrow{\mathcal{L}} s \cdot X(s), \text{ ROC} = R$$

* puede eliminarse un polo en $s=0$

Diferenciación en s

$$-t \cdot x(t) \xleftrightarrow{\mathcal{L}} \frac{dX(s)}{ds}, \text{ ROC} = R \quad \frac{1}{(s+a)} \rightarrow \frac{1}{(s+a)^2}$$

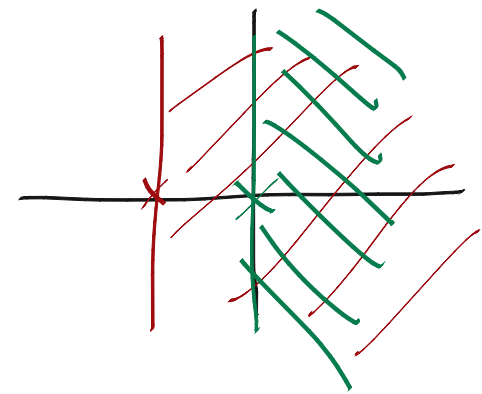
* a lo sumo aumenta la multiplicidad

Integral en el tiempo

$$\int_{-\infty}^t x(\tau) d\tau \xleftrightarrow{\mathcal{L}} \frac{1}{s} \cdot X(s), \text{ ROC } \supset \{R \cap \Pi_0^+\}$$

p
polo en 00.

$$\Pi_{\lambda}^+ = \{s \in \mathbb{C} / \operatorname{Re}(s) > \lambda\}$$



Teoremas del valor inicial y final

(H) $x(t) = 0, \forall t < 0$ y $x(t)$ no tiene singularidades en cero.
 $x(t)$ es derecha

(T) * $x(0^+) = \lim_{s \rightarrow \infty} s \cdot X(s)$

* $\lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} s \cdot X(s)$

Ejemplo 9.16 (9.4) $x(t) = u(t) \cdot e^{-2t} + u(t) \cdot e^{-t} \cdot \cos(3t)$

$$X(s) = \frac{2s^2 + 5s + 12}{(s^2 + 2s + 10)(s + 2)}$$

1) evalúo en t : $x(0^+) = 2$

2) $\lim_{s \rightarrow \infty} s \cdot X(s) = \lim_{s \rightarrow \infty} \frac{s(2s^2 + 5s + 12)}{(s^2 + 2s + 10)(s + 2)} = \frac{2s^3}{s^3} = 2$ 2

PROPIEDADES (APLICADAS A SISTEMAS)

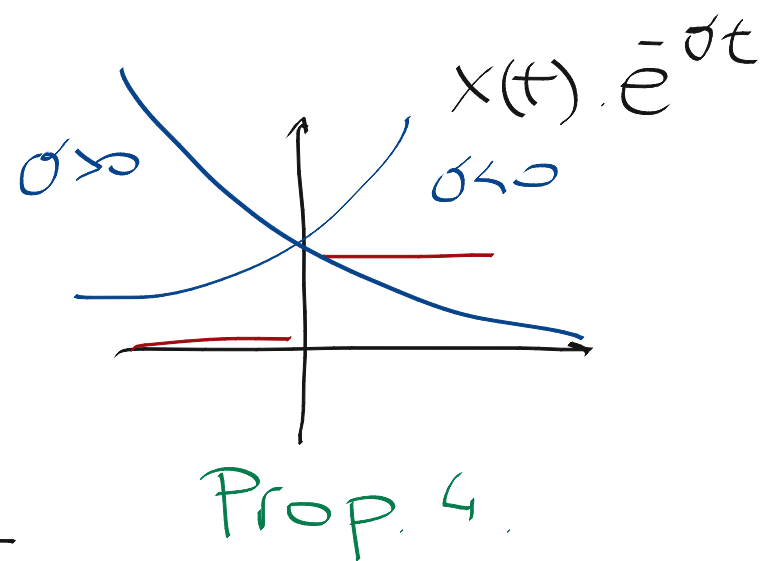
$$Y(s) = H(s) \cdot X(s) \leftarrow \text{función de transferencia}$$

$$s = j\omega \rightarrow Y(j\omega) = H(j\omega) \cdot X(j\omega)$$

$\uparrow$ resp. en frecuencia

Causalidad: $h(t) = 0, \forall t < 0$ (derecha)

$$H(s) = \int_{-\infty}^{\infty} h(t) \cdot e^{-st} dt = \int_0^{\infty} h(t) \cdot e^{-st} dt$$



1) $\textcircled{H}$ sistema causal $\textcircled{T}$ ROC es un Π^+

obs: al revés no pasa.

2) $\textcircled{H}$ ROC es un Π^+ $\textcircled{T}$ sistema causal
 $X(s)$ racional

Ejemplo 9.17

$$h(t) = u(t) \cdot e^{-t} \rightarrow \text{derecha} \rightarrow \text{causal} \checkmark$$

$$H(s) = \frac{1}{s+1}, \operatorname{Re}\{s\} > -1 \rightarrow \text{causal} \checkmark$$

Ejemplo 9.18

$$u(t) = e^{-|t|}, \text{ bilateral, } \underbrace{h(t) \neq 0, \forall t < 0}_{\text{no causal}} \times$$

$$H(s) = -\frac{2}{(s+1)(s-1)} = \frac{1}{s+1} - \frac{1}{s-1}, \underbrace{-1 < \operatorname{Re}(s) < 1}_{\text{no causal}} \times$$

Estabilidad

$$\mathcal{F}\{x(t) \cdot e^{-\sigma t}\} = \mathcal{L}\{x(t)\}$$

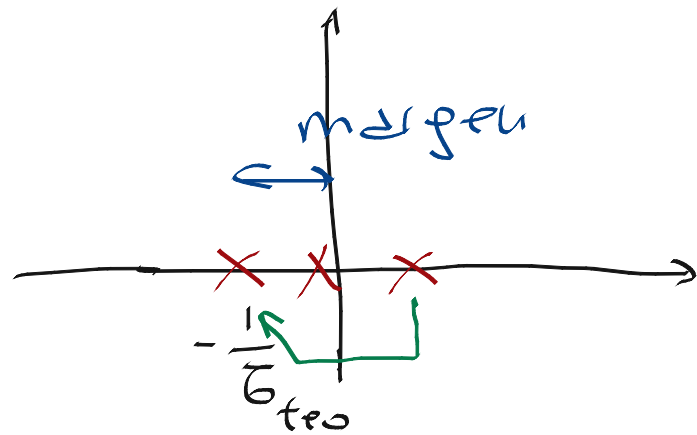
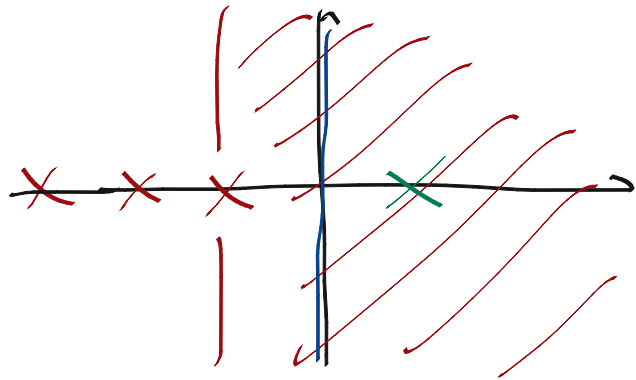
$\mathcal{F}$ es estable $\Leftrightarrow H(s)$ tiene una ROC $\supset L_0$

$$L_0 = \{s \in \mathbb{C} / \text{Re}(s) = \sigma\}$$

Lema:

Un sistema causal es estable $\Leftrightarrow$ sus polos $p_i \in \Pi_0^-$

$\sigma = \text{R.C.}$



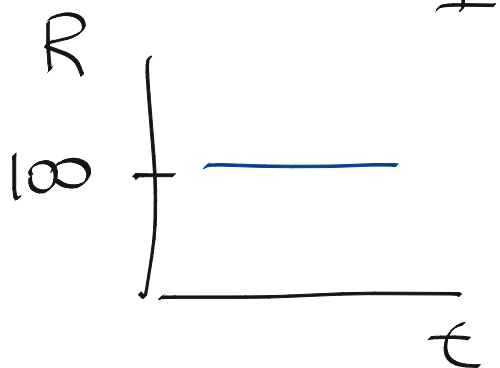
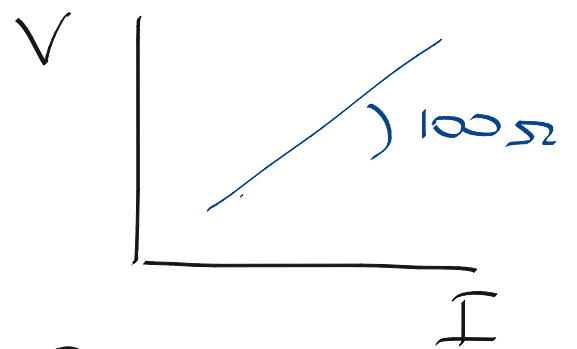
obs: * manera concreta de estudiar la estabilidad

* ¿qué tan estable es?
↳ margen de estabilidad

* Diseño de sistemas

↳ filtros: P.D.s.

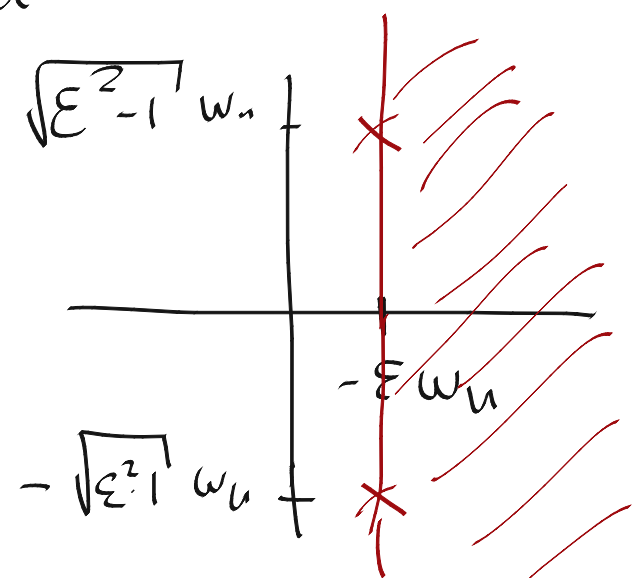
↳ modificación de sistemas: control (Mys)



Ejemplo 9.22: "Sist. de segundo orden"

$$p_i = -\varepsilon \omega_n \pm \sqrt{\varepsilon^2 - 1} \omega_n$$

si $\varepsilon < 0 \Rightarrow \operatorname{Re}\{p_i\} > 0$
 $\Rightarrow$ sist. inestable.



Obs: $\varepsilon = \sqrt{\frac{b}{2k_m}}$

Sistemas basados en ecuaciones diferenciales

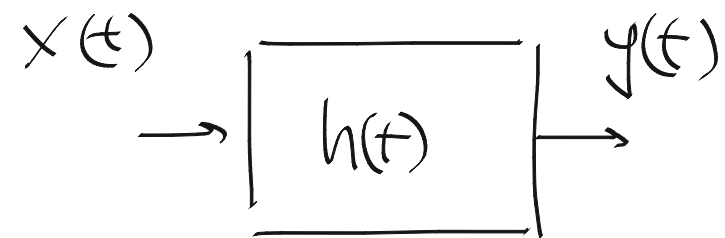
$$\underbrace{\sum_{n=0}^{N_p} d_n \cdot y^{(n)}(t)}_{\text{salida}} = \underbrace{\sum_{k=0}^{N_z} b_k \cdot x^{(k)}(t)}_{\text{entradas}}$$

$$\Rightarrow H(s) =$$

función
racional

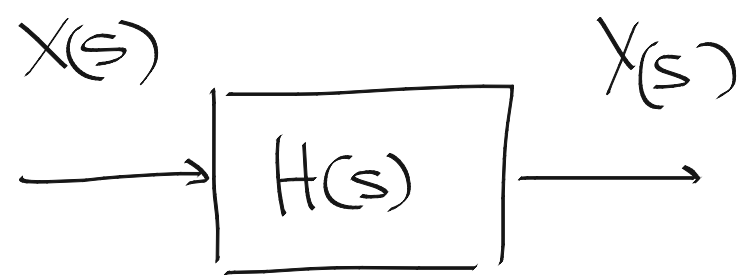
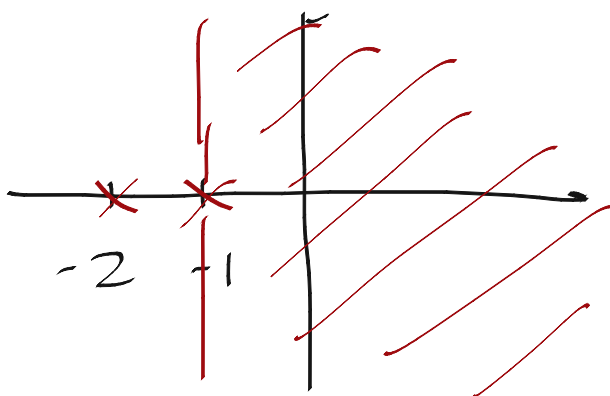
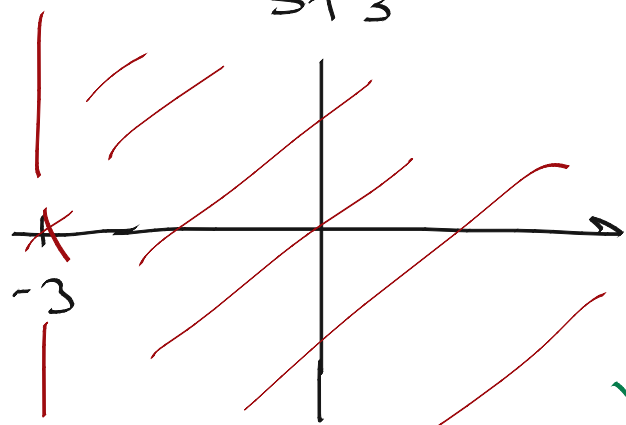
$$H(s) = \frac{\overbrace{\sum_{k=0}^{N_z} b_k \cdot s^k}^{\text{ceros}}}{\underbrace{\sum_{n=0}^{N_p} d_n \cdot s^n}_{\text{polos}}}$$

Ejemplo 9.25 $\begin{cases} x(t) = u(t) \cdot e^{-3t} \\ y(t) = u(t) \cdot (e^{-t} - e^{-2t}) \end{cases}$

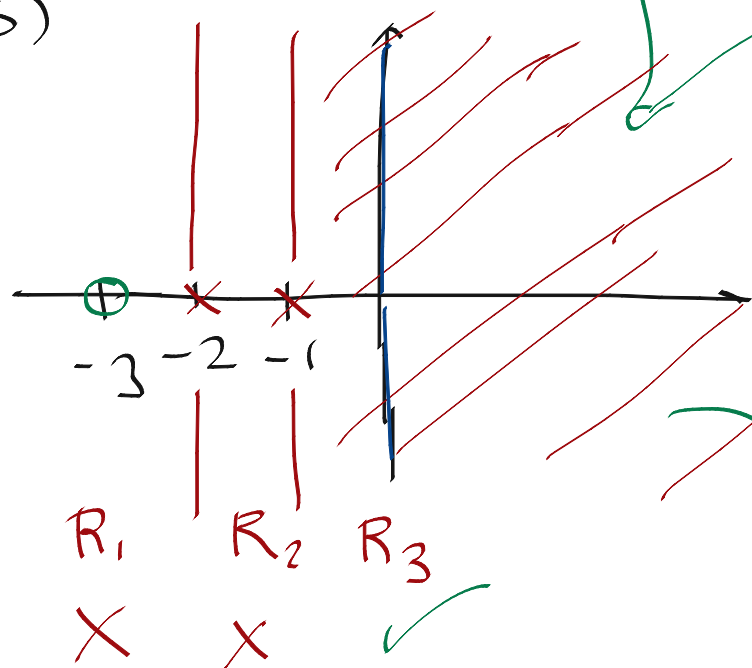


$$X(s) = \frac{1}{s+3}, \text{Re}(s) > -3$$

$$Y(s) = \frac{1}{s+1} - \frac{1}{s+2} = \frac{1}{(s+1)(s+2)}, \text{Re}(s) > -1$$



$H(s)$



prop. convolución

$\text{ROC} \supset R_1 \cap R_2$

$$\boxed{\text{Re}(s) > -1}$$

$$Y(s) = H(s) X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s+3}{(s+1)(s+2)}$$

obs: * identificación de sistemas

* $\text{ROC} \supset \Pi^+ \Rightarrow \text{causal}$

* $\text{Re}\{p_i\} < 0, \forall i \Rightarrow \text{estable}$

Clase 34

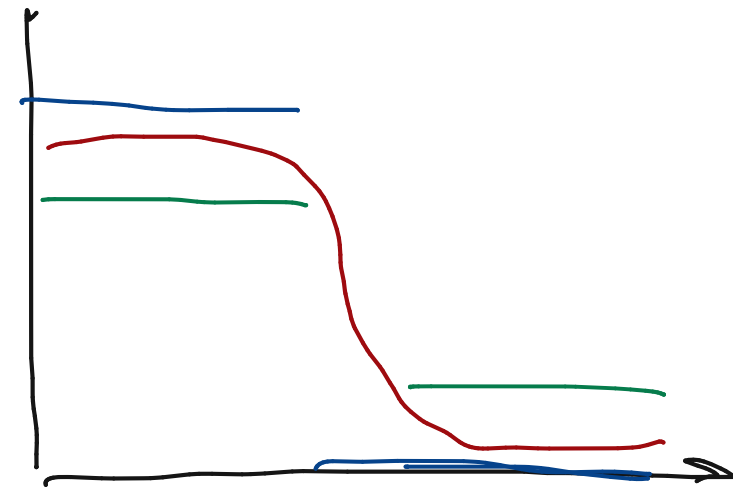
Transformada de Laplace

Aplicación: diseño de filtros

DISEÑO DE SISTEMAS (FILTROS)

Filtro Butterworth (pasabajos) (p 703)

$$|B(j\omega)|^2 = \frac{1}{1 + \left(\frac{j\omega}{j\omega_c}\right)^{2N}}$$

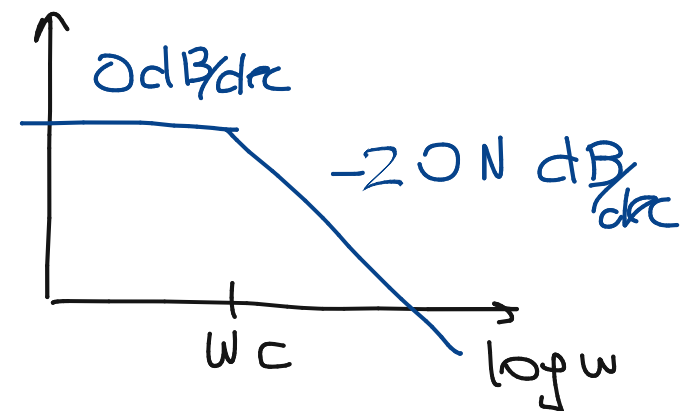


1) Bode

$$10 \log |B(j\omega)|^2 = -20 \log \left[1 + \left(\frac{j\omega}{j\omega_c}\right)^{2N} \right]$$

$$a) \omega \gg \omega_c \Rightarrow -20 \log \left(\frac{\omega}{\omega_c}\right)^{2N} = -20N \log\left(\frac{\omega}{\omega_c}\right)$$

$$b) \omega \ll \omega_c \Rightarrow -10 \log(1) = 0$$



2) Transferencia

* Hay infinitas $B(j\omega)$ que dan ese módulo

$$\rightarrow |B(j\omega)|^2 = B(j\omega) B^*(j\omega)$$

$$\rightarrow \text{Asumo que } b(t) \in \mathbb{R} \Rightarrow B^*(j\omega) = B(-j\omega)$$

$$B(j\omega) B(-j\omega) = \frac{1}{1 + \left(\frac{j\omega}{j\omega_c}\right)^{2N}} \quad \text{y sabemos que } s = j\omega$$

$$B(s) B(-s) = \frac{1}{1 + \left(\frac{s}{j\omega_c}\right)^{2N}} = P(s) \rightarrow \text{función racional con } 2N \text{ polos}$$

3) Polos $P(s)$

$$1 + \left(\frac{s}{\omega_c}\right)^{2N} = 0 \Rightarrow \frac{s}{\omega_c} = \sqrt[2N]{-1} \Rightarrow s_p = \sqrt[2N]{-1} \omega_c, \quad \omega_c = 1 e^{j\frac{\pi}{2}}$$

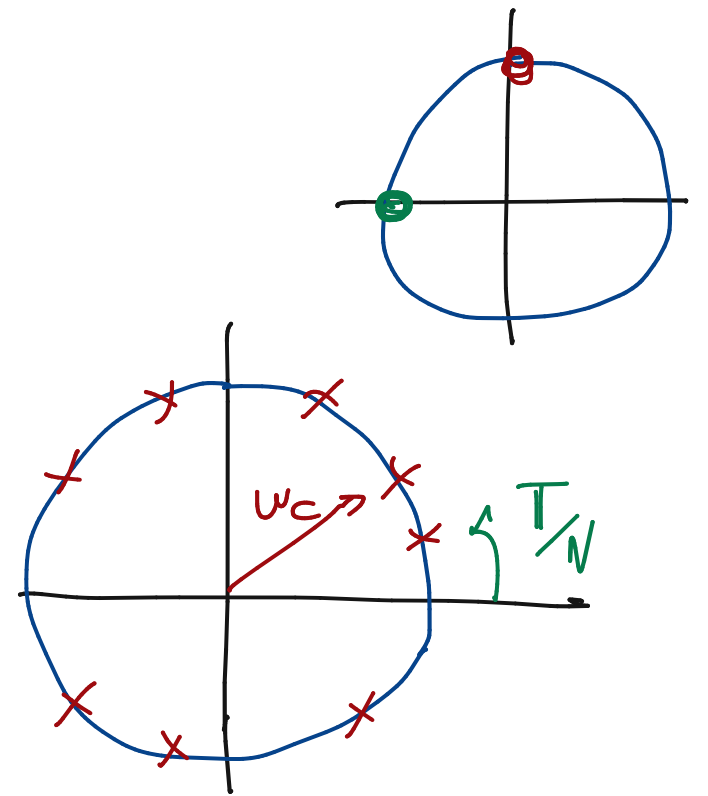
$$s_p = \sqrt[2N]{1 e^{j(\pi+2k\pi)}} e^{j\frac{\pi}{2}} \omega_c$$

$$s_p = \omega_c e^{j\left(\frac{\pi}{2} + \frac{\pi}{2N} + k \frac{\pi}{N}\right)}$$

obs * hay $2N$ polos equispaciados

* módulo ω_c y ángulo $\frac{\pi}{N}$

* nunca cae en $j\omega$ y en 0 para N impar



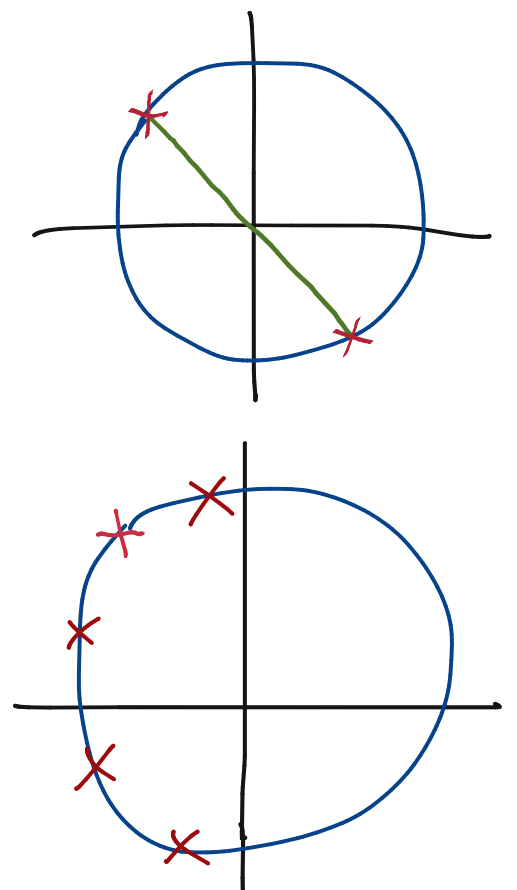
4) Polos de $B(s)$

* Si s_p es polo de $B(s) \Rightarrow -s_p$ es polo de $B(-s)$

* tengo que elegir N polos

* si fuere necesario sea causal y estable
 $\hookrightarrow$ es decir $\text{Re}(s_p) < 0$

* con los polos puedo armar el $B(s)$

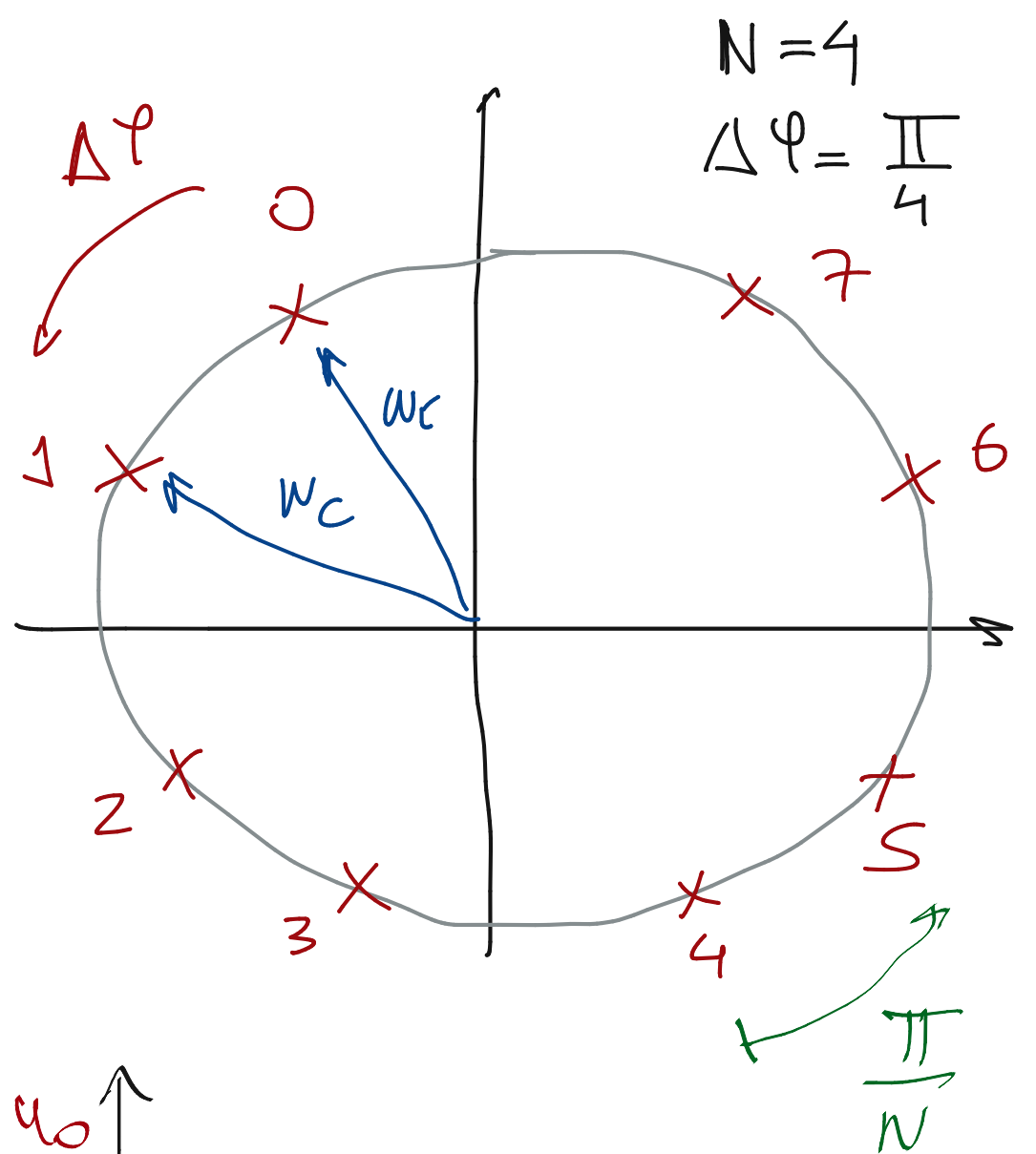


5) Transferencia (exp. algebra)

* polos $\rightarrow \varphi_0 = \frac{\pi}{2} + \frac{\pi}{2N}$, $\Delta\varphi = \frac{\pi}{N}$

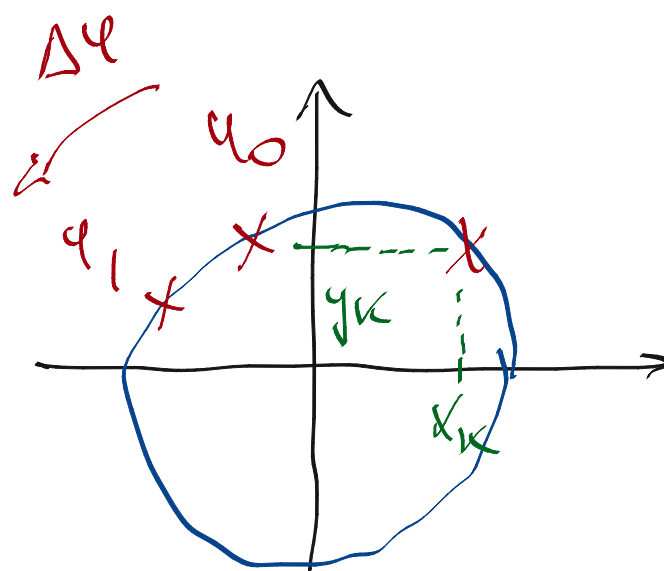
$\rightarrow \varphi_k = \varphi_0 + k \Delta\varphi$

$\rightarrow \boxed{s_p = \Omega_c e^{j\varphi_k}}_{k=0 \dots 3}$



* $s_p^k = x_k + j y_k$

$\boxed{\begin{aligned} x_k &= \Omega_c \cos \varphi_k \\ y_k &= \Omega_c \sin \varphi_k \end{aligned}}$



*

$$\mathcal{B}(s) = \frac{1}{\prod_{k=0}^{N-1} (s - s_p^k)}$$

Ejemplo

$$\delta_1 = 0.8, \delta_2 = 0.15$$

$$f_p = 10 \text{ Hz}, f_s = 30 \text{ Hz}$$

$$\omega_p = 2\pi 10 \text{ rad/s}, \omega_s = 2\pi 30 \text{ rad/s}$$

$$\textcircled{1} \quad |B(j\omega_p)|^2 \geq \delta_1^2$$

$$\textcircled{2} \quad |B(j\omega_s)|^2 \leq \delta_2^2$$

Obs

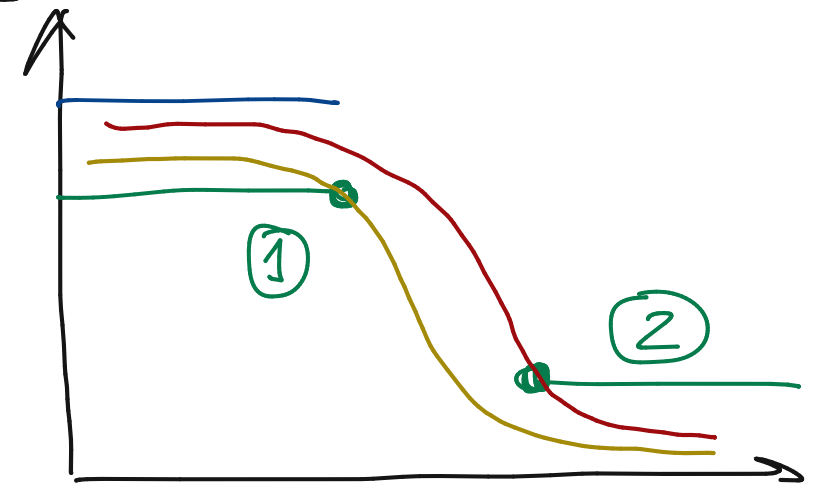
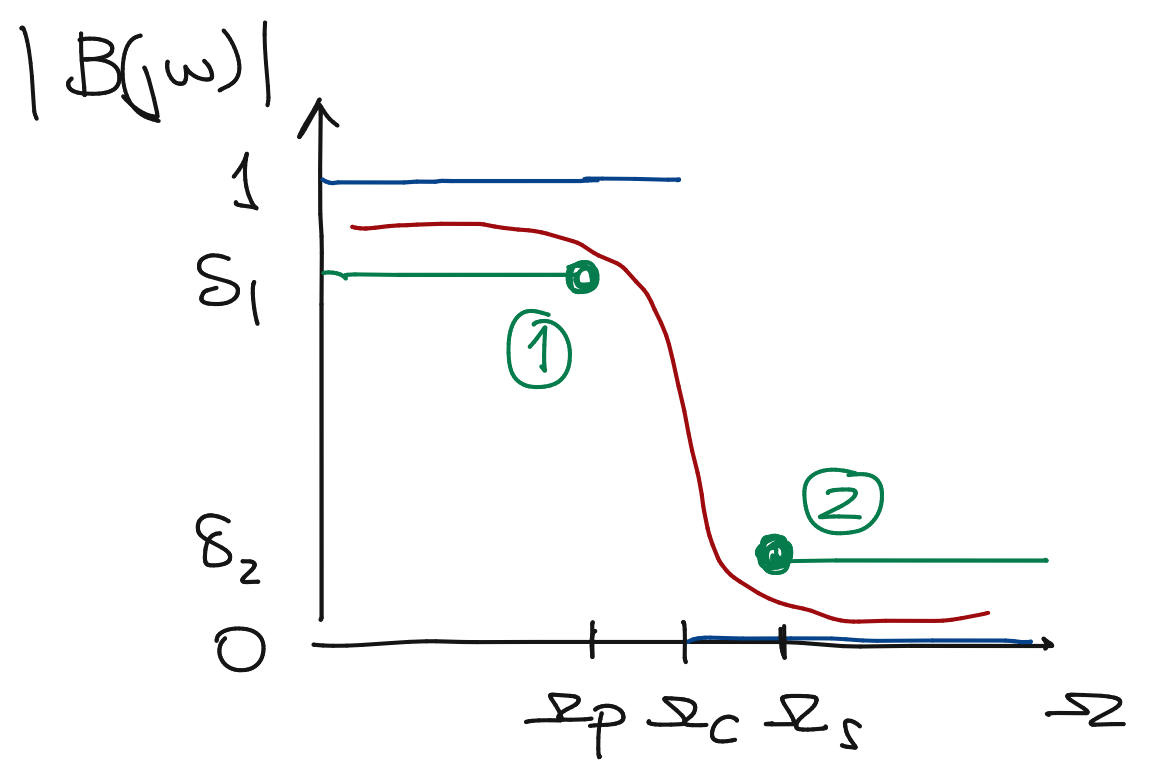
* incógnitas N, ω_c

* Sistema de 2x2 ecuaciones

pasos

* cálculo N

* cálculo ω_c



Diseño

* cálculo de N

① impongo la igualdad $|B(j\omega_p)|^2 = \frac{1}{1 + \left(\frac{\omega_p}{\omega_c}\right)^{2N}} = \delta_1^{-2}$

$$1 + \left(\frac{\omega_p}{\omega_s}\right)^{2N} = \delta_1^{-2} \Rightarrow 2N \log\left(\frac{\omega_p}{\omega_s}\right) = \underbrace{\log(\delta_1^{-2} - 1)}_{\text{cte}} \xrightarrow{\text{defino}} d_1 = \log(\delta_1^{-2} - 1)$$

$$2N \log\left(\frac{\omega_p}{\omega_c}\right) = d_1 \quad \textcircled{A}$$

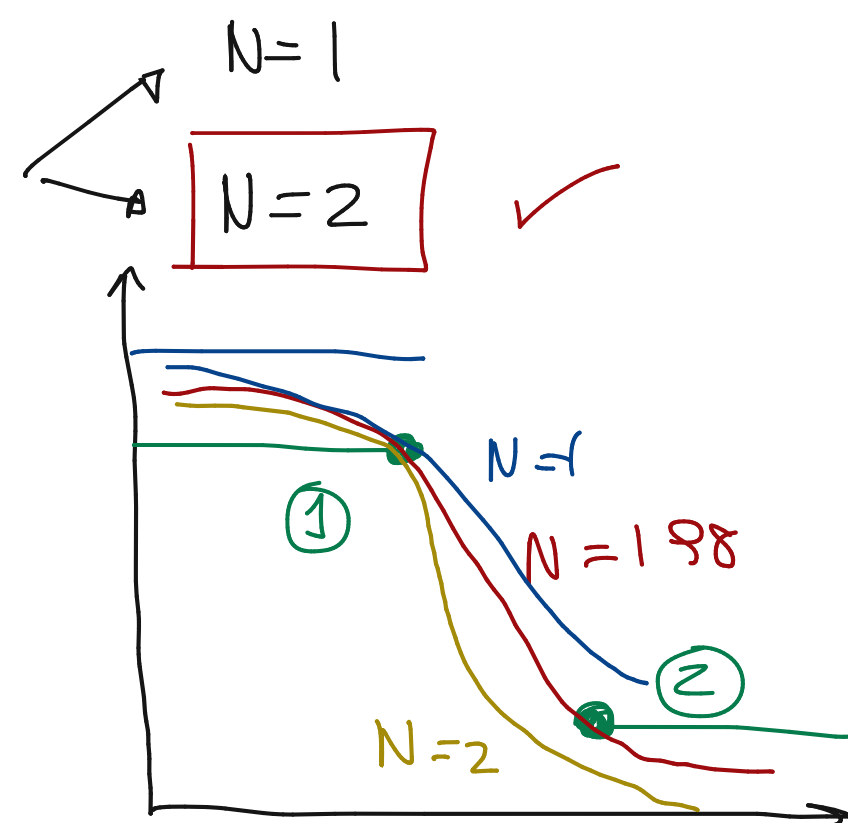
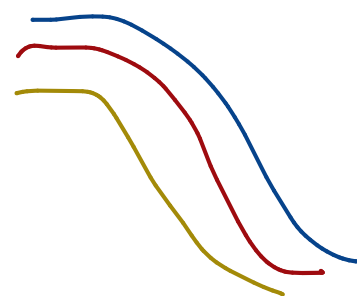
② $2N \log\left(\frac{\omega_s}{\omega_c}\right) = d_2 \quad \textcircled{B}$

$$\textcircled{A} - \textcircled{B} = 2N \left(\log \omega_p - \cancel{\log \omega_c} - \log \omega_s + \cancel{\log \omega_c} \right) = d_1 - d_2$$

$$2N \log\left(\frac{\omega_p}{\omega_s}\right) = d_1 - d_2$$

$$N = \frac{d_1 - d_2}{2 \log\left(\frac{\omega_p}{\omega_s}\right)} \Rightarrow N = 198$$

obs $|B(j\omega)|$ decrece con N



* Cálculo de Σ_c (lo hago en ⑦)

$$2N \log\left(\frac{\Sigma_p}{\Sigma_c}\right) = d_1 \Rightarrow \log\left(\frac{\Sigma_p}{\Sigma_c}\right) = \frac{d_1}{2N}$$

$$\log\left(\frac{\Sigma_c}{\Sigma_p}\right) = -\frac{d_1}{2N} \Rightarrow$$

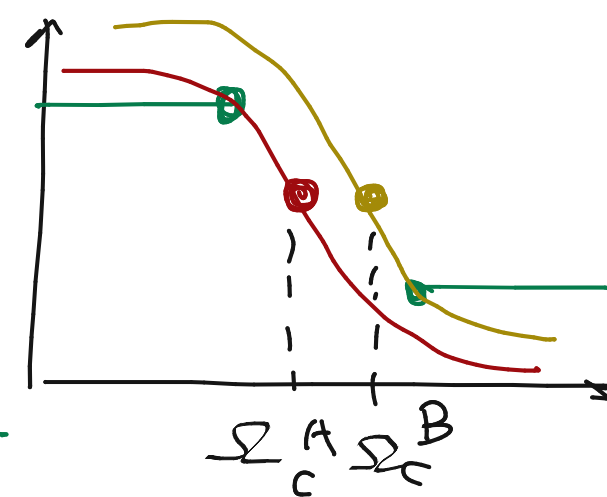
$$\Sigma_c = \Sigma_p e^{-\frac{d_1}{2N}}$$

$$\Sigma_c = 72 \text{ SS rads}$$

$$\rightarrow f_c = 11 \text{ SS Hz}$$

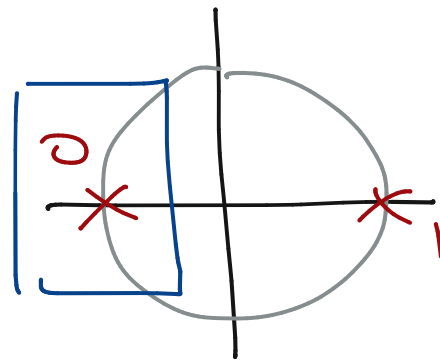
obs si resuelvo en ③

$$f_c = 1168$$



Transfencia

1) $N=1$, $s_p' = \omega_c e^{j\pi} = -\omega_c$
 $s_p^2 = \omega_c e^{j\pi+\pi}$



$$B(s) = \frac{1}{1 + \frac{s}{\omega_c}}$$

$$\frac{1}{1 + \tau s}$$

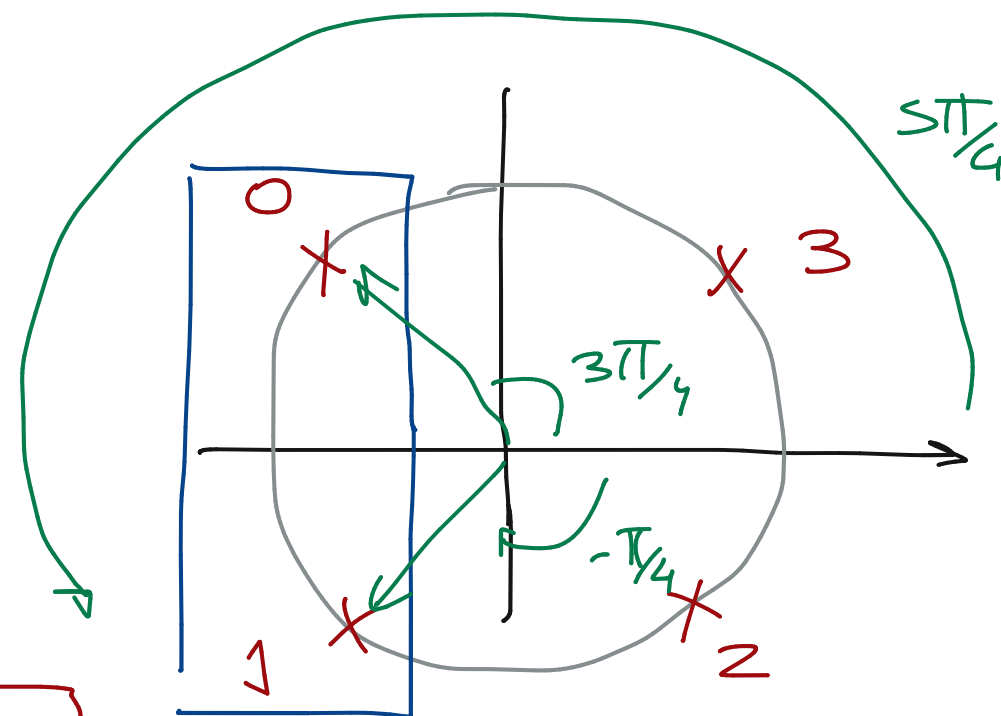
$$\Rightarrow \tau = \frac{1}{\omega_c}$$

2) Sistema de segundo orden $N=2$

$$N=2, \Delta\varphi = \frac{\pi}{2}, \varphi_0 = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$$

$$s_p^0 = \omega_c e^{j\varphi_0} = \omega_c e^{j\frac{3\pi}{4}}$$

$$s_p^1 = \omega_c e^{j(\varphi_0 + \frac{\pi}{2})} = \omega_c e^{j\frac{5\pi}{4}}$$



$$B(s) = \frac{1}{(s + \omega_c e^{j\frac{3\pi}{4}})(s + \omega_c e^{j\frac{5\pi}{4}})}$$

$$e^{j\frac{3\pi}{4}} + e^{j\frac{5\pi}{4}}$$

$$e^{j\frac{3\pi}{4}} + e^{-j\frac{3\pi}{4}}$$

$$B(s) = \frac{1}{s^2 + 2\omega_c \underbrace{\cos\left(\frac{3\pi}{4}\right)}_{\varepsilon} s + \omega_c^2}$$

$$\varepsilon = \frac{\sqrt{2}}{2} \approx 0.7 \rightarrow \text{subamortiguado}$$

Implementación

Ejemplo

* desarrollo

$$\begin{array}{ccc} & -1 & -2 \\ & \downarrow & \downarrow \\ (s+1)(s+2) & = & s^2 + 3s + 2 \\ \downarrow & \downarrow & \downarrow \\ [1 \ 1] & [1 \ 2] & = [1 \ 3 \ 2] \end{array}$$

convolve

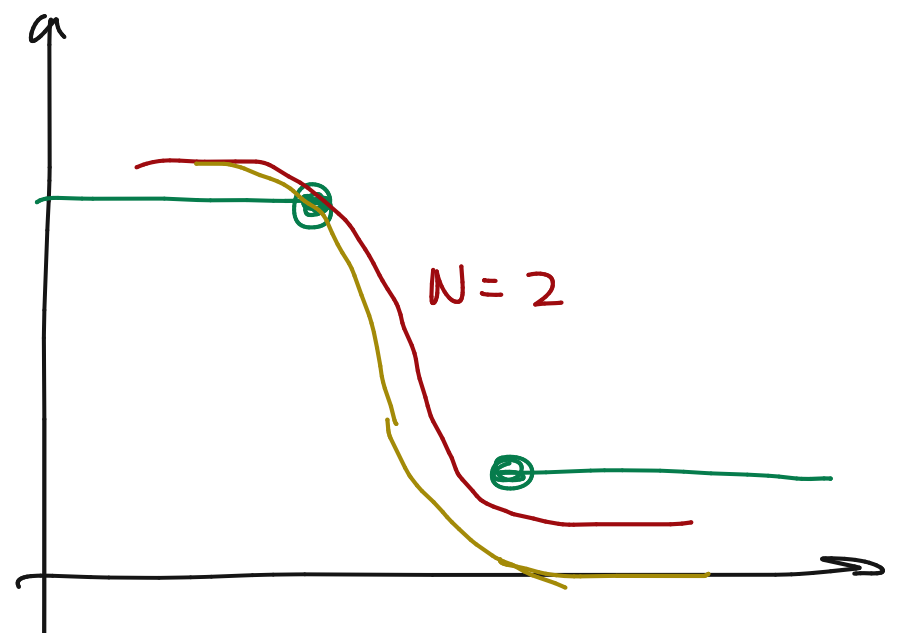
* raíces

$$[1 \ 3 \ 2] \rightarrow [-1 \ -2]$$

roots

Ejemplo

$$B(s) = \frac{1}{s^2 + 10s + 26}$$



Implementación futura:

*

Butter ()

RESUME

N

La place

- * generalización de $\mathcal{F}$ → sistemas más generales
- * completa el análisis → estabilidad y causalidad
→ diseño
- * DFC + ROC estudio completo del sistema

RESUME

Curso

N

"Aprender a describir señales y sistemas"

- * Definición de señales y sistemas
- * Estudio y caracterización de sistemas
- * Análisis de SLITS en el tiempo
- * Análisis espectral de señales periódicas
 - tiempo continuo y discreto
- * Análisis espectral de señales **no**-periódicas
 - tiempo continuo y discreto
- * Muestreo relación entre TC y TD
- * Análisis general (Laplace)
 - tiempo continuo
 - sistemas muestreados
 - respuestas transitorias
- * Análisis combinado tiempo y frecuencia
 - reforzamientos de filtros reales
 - diseño de sistemas

RESUME

Futuro

N

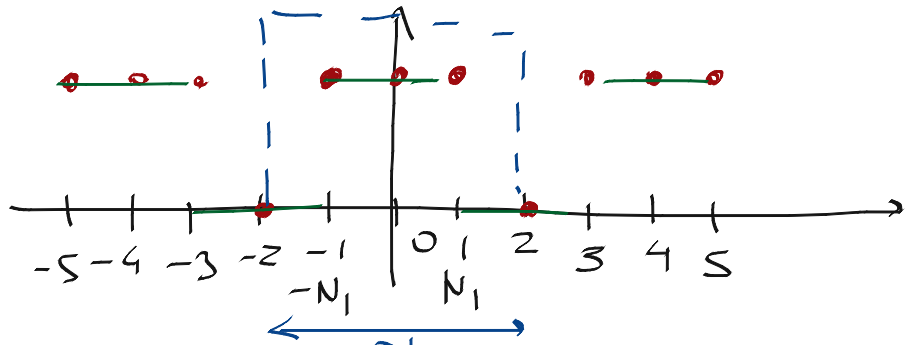
- * Señales no estacionarias transformadas Tiempo $\mathcal{PDS}$
Frecuencia
- * Señales no determinísticas procesos estocásticos $\mathcal{PDS}$
- * sistemas no lineales aproximación **CONTROL**
- * Completar el análisis para tiempo discreto Trans $\mathcal{Z}$ $\mathcal{PDS}$
- * Implementación Digital PC, sistemas embebido, E Digital $\mathcal{PDS}$
- * Procesamiento de señales Diseño de filtros $\mathcal{PDS}$
- * Control → Diseño de sistemas
→ Diseño óptimo **CONTROL**
- * Modelos y simulación → Modelado
→ Identificación **MSC**
→ Simulación

Fin

Ejemplo 3.12

$$N=4, N_1=1$$

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} x[n] \cdot e^{-jkw_0 n}$$



$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} e^{-jkw_0 n} = \frac{1}{N} \sum_{m=0}^{2N_1} e^{-jkw_0 (m-N_1)} = \frac{1}{N} \sum_{m=0}^{2N_1} e^{-jkw_0 m} e^{+jkw_0 N_1}$$

$$a_k = \frac{1}{N} \cdot e^{jkw_0 N_1} \sum_{m=0}^{2N_1} \underbrace{(e^{-jkw_0})^m}_r = \frac{1}{N} \cdot e^{jkw_0 N_1} \frac{1-r^{2N_1+1}}{1-r}$$

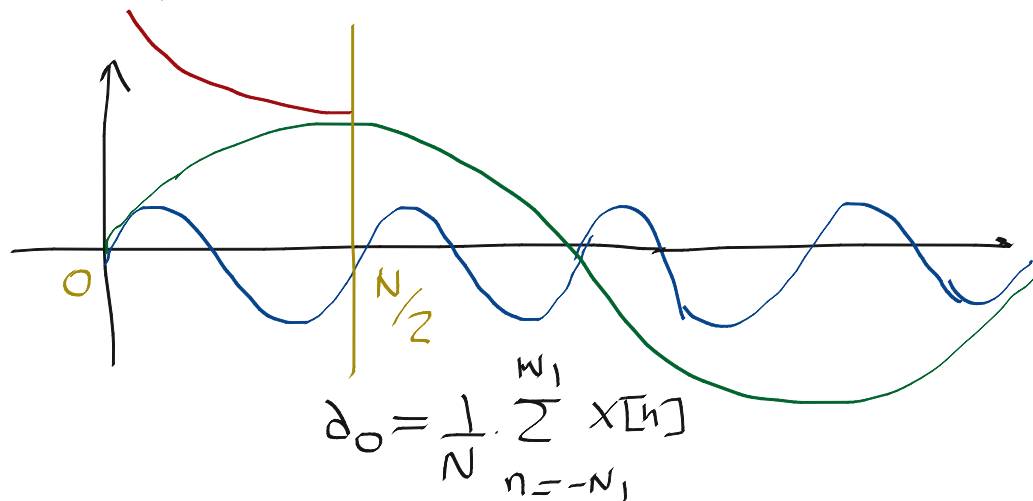
$$a_k = \frac{1}{N} \cdot e^{jkw_0 N_1} \frac{1 - e^{-jkw_0 (2N_1+1)}}{1 - e^{-jkw_0}} = \frac{1}{N} \cdot \frac{e^{jkw_0 N_1} - e^{-jkw_0 (N_1+1)}}{(1 - e^{-jkw_0})} \cdot e^{j\frac{kw_0}{2}}$$

$$a_k = \frac{1}{N} \frac{e^{jkw_0 (N_1+\frac{1}{2})} - e^{-jkw_0 (N_1+\frac{1}{2})}}{e^{j\frac{kw_0}{2}} - e^{-j\frac{kw_0}{2}}} \cdot \frac{z_i}{z_j}$$

$$a_k = \frac{1}{N} \frac{\sin \left[\left(N_1 + \frac{1}{2} \right) \omega_0 \cdot k \right]}{\sin \left[\frac{\omega_0}{2} \cdot k \right]}$$

$$d_k = \frac{1}{4} \cdot \frac{\sin\left[\frac{3}{2} \cdot \frac{2\pi}{4} k\right]}{\sin\left[\frac{2\pi}{4 \cdot 2} k\right]} = \frac{1}{4} \cdot \frac{\sin\left[\frac{3\pi}{4} k\right]}{\sin\left[\frac{\pi}{4} k\right]}$$

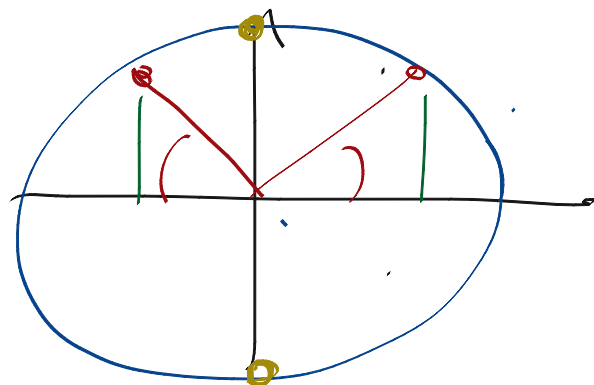
k	*	*	d_k
0	—	—	$\frac{3}{4}$
1	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{4}$
2	—	—	$-\frac{1}{4}$
3	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{4}$
4	—	—	$\frac{1}{4}$
5	—	—	$\frac{3}{4}$
6	—	—	$\frac{3}{4}$



$$d_0 = \frac{1}{N} \sum_{n=-N_1}^{N_1} x[n]$$

$$d_0 = \frac{1}{N} \sum_{n=-N_1}^{N_1} 1 = \frac{2N_1 + 1}{N} = \frac{3}{4}$$

$$\frac{9\pi}{4} - \frac{8\pi}{4} = \frac{\pi}{4}$$



$$\sin\left[\frac{\omega_0}{2} k\right] = \sin\left[\frac{2\pi}{N \cdot 2} k\right]$$

$$\sin\left[\frac{\pi}{N} k\right]$$

$$\frac{\pi}{2} = \frac{\pi}{N} k$$

$$k = \frac{N}{2}$$

Anexo