Clase 18

Transformada de Fourier

Introducción

RESUMEN FOURIER

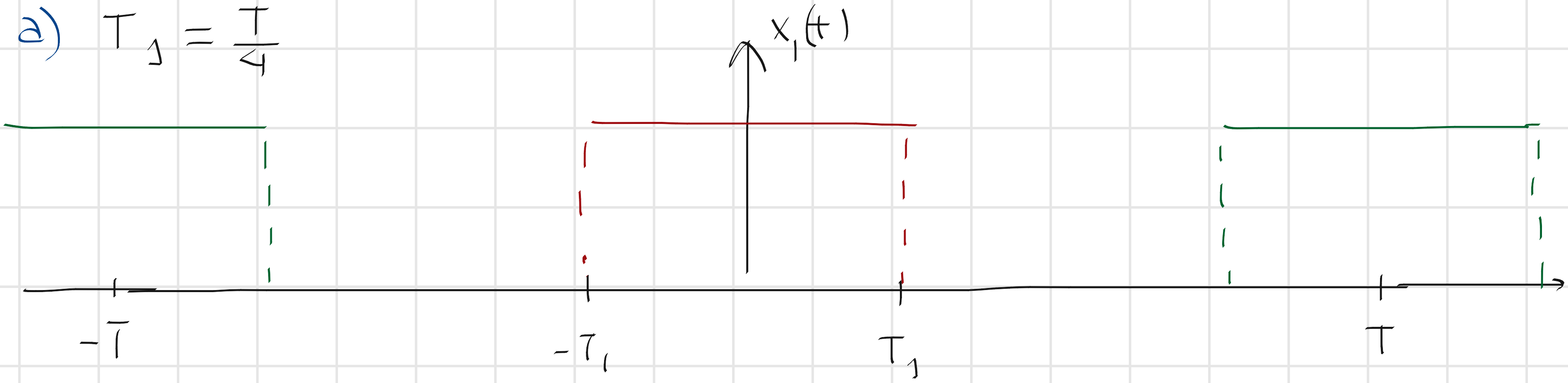
- Descripción alternativa de sistemas
- Exp. son funciones propias de SLIT
- Toda señal periódica admite una desc. como S.F.
- descomposición y superposición
- entrada periódica ⇒ salida periódica
- Resp. en frecuencia
- Estudiamos sistemas: filtros
- Aplicaciones:
  - → conformadores vs selectivos
  - → déf. parámetros
  - → sistemas comunes (op. matemáticas)
  - → sistemas reales (basados en ec.d.f)
  - → Aplicaciones reales

REPRESENTACIÓN DE SEÑALES APERIÓDICAS CONTINUAS

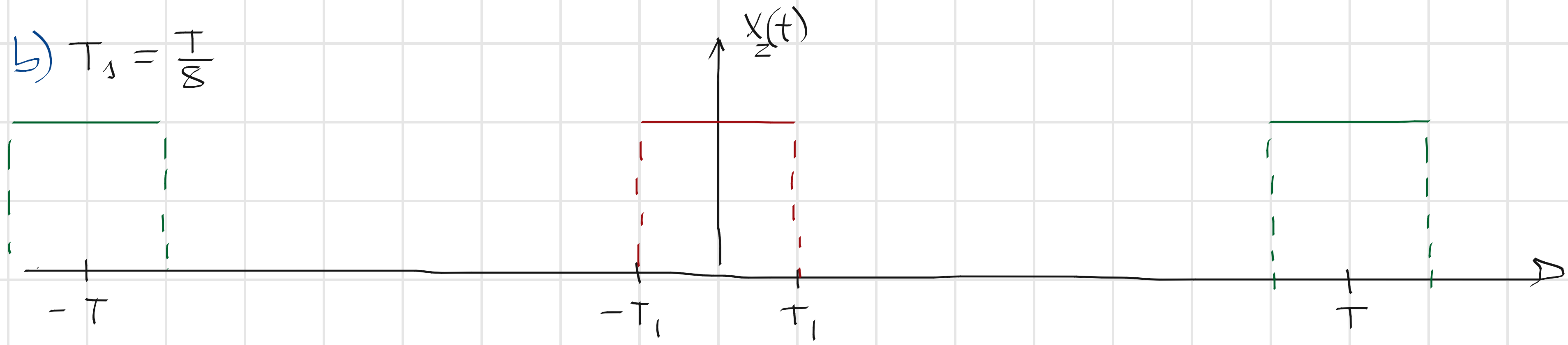
Repaso ej 3.5

"T fijo, T_1 variable"

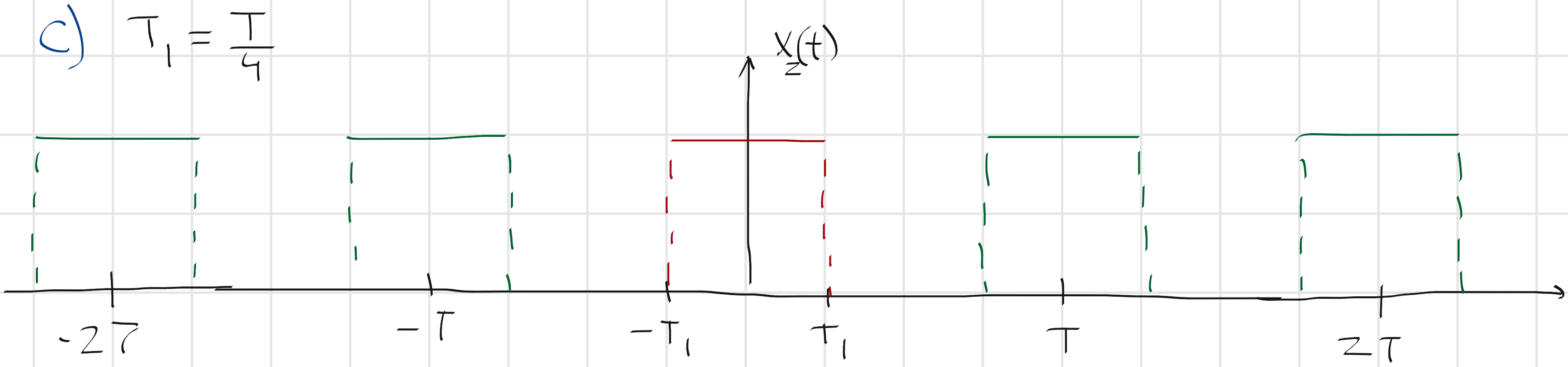
a) $T_1 = \frac{T}{4}$



b) $T_1 = \frac{T}{8}$



c) $T_1 = \frac{T}{4}$



Ejemplo 3.5 (cont)

$$a_k = \begin{cases} \dfrac{2}{T} \cdot \dfrac{\operatorname{sen}[k\omega_0 T_1]}{k\omega_0}, & k \neq 0 \\ \dfrac{2T_1}{T}, & k = 0 \end{cases} \qquad \boxed{\omega_0 = \frac{2\pi}{T}}$$

a) $T_1 = 2$, $T = 8 \Rightarrow \omega_0 = \frac{\pi}{4}$

$$a_k = \frac{1}{4} \cdot \frac{\operatorname{sen}\left[\frac{\pi}{2}k\right]}{k \cdot \frac{\pi}{4}} = \frac{\operatorname{sen}\left[\frac{\pi}{2}k\right]}{\pi k}, \quad a_0 = \frac{1}{2}$$

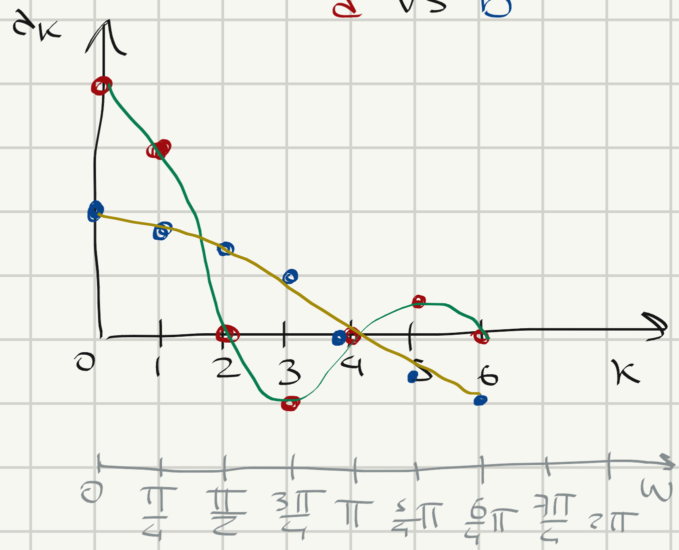
b) $T_1 = 1$, $T = 8 \Rightarrow \omega_0 = \frac{\pi}{4}$

$$a_k = \frac{1}{4} \frac{\operatorname{sen}\left[\frac{\pi}{4}k\right]}{\frac{\pi}{4}k} = \frac{\operatorname{sen}\left[\frac{\pi}{4}k\right]}{\pi k}, \quad a_0 = \frac{1}{4}$$

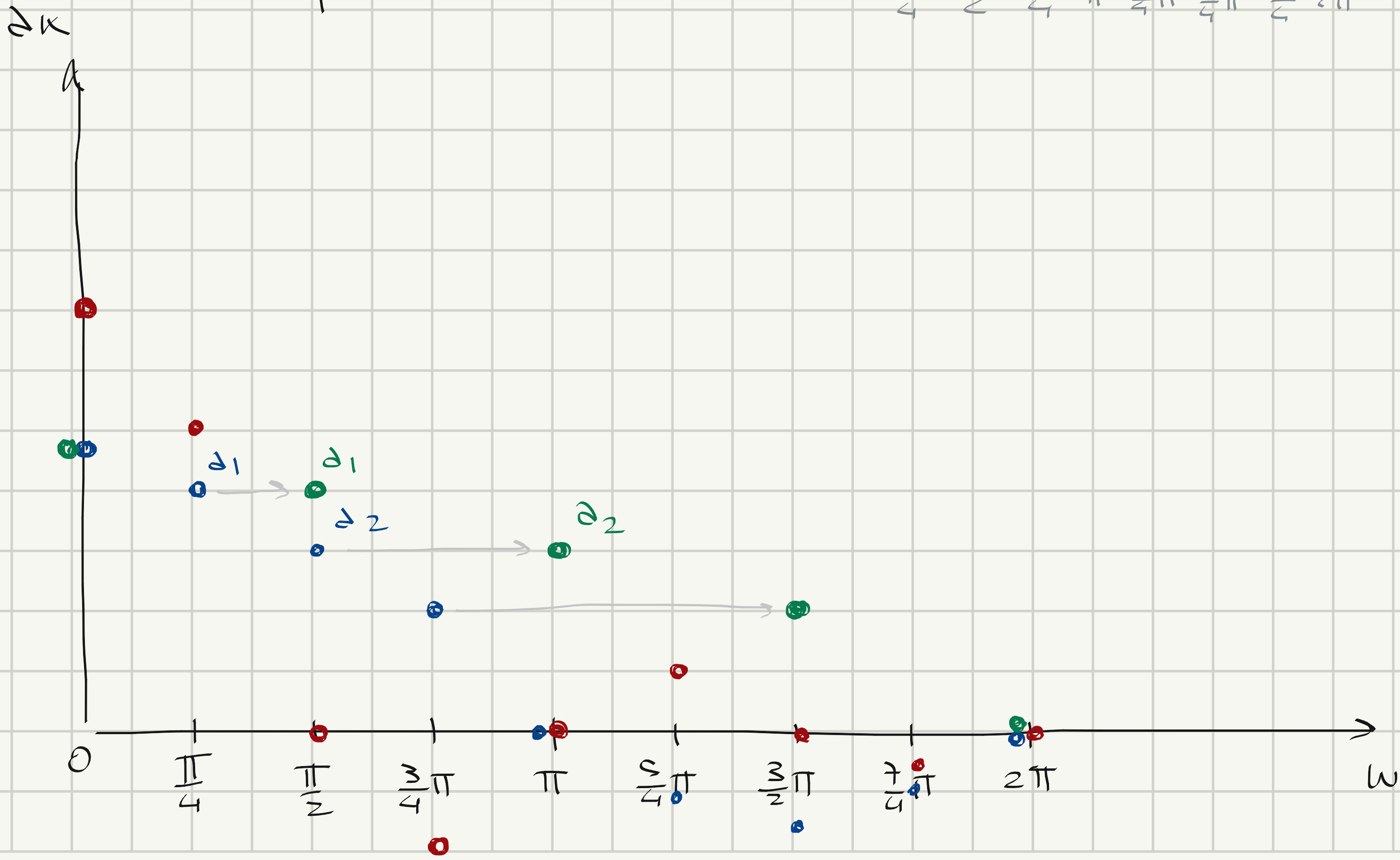
c) $T_1 = 1$, $T = 4 \Rightarrow \omega_0 = \frac{\pi}{2}$

$$a_k = \frac{\operatorname{sen}\left[\frac{\pi}{2}k\right]}{\pi k}, \quad a_0 = \frac{1}{2}$$

a vs b



Comparacion: a b c



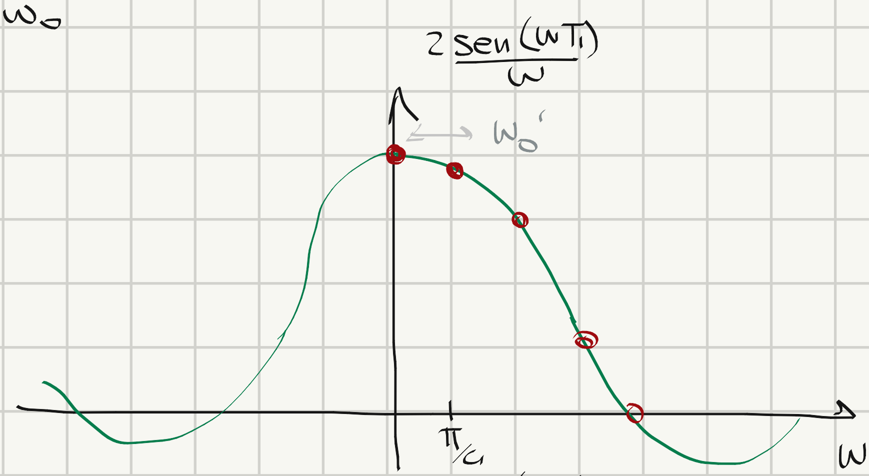
$$a_k = \frac{2}{T}\,\frac{\text{sen}\,[k\omega_0 T_1]}{k\omega_0} \longrightarrow Ta_k = 2\cdot\frac{\text{sen}\,[k\omega_0 T_1]}{\underbrace{k\omega_0}_{\omega}}$$

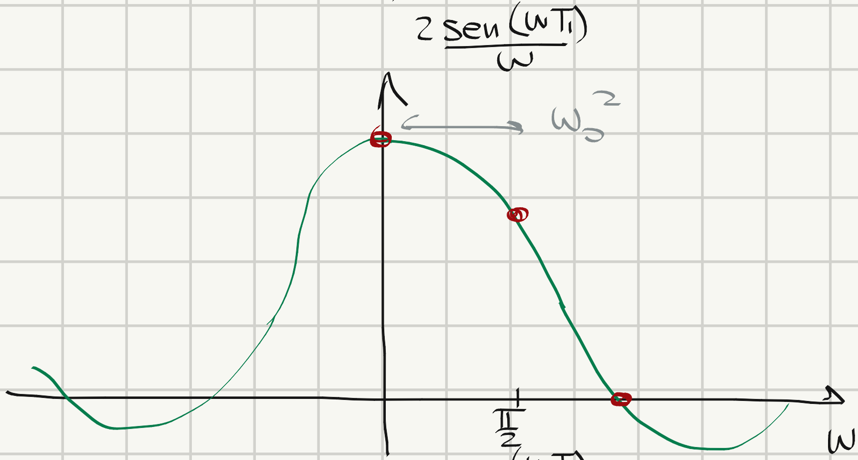
$$Ta_k = 2\,\frac{\text{sen}(\omega T_1)}{\omega}\Bigg|_{\omega = k\omega_0}$$

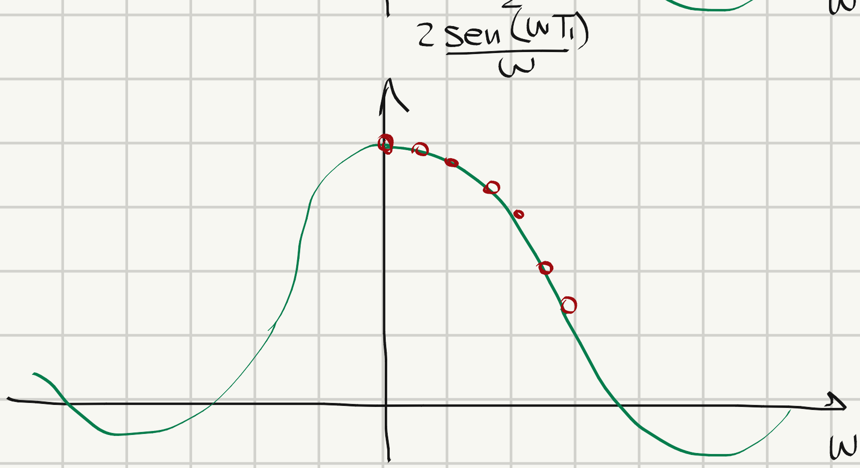
b) $T=8$, $\omega_0^1 = \frac{\pi}{4}$

c) $T=4$, $\omega_0^2 = \frac{\pi}{2}$

$T \to \infty$, $\omega_0 \to 0$

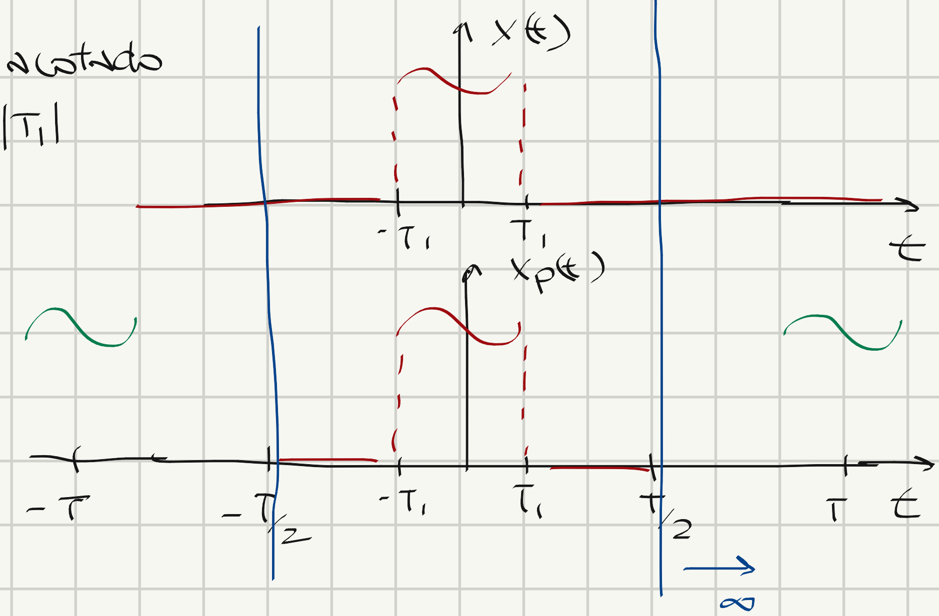






Idea: $x(t)$ soporte acotado

$x(t) = 0 \quad , \quad \forall t > |T_1|$



$x_p(t) = x(t) \quad , \quad |t| < T/2$

periódica T

$$\lim_{T \to \infty} x_p(t) = x(t)$$

a)

$$x_p(t) = \sum_{k=-\infty}^{\infty} a_k \cdot e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} x_p(t) \cdot e^{-jk\omega_0 t}$$

$$T a_k = \int_{-T/2}^{T/2} x_p(t) \cdot e^{-j \overset{\sim \omega}{k\omega_0} t} dt \Rightarrow \lim_{T \to 0} T a_k = \underbrace{\int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt}_{\text{función de } \omega} = X(j\omega)$$

$$\boxed{X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \cdot dt}$$ ec. análisis

$$\boxed{a_k = \frac{1}{T} \cdot X(j\omega) \Big|_{\omega = k\omega_0}}$$

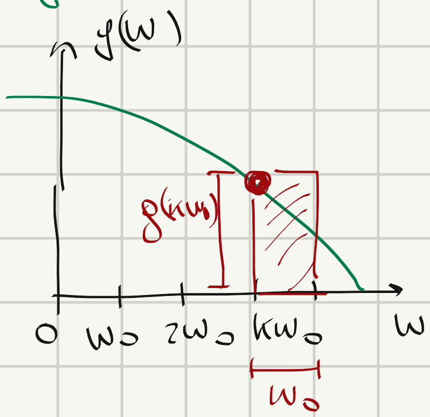
b)

$$x_p(t) = \sum_{k=-\infty}^{\infty} a_k \cdot e^{jk\omega_0 t}$$

$$\omega_0 = \frac{2\pi}{T} \rightarrow \frac{1}{T} = \frac{\omega_0}{2\pi}$$

$$\lim_{T\to\infty} \sum_{k=-\infty}^{\infty} \frac{1}{T} \cdot X(jk\omega_0) \cdot e^{jk\omega_0 t} = \lim_{T\to\infty} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \underbrace{X(j\overset{\omega}{\overline{k\omega_0}}) \cdot e^{j\overset{\omega}{\overline{k\omega_0}} t}}_{g(\omega)} \cdot \omega_0$$

$$\lim_{T\to\infty} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \underbrace{g(k\omega_0) \cdot \omega_0}_{\text{área}} = \int_{-\infty}^{\infty} g(\omega) \cdot d\omega$$



$$\boxed{x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) \cdot e^{j\omega t} \cdot d\omega}$$

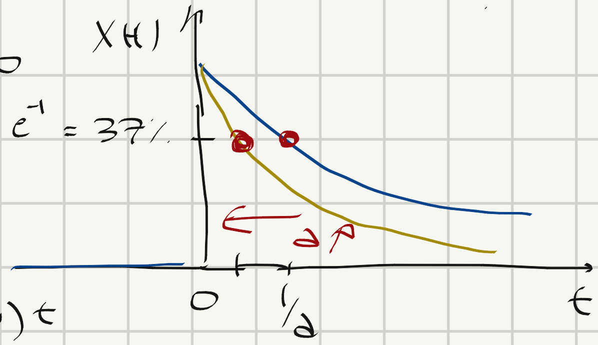
ec. síntesis

Definición

* ec. análisis: $$\boxed{X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \cdot dt}$$

* ec. síntesis: $$\boxed{x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) \cdot e^{j\omega t} \cdot d\omega}$$

Ejemplo 4.1 $\quad x(t) = u(t)\cdot e^{-at}, \; a>0$



$$X(j\omega)=\int_{-\infty}^{\infty} x(t)\cdot e^{-j\omega t}\cdot dt$$

$$X(j\omega)=\int_0^\infty e^{-at}\cdot e^{-j\omega t}\cdot dt = \int_0^\infty e^{-(a+j\omega)t}\cdot dt$$

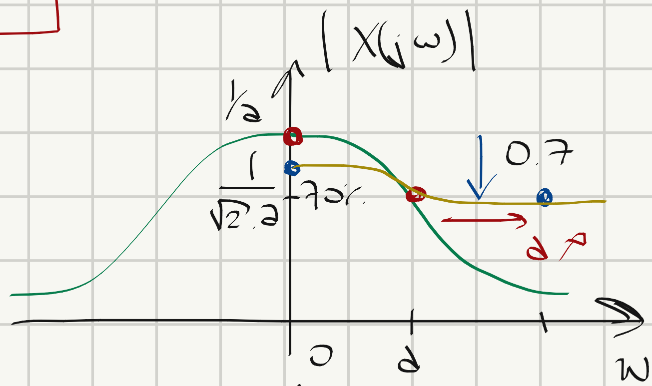
$$X(j\omega) = -\frac{e^{-(a+j\omega)t}}{a+j\omega}\Bigg|_0^\infty = \boxed{\frac{1}{a+j\omega}}$$

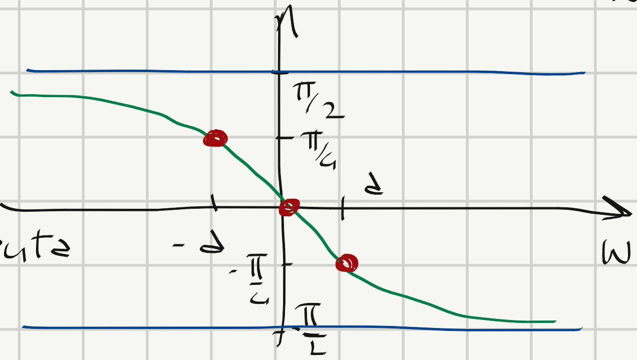
$\to 0$

$$\frac{1}{1+j\omega\tau} \qquad \tau \sim \frac{1}{a}$$

$$|X(j\omega)| = \boxed{\frac{1}{\sqrt{a^2+w^2}}}$$

$$\Theta(w) = \angle X(j\omega) = \boxed{-\operatorname{atan}\left(\frac{w}{a}\right)}$$





obs:

* es la T.F. de una función
* si fuera señal diría que es lenta
* si fuera sistema es:
  → pasabajos
  → causal

CONVERGENCIA

Condición simple

$$\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\cdot e^{j\omega t}\cdot dt \qquad e(t) = \hat{x}(t) - x(t)$$

Si $x(t)$ de energía finita

$$\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty \Rightarrow \int_{-\infty}^{\infty} |e(t)|^2 = 0$$

→ $\hat{x}(t)$ converge

→ $\hat{x}(t) = x(t)$ en casi todo punto

obs: $\displaystyle\int_1^{\infty} \frac{1}{t^2}\cdot dt \underset{\alpha>2}{>} \int_1^{\infty} \frac{1}{t^{\alpha}}\cdot dt$

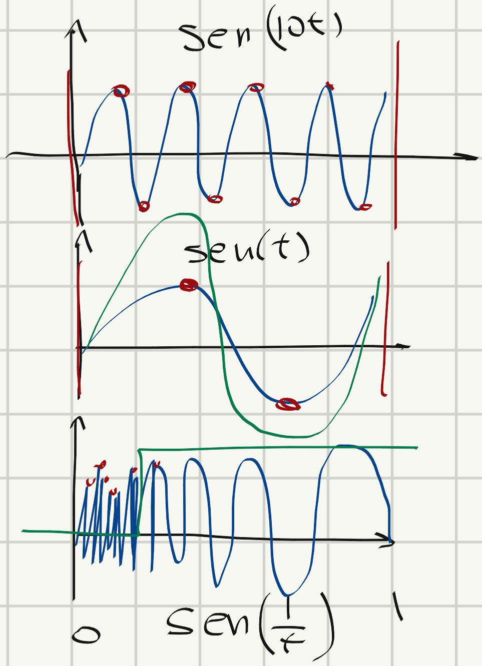
Condiciones de Dirichlet

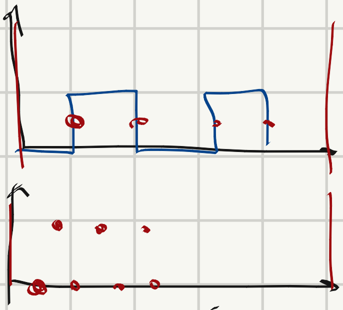
1) $x(t)$ absolutamente integrable: $\displaystyle\int_{-\infty}^{\infty} |x(t)|\cdot dt < \infty$

90

2) $x(t)$ tenga un número finito de extremos en un intervalo finito

3) $x(t)$ tenga un número finito de discontinuidades finitas en un intervalo finito.





función dirichlet.

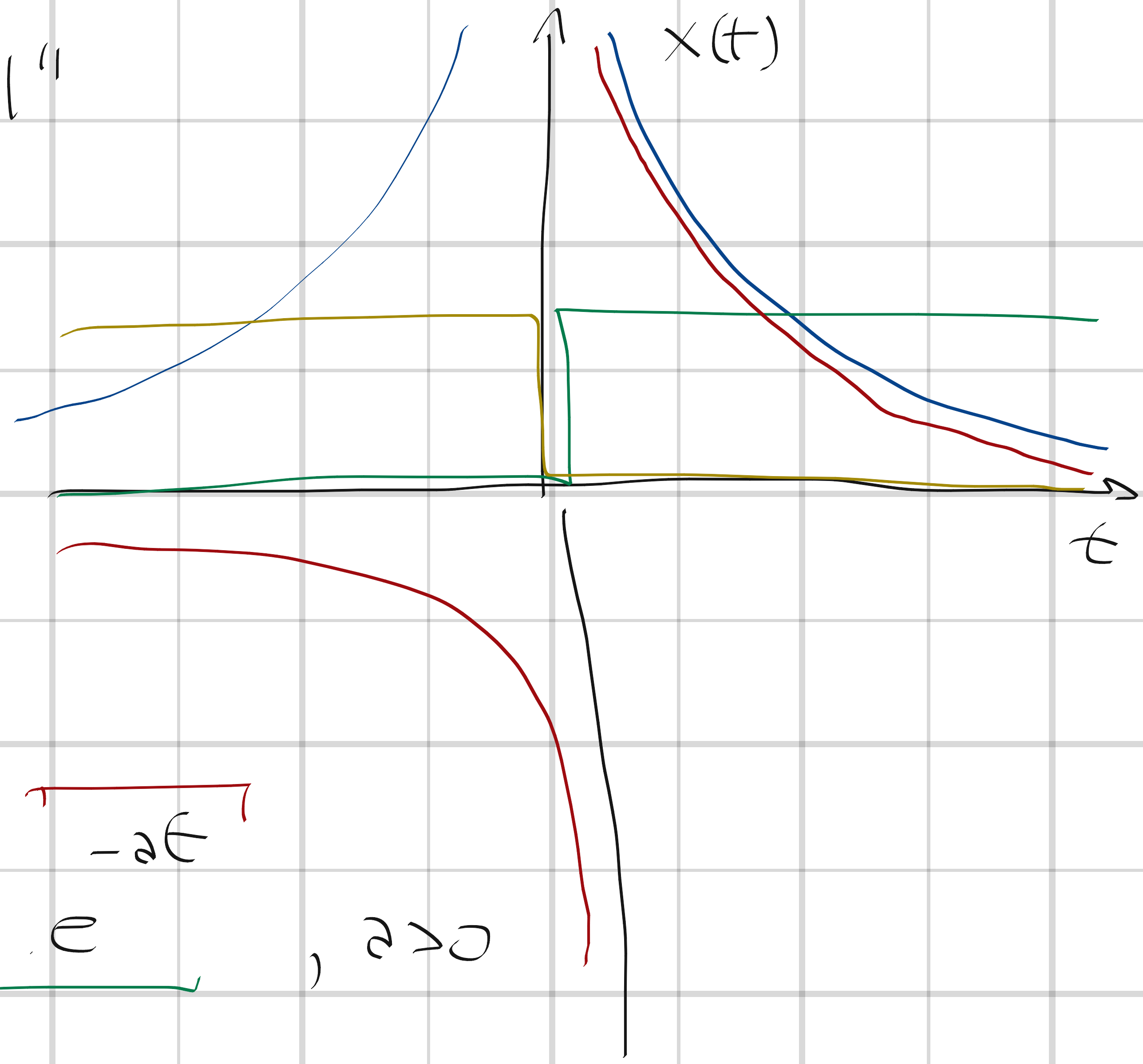
$$\begin{cases} 1 & t \in \mathbb{Q} \\ 0 & t \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Practico 10

Transformada de Fourier

Ejemplos de cálculo

Ejemplo 4.2: "exponencial bilateral"



1) es lenta o rápida?
2) cómo se compara con la anterior
3) escribir la función

$$x(t) = \underbrace{u(-t)\, e^{at}} + \underbrace{u(t)\, e^{\overline{-at}}} \quad , a>0$$

$$x(t) = \begin{cases} e^{-at} & t>0 \\ e^{at} & t<0 \end{cases}$$

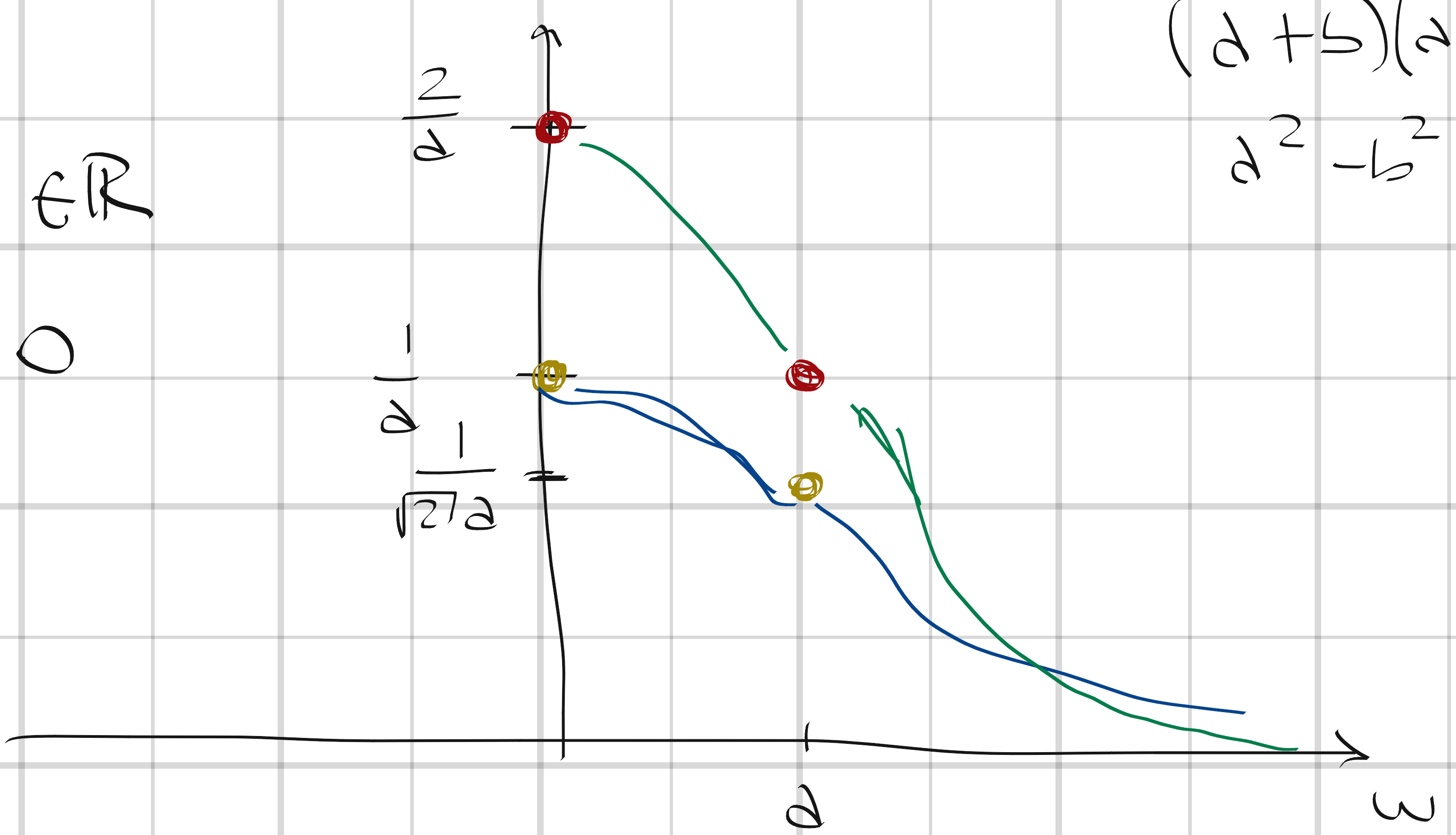
$$x(t) = e^{-a|t|}$$

4) calcular $X(j\omega)$

$$\underline{e^{at}} \cdot \underline{e^{-j\omega t}} \qquad \underline{e^{-at}} \cdot \underline{e^{-j\omega t}}$$

$$\frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{2a}{a^2+\omega^2}$$

$$(a+b)(a-b)$$
$$a^2-b^2$$



Obs:
- $X(j\omega) \in \mathbb{R}$
- fase $= 0$

Ejemplo 4.3

$$x(t) = \delta(t)$$

$$X(jw) = \int_{-\infty}^{\infty} x(t) \cdot e^{-jwt} \cdot dt = \int_{-\infty}^{\infty} \delta(t) \cdot e^{-jwt} dt = e^{-jwt}\Big|_0 = \boxed{1}$$

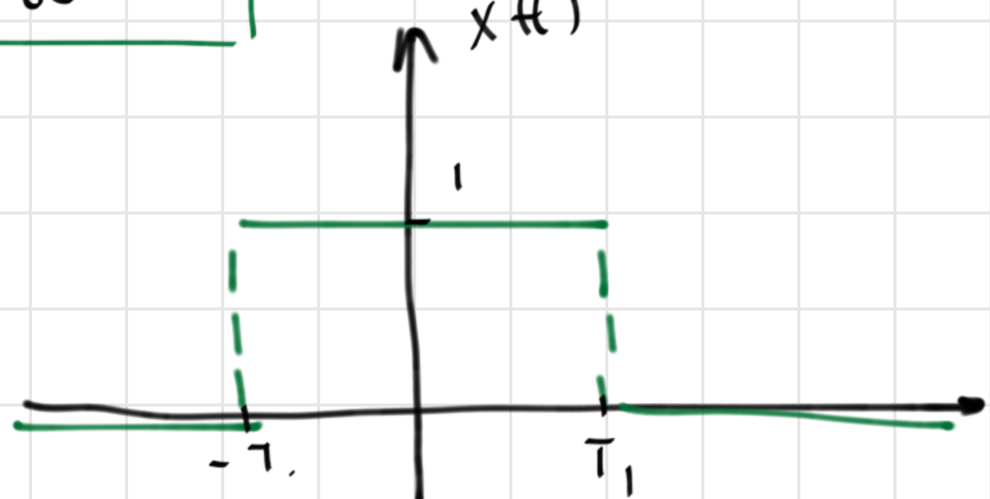
Ejemplo 4.4

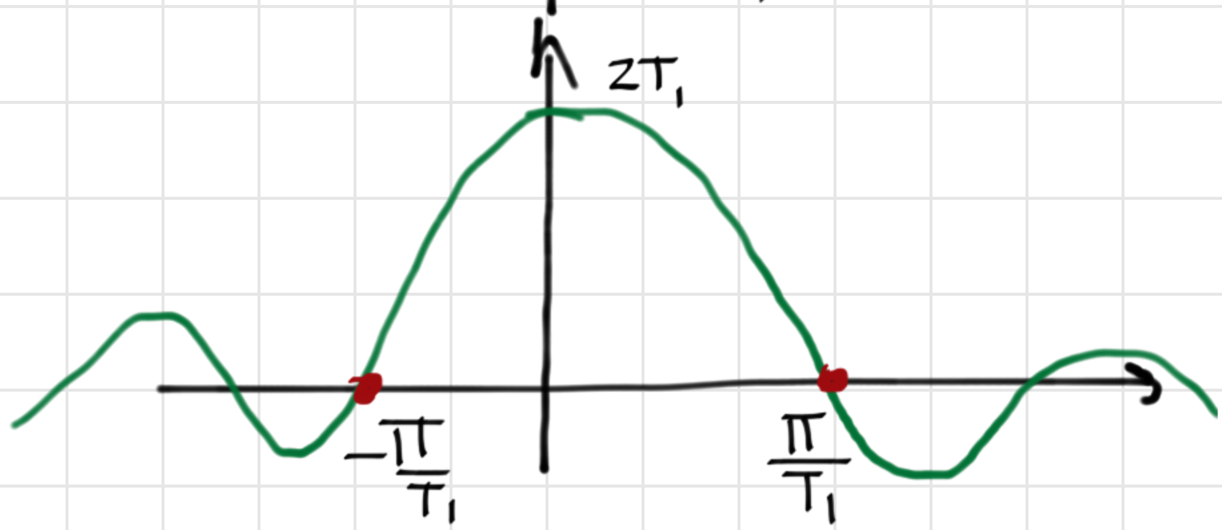
(solución)

$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & |t| \geq T_1 \end{cases}$$

$$X(jw) = \int_{-\infty}^{\infty} x(t) \cdot e^{-jwt} dt = \int_{-T_1}^{T_2} e^{-jwt} \cdot dt = \frac{e^{-jwt}}{-jw}\Big|_{-T_1}^{T_1}$$

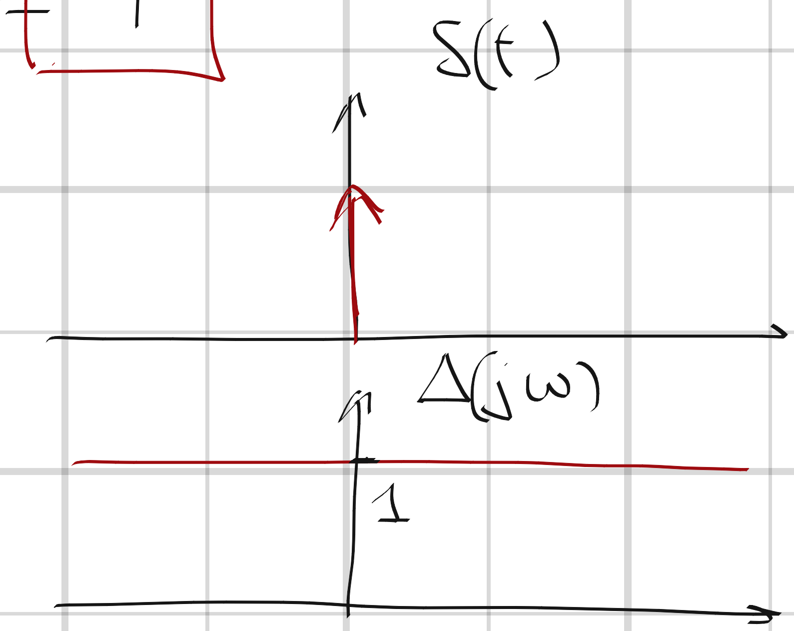
$$X(jw) = \frac{e^{jwT_1} - e^{-jwT_1}}{jw} = \boxed{\frac{2\,\mathrm{sen}(wT_1)}{w}}$$



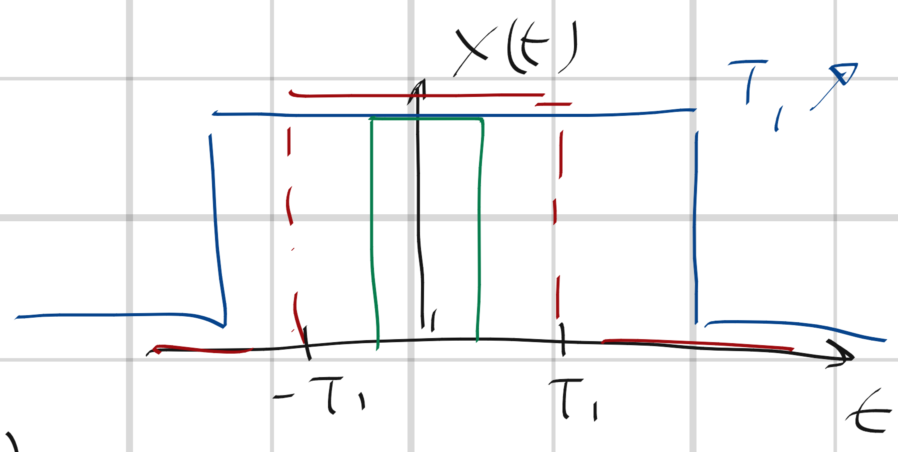


Ejemplo 4.3 $\quad x(t) = \delta(t)$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) \cdot e^{-j\omega t}\, dt = e^{-j\omega t}\Big|_{t=0} = \boxed{1}$$



Ejemplo 4.4
(guiado en clase)



1) escribir $x(t)$

2) calcular $X(j\omega)$

3) dibujar $|X(j\omega)|$

Victoria/Lucía $\quad \boxed{2\,\dfrac{\operatorname{sen} \omega T_1}{\omega T_1} \cdot T_1}$

Manuel $\quad \dfrac{\operatorname{sen}(\omega T_1)}{\omega}$

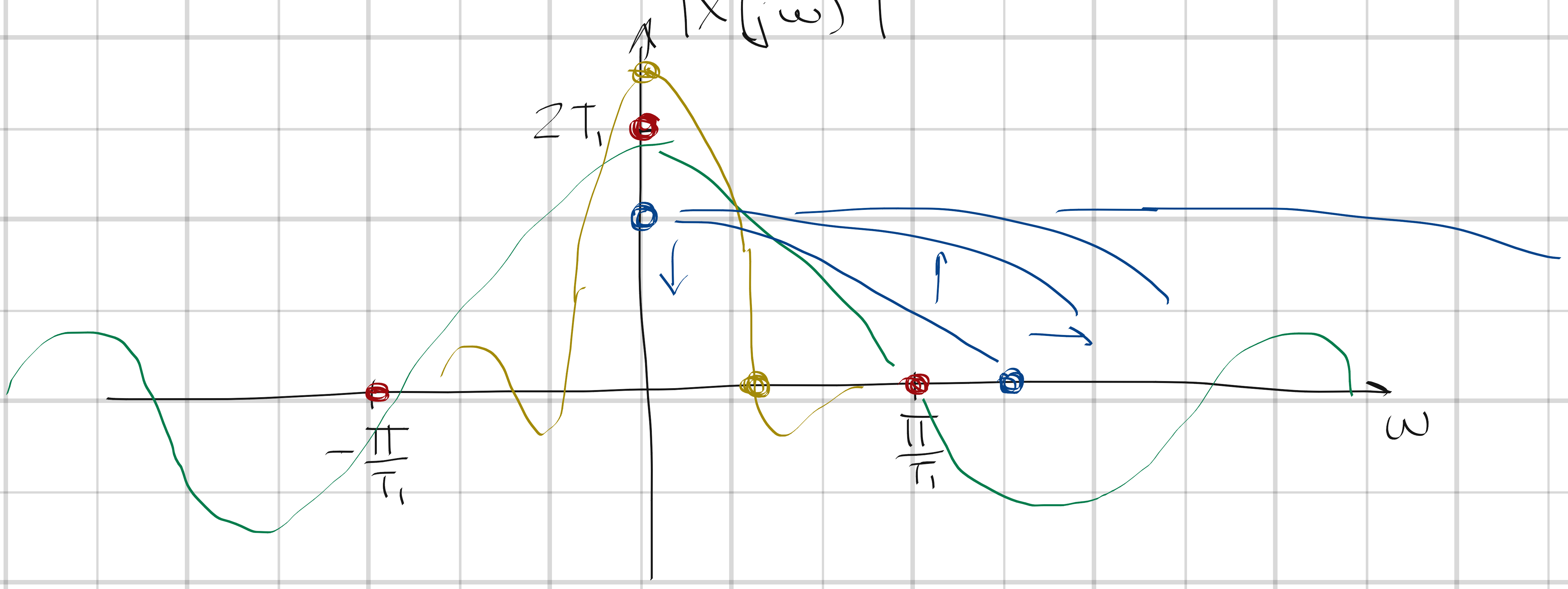
Tamara $\quad X(j\omega) = -2\,\dfrac{\operatorname{sen}(\omega T_1)}{\omega T_1}$

$X(j\omega) = 2 T_1 \cdot \operatorname{senc}(T_1 \omega)$

ptos característicos: $\omega = 0$

$\omega_z \,/\, |X(j\omega_z)| = 0$

$|X(j\omega)|$



Convergencia

$$\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2 \cdot \frac{\operatorname{sen}(wT_1)}{w} e^{jwt} dw$$

$$|\hat{x}(t)|^2 \leq \frac{1}{2\pi}\int_{-\infty}^{\infty} 2 \left|\frac{\operatorname{sen}(wT_1)}{w}\right|^{2 \leq 1} \cdot \left|e^{jwt}\right|^{=1} dw \leq \frac{2}{\pi}\int_{0\,B}^{\infty\,A} \frac{\operatorname{sen}^2(wT_1)}{w^2} dw$$

A) $\frac{2}{\pi}\int_1^{\infty} \frac{\operatorname{sen}^2(wT_1)}{w^2} dw \leq \int_1^{\infty} \frac{1}{w^2} dw < \infty$

B) $\frac{2}{\pi}\int_0^1 \frac{\operatorname{sen}^2(wT_1)}{w^2} dw$ finito!!

* ¿a cuanto converge?

$$\hat{X}(T_1) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\frac{\operatorname{sen}(wT_1)}{w} \cdot e^{jwT_1} \cdot dw = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{2jw}\left(e^{j2wT_1} - 1\right)$$

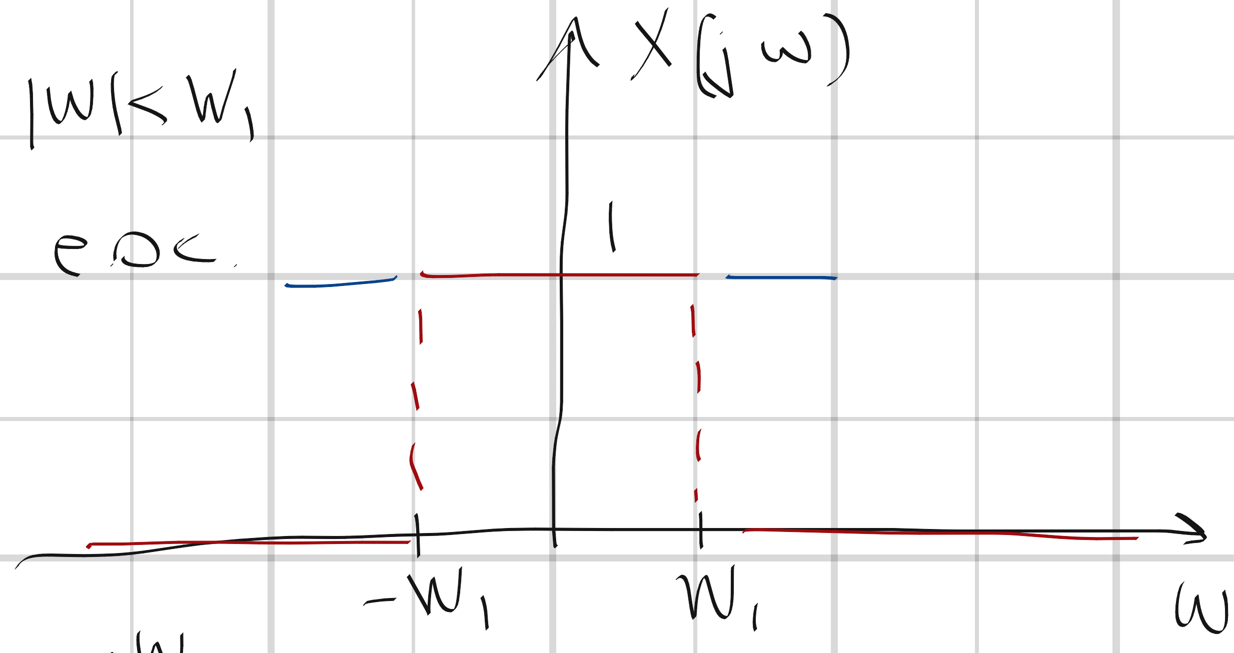
$$\hat{x}(T_1) = \frac{1}{2j\pi} \cdot \left(\int_{-\infty}^{\infty} \frac{e^{jwT_1}}{w} - \int_{-\infty}^{\infty} \frac{1}{w}\right)$$

$$\int_{-\infty}^{\infty} \frac{1}{w} \cdot e^{j2wT_1} dw$$

$\boxed{\text{sin terminar}}$

Ejemplo 4.5

$$X(j\omega) = \begin{cases} 1 & |\omega| < W_1 \\ 0 & \text{e.o.c.} \end{cases}$$

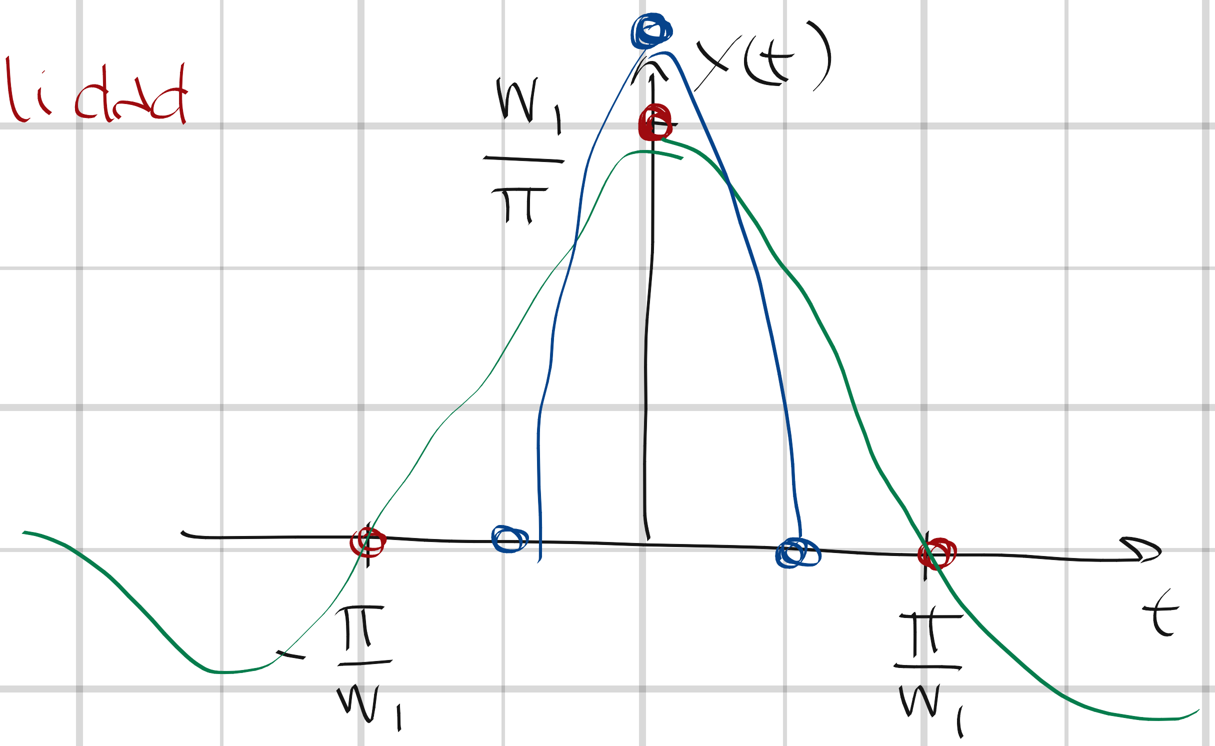


$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$$x(t) = \frac{1}{2\pi}\int_{-W_1}^{W_1} 1 \cdot e^{j\omega t}\, d\omega = \frac{1}{2\pi}\frac{e^{j\omega t}}{jt}\Bigg|_{-W_1}^{W_1} = \frac{1}{2\pi}\frac{e^{jW_1 t} - e^{-jW_1 t}}{jt} = \frac{W_1}{W_1}\frac{\operatorname{sen}(W_1 t)}{\pi t}$$

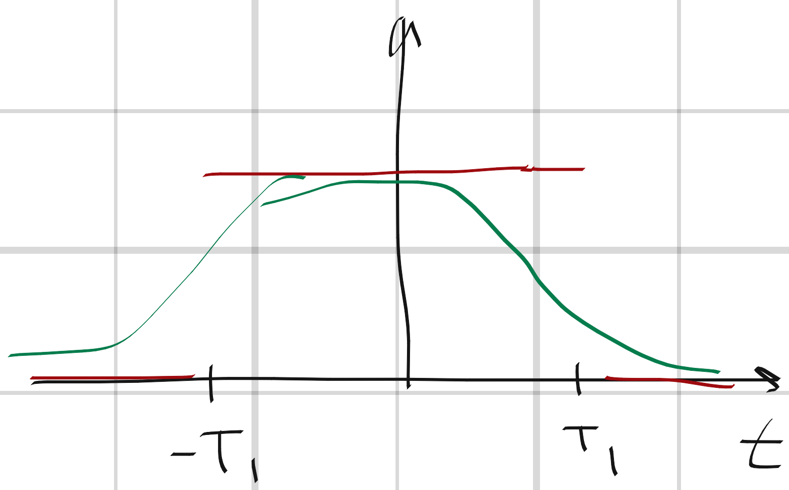
$$\boxed{x(t) = \frac{W_1}{\pi}\operatorname{senc}(W_1 t)}$$

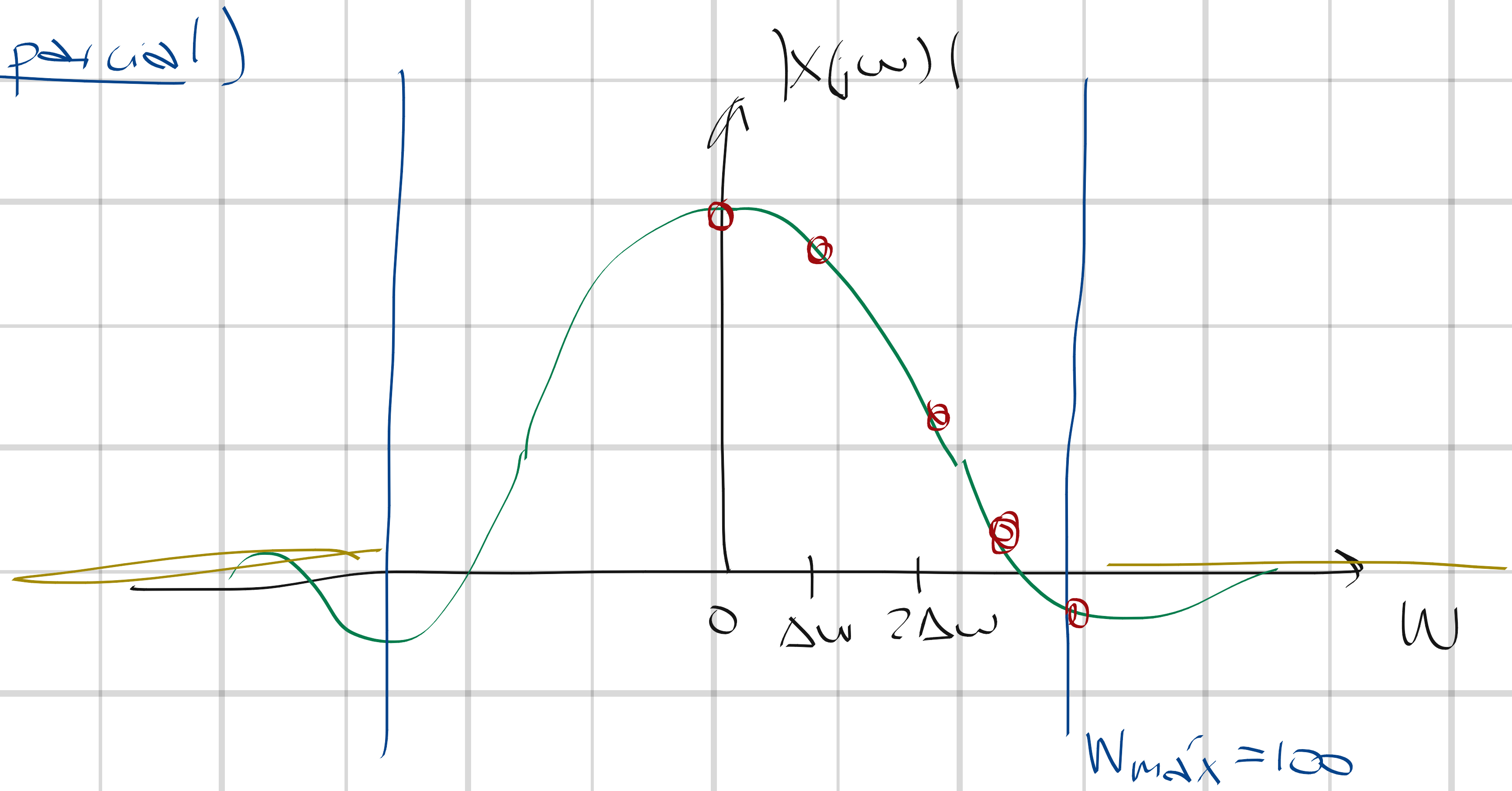
Dualidad

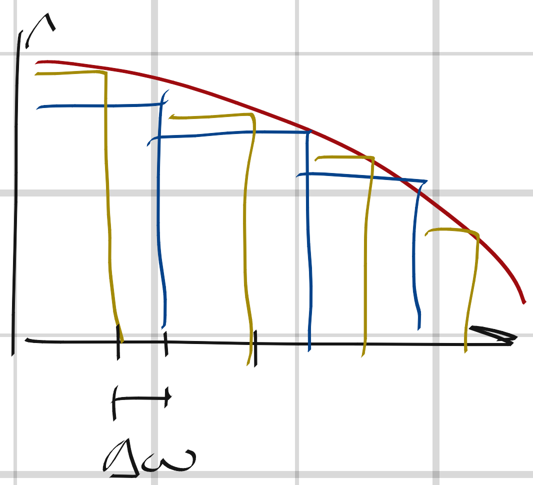


obs: * dualidad (con el 4.4)

Ejemplo 4.4 (rec. parcial)







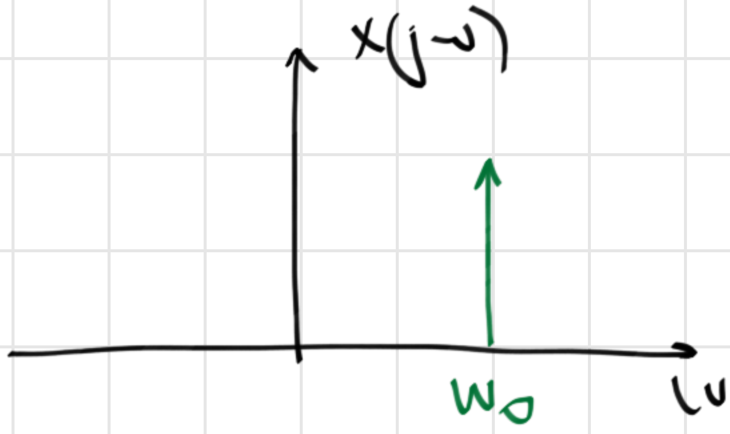
Clase 19

Transformada de Fourier

Señales periódicas y propiedades

TRANSFORMADA DE SEÑALES PERIÓDICAS

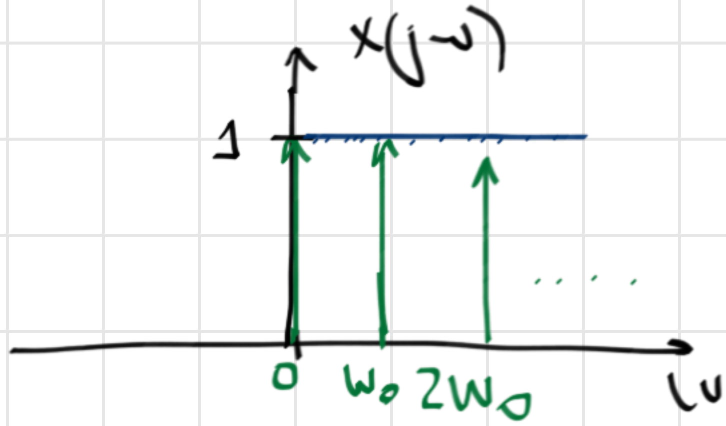
a) $X(j\omega) = 2\pi \cdot \delta(\omega - \omega_0)$



$$x(t) = \frac{1}{\cancel{2\pi}} \int_{-\infty}^{\infty} \cancel{2}\pi \cdot \delta(\omega-\omega_0) e^{j\omega t} \cdot d\omega$$

$$x(t) = e^{j\omega t}\Big|_{\omega=\omega_0} = \boxed{e^{j\omega_0 t}}$$

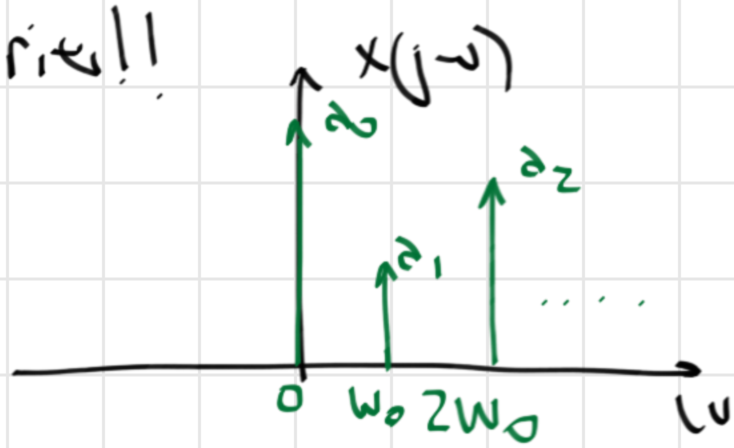
b) $X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi \cdot \delta(\omega - k\omega_0)$



$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} \cdot e^{jn\omega_0 t}}$$

← es una Serie de Fourier!!

b) $X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \cdot \delta(\omega - k\omega_0)$

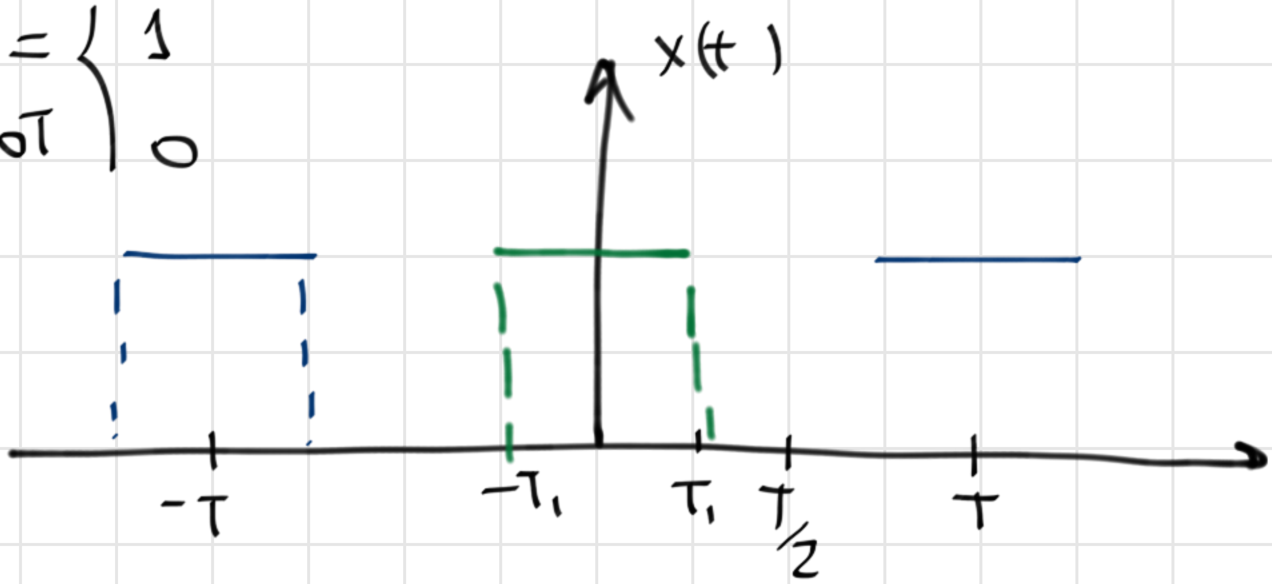


$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k \cdot e^{jn\omega_0 t}}$$

← es una Serie de Fourier!!

ec. síntesis

Ejemplo 4.6 $x(t) = \begin{cases} 1 \\ 0 \end{cases}$ periodo T



$$a_k = \frac{\text{sen}(k w_0 T_1)}{\pi k}$$

$$X(jw) = \sum_{k=-\infty}^{\infty} 2 \frac{\text{sen}(k w_0 T_1)}{k} \cdot \delta(w - k w_0)$$

$$X(jw) = \sum_{-\infty}^{\infty} 2 \frac{\text{sen}(\frac{\pi}{2} k)}{k} \cdot \delta(w - k w_0)$$

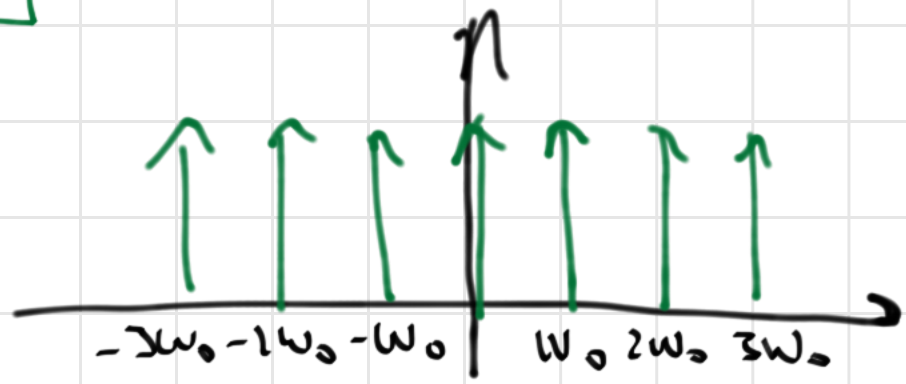
completar con discretización

Ejemplo 4.8

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\cdot e^{-j\omega_0 t}\, dt = \boxed{\frac{1}{T}}$$

$$X(j\omega) = 2\pi \sum_{n=-\infty}^{\infty} \frac{1}{T}\cdot \delta(\omega - k\omega_0)$$



$$\boxed{X(j\omega) = \omega_0 \cdot \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0)}$$

PROPIEDADES DE LA TRANSFORMADA

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega) \;,\; y(t) \xleftrightarrow{\mathcal{F}} Y(j\omega) \;,\; z(t) \xleftrightarrow{\mathcal{F}} Z(j\omega)$$

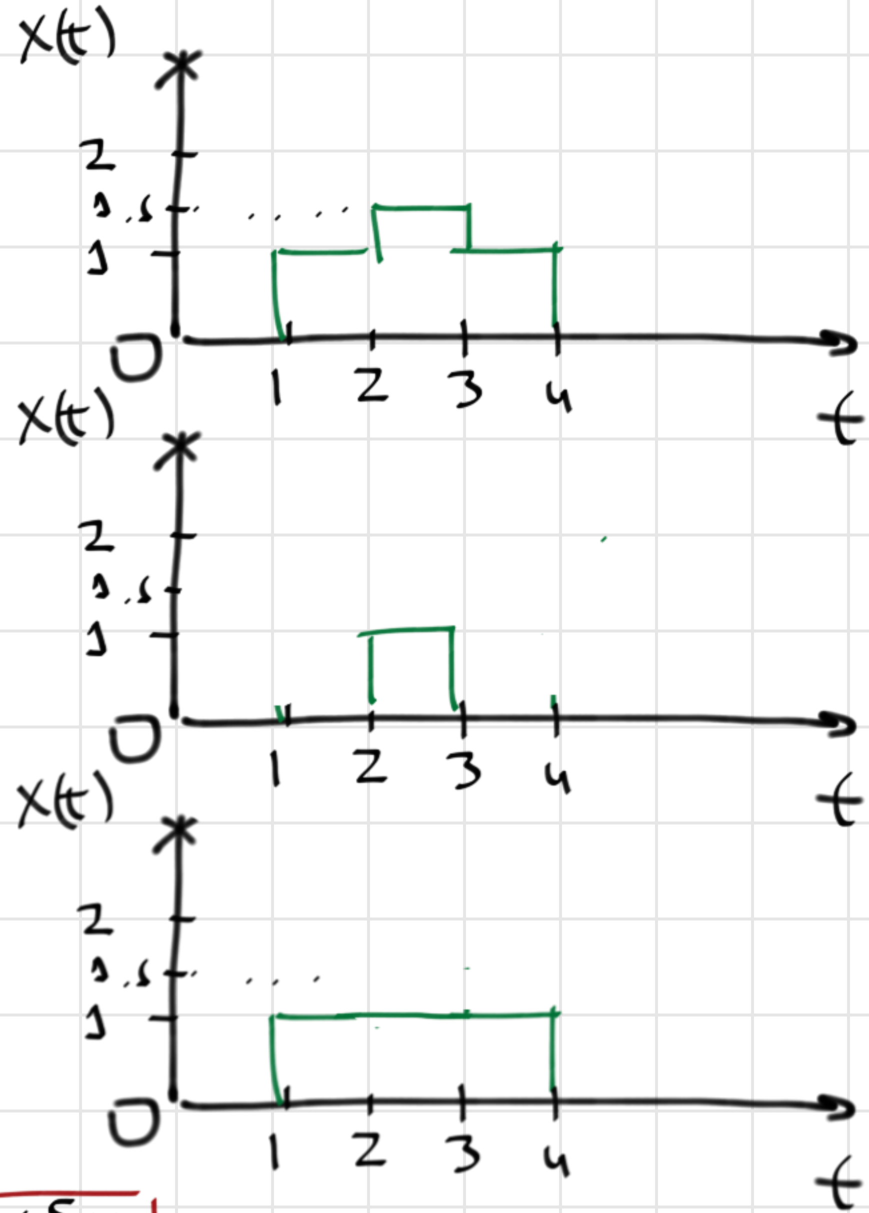
Linealidad

$$a\,x(t) + b\,y(t) \xleftrightarrow{\mathcal{F}} a \cdot X(j\omega) + b\,Y(j\omega)$$

Desplazamiento temporal

$$x(t-t_0) \xleftrightarrow{\mathcal{F}} e^{-j\omega t_0} \cdot X(j\omega)$$

Ejemplo 4.9 (p. 302)



$$x(t) = \frac{1}{2} x_1\left(t-\frac{5}{2}\right) + x_2\left(t-\frac{5}{2}\right)$$

$$P(j\omega) = \frac{2 \cdot \operatorname{sen}(\omega T_1)}{\omega}$$

$$X_1(j\omega) = \frac{2 \cdot \operatorname{sen}\left(\frac{\omega}{2}\right)}{\omega}$$

$$X_2(j\omega) = \frac{2 \cdot \operatorname{sen}\left(\frac{3}{2}\omega\right)}{\omega}$$

$$\boxed{X(j\omega) = \frac{2}{\omega} \cdot \left(\frac{1}{2}\operatorname{sen}\left(\frac{\omega}{2}\right) + \operatorname{sen}\left(\frac{3}{2}\omega\right)\right) e^{j\frac{5}{2}\omega}}$$

Conjugación

a) $x^*(t) \xleftrightarrow{\mathcal{F}} X^*(-jw)$

$$X^*(jw) = \int_{-\infty}^{\infty} x^*(t) \cdot e^{jwt} \Rightarrow X(-jw) = \int_{-\infty}^{\infty} x^*(t) \cdot e^{-jwt} \cdot dt = \mathcal{F}\{x^*(t)\}$$

b) $x(t)$ real $\quad X(jw) = X^*(-jw)$

c) * $Re\{X(jw)\} = Re\{X^*(-jw)\} = Re\{X(-jw)\} \Rightarrow \boxed{Re\{X(jw)\}\ PAR}$

* $Im\{X(jw)\} = Im\{X^*(-jw)\} = -Im\{X(-jw)\} \Rightarrow \boxed{Im\{X(jw)\}\ IMPAR}$

* $|X(jw)|$ PAR

* $\angle X(jw)$ IMPAR

d) $x(t)$ real y par $\quad X(-jw) = \int_{-\infty}^{\infty} x(t) \cdot e^{+jwt} \cdot dw \overset{\tau = -t}{=} + \int_{-\infty}^{\infty} x(-\tau) \cdot e^{-jw\tau} d\tau$

$$X(-jw) = \int_{-\infty}^{\infty} x(\tau) \cdot e^{-jw\tau} \cdot dw = X(jw) \Rightarrow \boxed{X(jw)\ PAR}$$

$$X(jw) = X^*(-jw) = X^*(jw) \Rightarrow \boxed{X(jw)\ REAL}$$

e) $x(t)$ real y par $\Rightarrow \boxed{X(jw) \text{ imag e impar}}$

f) $x(t) = x_{par}(t) + x_{impar}(t)$

$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$

$x_{par} \overset{\mathcal{F}}{\longleftrightarrow} Re\{X(j\omega)\}$

$x_{impar} \overset{\mathcal{F}}{\longleftrightarrow} Im\{X(j\omega)\}$

Ejemplo 4.10

$$x(t) = e^{-a|t|}, \quad a>0 \quad (\text{idem } 4.2)$$

$4.1 \Rightarrow \quad e^{-at} \cdot u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{a+j\omega}$

$x(t)$ es par $\Rightarrow x(t) = u(t) \cdot e^{-at} + u(-t) \cdot e^{at}$

$$x(t) = 2 \cdot \left(\frac{\overbrace{u(t) \cdot e^{-at}}^{x_1(t)} + \overbrace{u(-t) e^{at}}^{x_1(-t)}}{2} \right) = 2\, Par\{u(t) \cdot e^{-at}\}$$

$$X(j\omega) = 2 \cdot Par\{X_1(j\omega)\} = \boxed{2 \cdot \frac{a}{a^2 + \omega^2}}$$

Clase 20

Transformada de Fourier

Propiedades (cont)

Diferenciación en t

$$\frac{d\,x(t)}{dt} = \underbrace{\frac{d}{dt}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega\right)}_{\text{ec. síntesis}}$$

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

$$\frac{d\,x(t)}{dt} \overset{\mathcal{F}}{\longleftrightarrow} ?$$

$$\frac{d\,x(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{j\omega\, X(j\omega)}_{G(j\omega)}\, e^{j\omega t}\, d\omega$$

$$\boxed{\frac{d\,x(t)}{dt} \overset{\mathcal{F}}{\longleftrightarrow} j\omega\, X(j\omega)}$$

Integración

$$\boxed{\int_{-\infty}^{t} x(\tau)\, d\tau \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega}\, X(j\omega) + \pi\, \underbrace{X(j\,0)}_{\to \text{ continua}}\, \delta(\omega)}$$

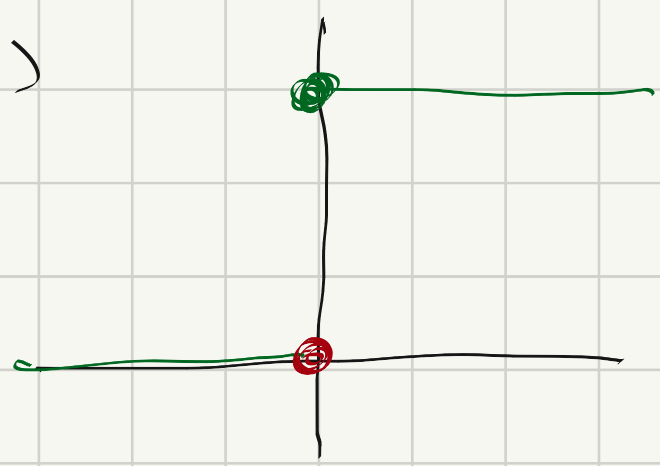
Ejemplo 4.11

$$g(t) = \delta(t) \overset{\mathcal{F}}{\longleftrightarrow} G(j\omega) = 1$$

$G(j\,0)$

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau \Rightarrow \mathcal{F}\{u(t)\} = \frac{1}{j\omega}\cdot 1 + \pi \cdot 1 \cdot \delta(\omega)$$

$$\boxed{u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega} + \pi\,\delta(\omega)}$$



verif.

$$\delta(t) = \frac{d\,u(t)}{dt} \overset{\mathcal{F}}{\longleftrightarrow} j\omega\left(\frac{1}{j\omega} + \pi\,\delta(\omega)\right) = 1 + \pi\, j\omega\, \delta(\omega)$$

$$= 1 + \pi\, \cancel{j\,0}\, \delta(\omega) \quad \to 0$$

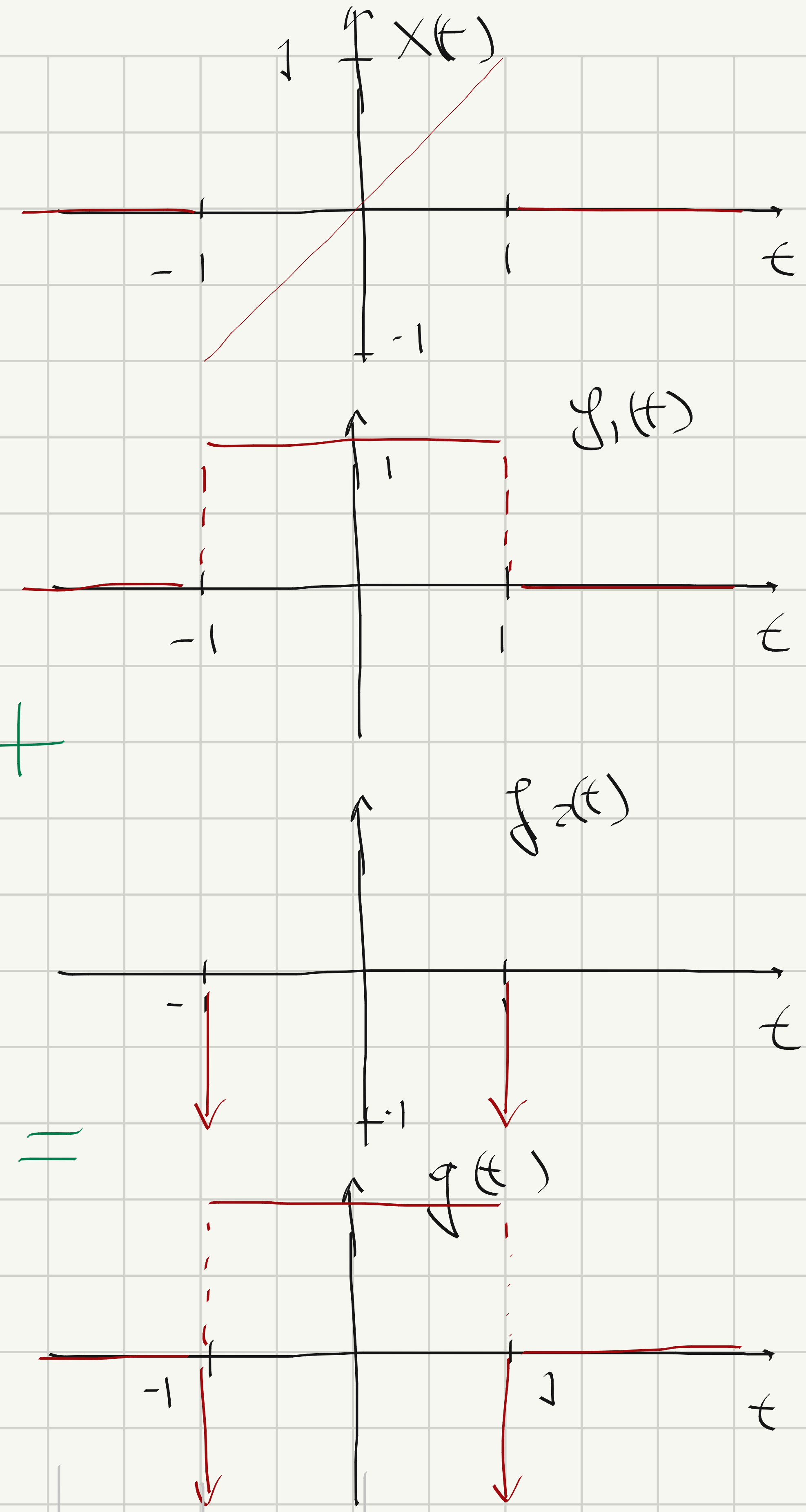
$$\boxed{\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} 1}$$

Ejemplo 4.12

$X(j\omega)$? $\qquad g(t) = \dfrac{d\,x(t)}{dt}$

$$G(j\omega) = \underbrace{2\,\frac{\operatorname{sen}(\omega)}{\omega}}_{G_1} - \underbrace{e^{-j1\omega} - e^{-j(-1)\omega}}_{G_2}$$

$$\boxed{G(j\omega) = 2\,\frac{\operatorname{sen}(\omega)}{\omega} - 2\cos(\omega)}$$

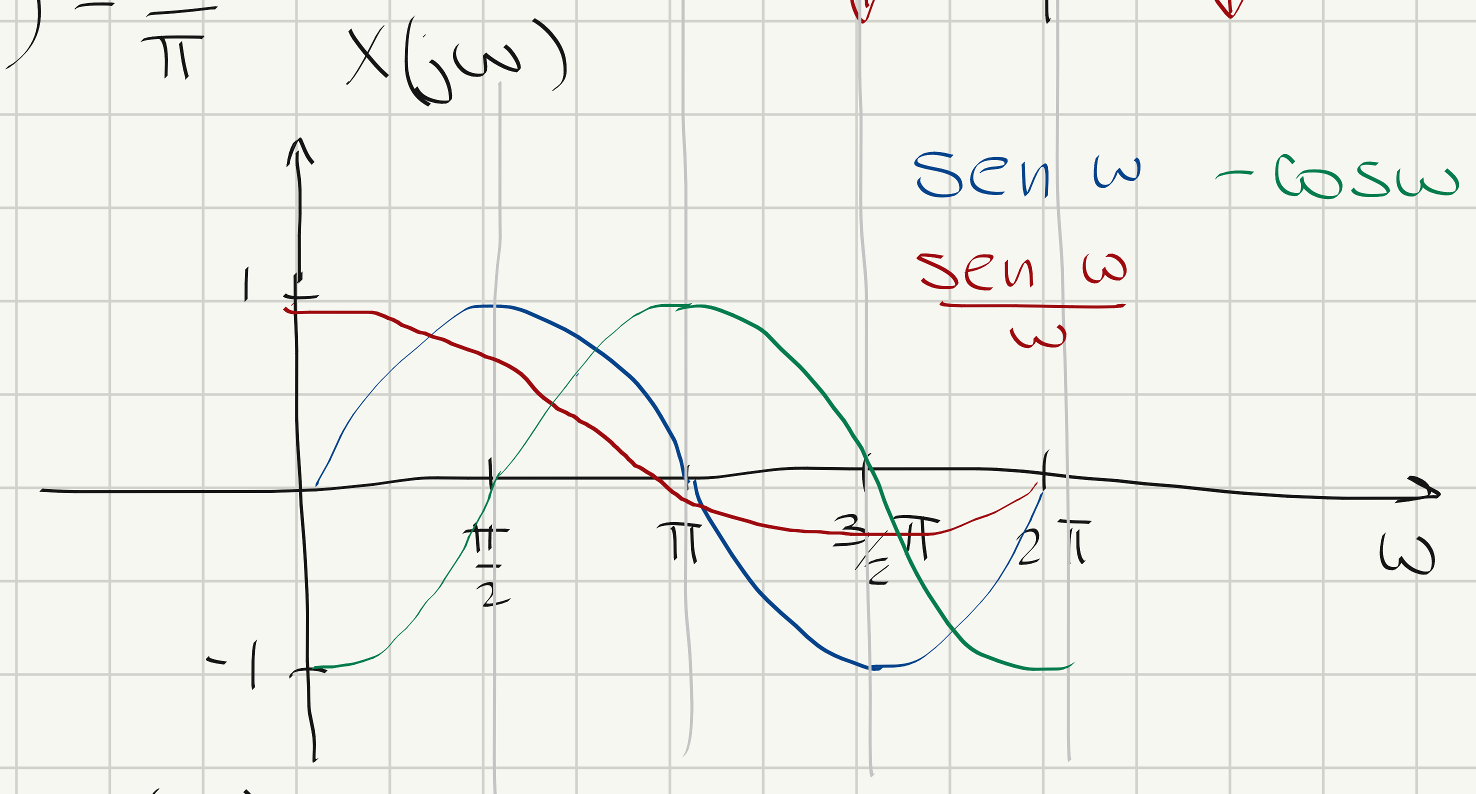


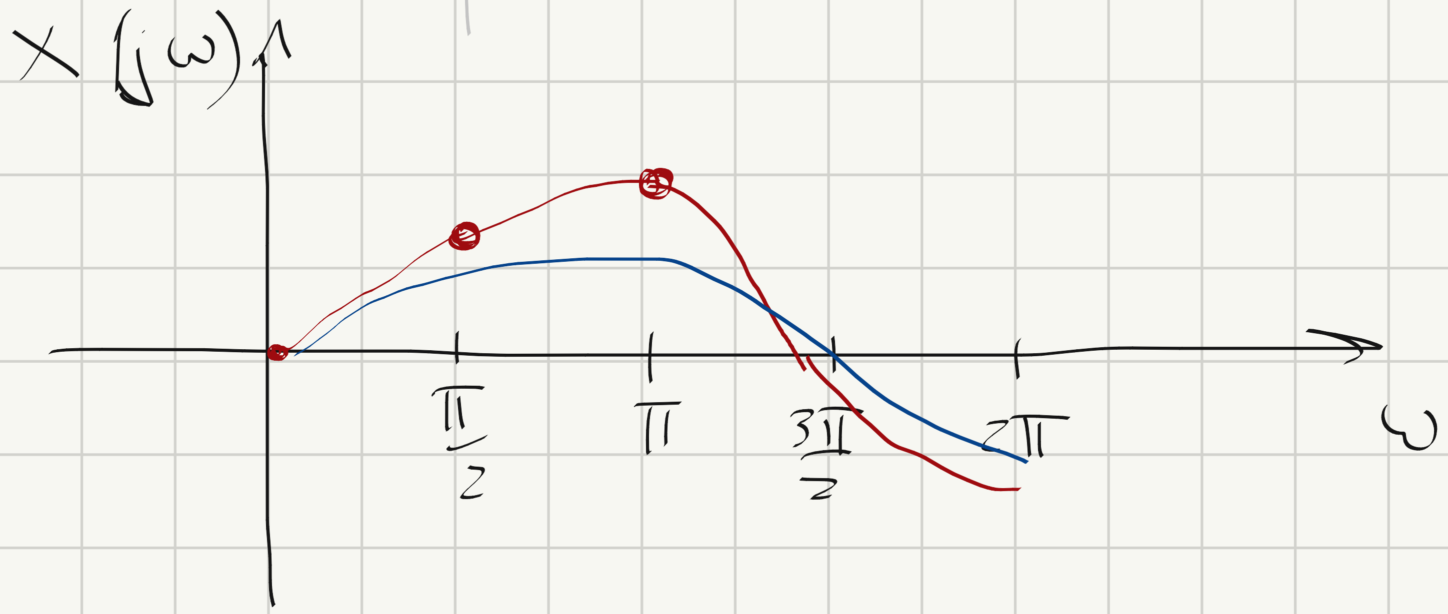
$$X(j\omega) = \frac{1}{j\omega}\,2\left(\frac{\operatorname{sen}(\omega)}{\omega} - \cos\omega\right) + \pi\,\overset{0}{G(j0)}\,\delta(\omega) +$$

$$\boxed{X(j\omega) = \frac{2}{j\omega}\left(\operatorname{senc}(\omega) - \cos\omega\right)}$$

$$\left|X\left(\frac{\pi}{2}\right)\right| = \frac{2}{\pi/2}\left(\frac{1}{\pi/2} - 0\right) = \frac{8}{\pi^2} \simeq 0.81$$

$$|X(\pi)| = \frac{2}{\pi}\left(0 - (-1)\right) = \frac{2}{\pi} \simeq 0.64$$





Escalamiento temporal (compresión / expansión)

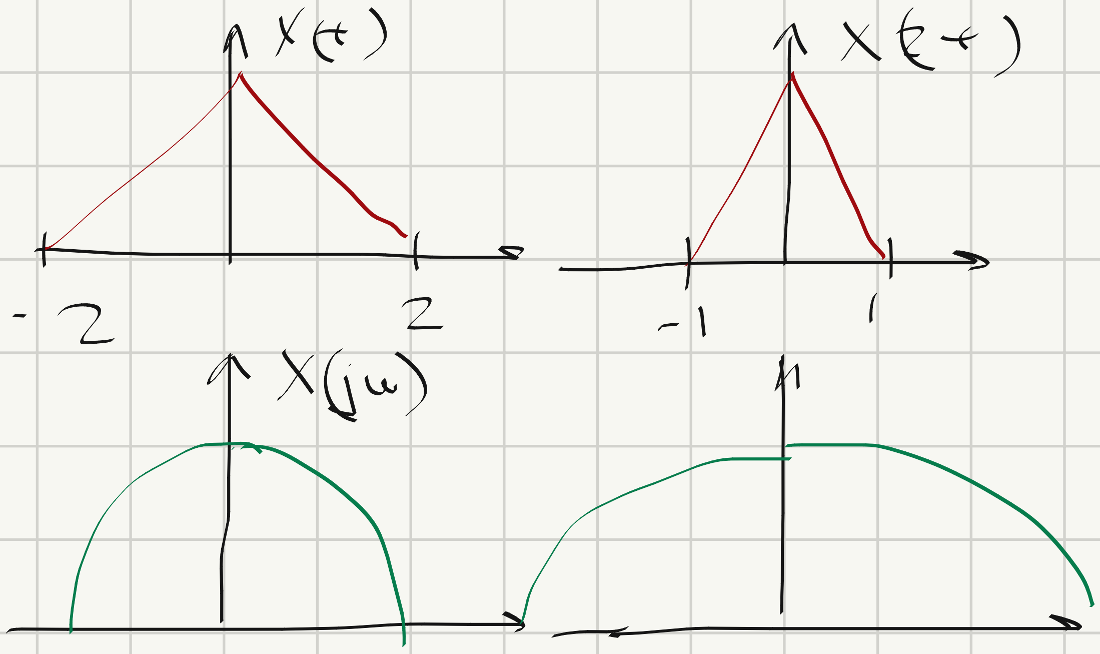
$$\mathcal{F}\{x(at)\} = \int_{-\infty}^{\infty} x(at)\, e^{-j\omega t}\, dt \overset{c.v.\ \tau = at}{=} \int_{\text{"}-\infty\text{"}}^{\text{"}\infty\text{"}} x(\tau)\, e^{-j\omega \frac{\tau}{a}}\, \frac{d\tau}{a}$$

ec. análisis

a) $a > 0$, $\mathcal{F}\{x(at)\} = \frac{1}{a}\int_{-\infty}^{\infty} \underline{x(\tau)}\; \underline{e^{-j\frac{\omega}{a}\tau}}\; \underline{d\tau} = \frac{1}{a} X\left(j\frac{\omega}{a}\right)$

b) $a < 0$, $\mathcal{F}\{x(at)\} = -\frac{1}{a}\int_{-\infty}^{\infty} x(\tau)\, e^{-j\frac{\omega}{a}\tau}\, d\tau = -\frac{1}{a} X\left(j\frac{\omega}{a}\right)$

$$\boxed{x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(j\frac{\omega}{a}\right)}$$



Corolario: $a = -1 \rightarrow$ inversión temporal

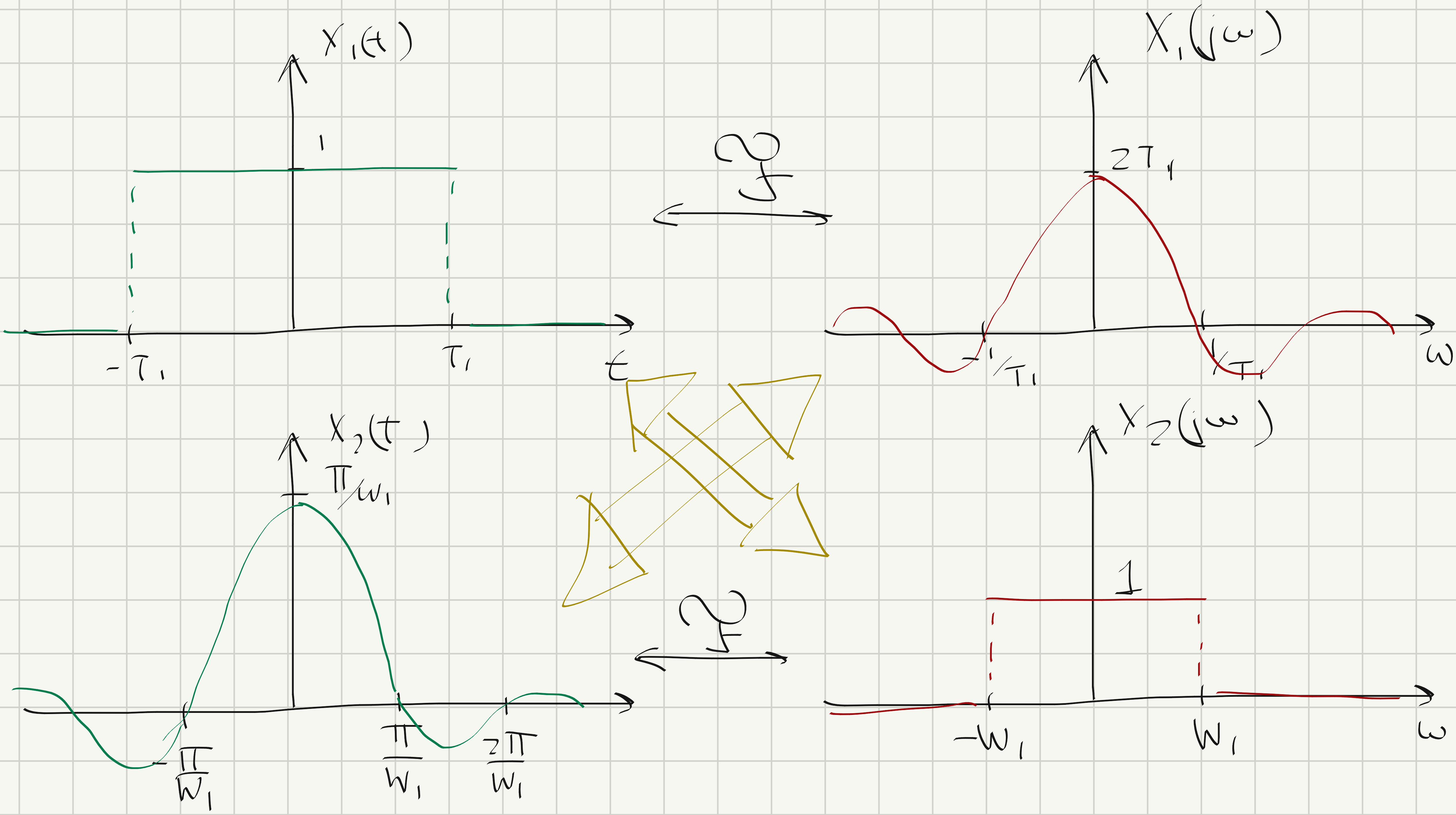
$$\mathcal{F}\{x(-t)\} = X(-j\omega)$$

Dualidad

senc

ej 4.4: $x_1(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & \text{e.o.c.} \end{cases} \xleftrightarrow{\mathcal{F}} X_1(j\omega) = 2T_1 \dfrac{\operatorname{sen}(T_1\omega)}{T_1\omega}$

ej 4.5: $x_2(t) = \dfrac{W_1}{\pi} \dfrac{\operatorname{sen}(W_1 t)}{W_1 t} \xleftrightarrow{\mathcal{F}} X_2(j\omega) = \begin{cases} 1 & |\omega| \leq W_1 \\ 0 & \text{e.o.c.} \end{cases}$



Ejemplo 4.13: (p 310)

$g(t) = \frac{1}{1+t^2}$, $G(j\omega)$?

$$g(t) = \frac{1}{1+t^2} \xrightarrow{\mathcal{F}} G(j\omega) = ?$$

$$x(t) = \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \frac{1}{1+\omega^2} = X(j\omega)$$

1 2 3

① $\rightarrow X(j\omega) = \frac{1}{1+\omega^2}$ $\mathcal{F}^{-1}$

② $\rightarrow x(t) = \frac{1}{2} e^{-|t|}$

ej 4.2, $a=1$

③ síntesis: $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$

$$\pi \cdot \cancel{\frac{1}{2}} e^{-|t|} = \frac{1}{\cancel{2\pi}} \int_{-\infty}^{\infty} \frac{1}{1+\omega^2}\, e^{j\omega t}\, d\omega \cdot \cancel{\pi}$$

C.V. $\tau = -t$

$$\pi\, e^{-|\tau|} = \int_{-\infty}^{\infty} \frac{1}{1+\omega^2}\, e^{-j\omega\tau}\, d\omega$$

C.V. $\tau \leftrightarrow \omega$

$$\pi\, e^{-|\omega|} = \int_{-\infty}^{\infty} \frac{1}{1+\tau^2}\, e^{-\tau\omega}\, d\tau \quad \leftarrow \text{ec. análisis}$$

$\mathcal{F}$

$$\boxed{\pi\, e^{-|\omega|} \overset{\mathcal{F}^{-1}}{\longleftrightarrow} \frac{1}{1+\tau^2}}$$

obs:

$$g(t) = \frac{1}{1+t^2} \overset{\mathcal{F}}{\longleftrightarrow} \pi \, e^{-|w|} = G(jw)$$

$$x(t) = \frac{1}{2} e^{-|t|} \overset{\mathcal{F}}{\longrightarrow} \frac{1}{1+w^2} = X(jw)$$

① ② ③

* Sirve para aprovechar cuentas
* Sirve para deducir nuevas propiedades
  → puedo demostrar "la mitad"

Diferenciación (en f)

$$jt\,x(t) \;?\; \overset{\mathcal{F}}{\longleftrightarrow} \frac{d\,X(j\omega)}{d\omega}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\,.\,e^{-j\omega t}\,.\,dt$$

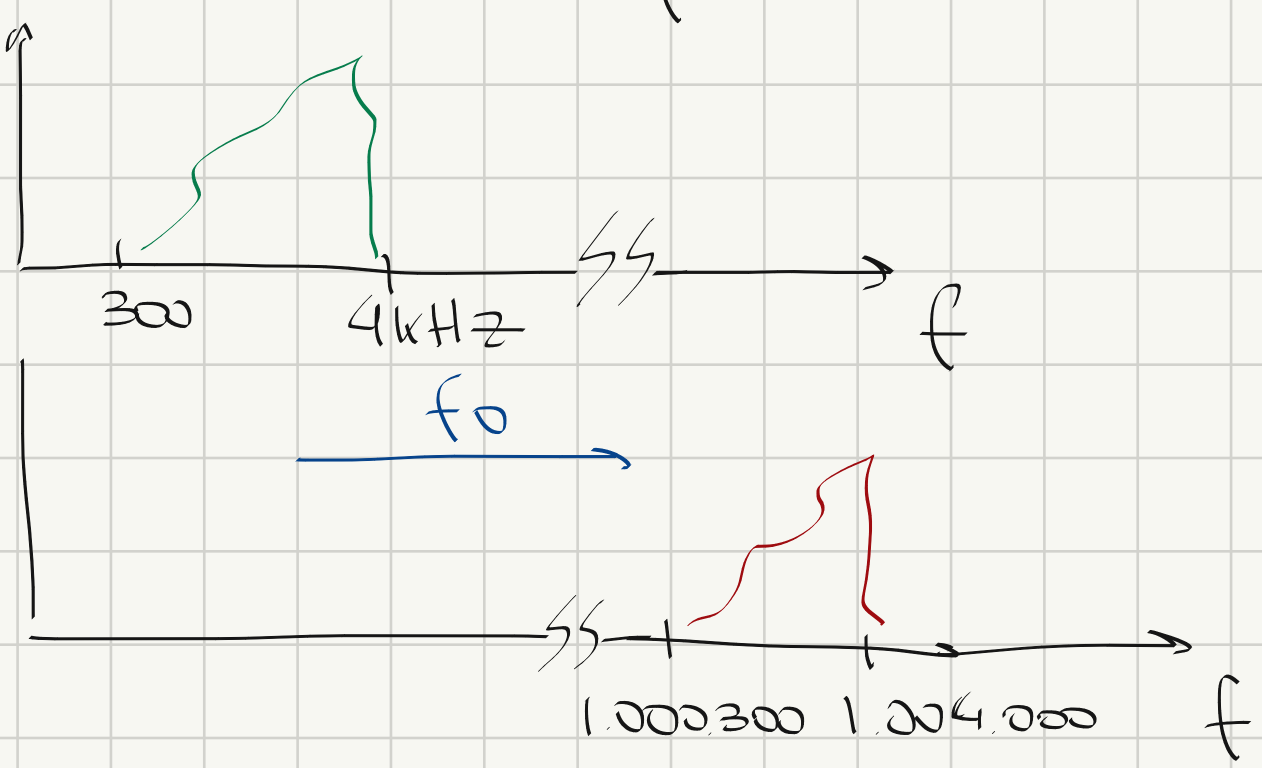
$$\frac{d\,X(j\omega)}{d\omega} = \int_{-\infty}^{\infty} \underbrace{-jt\,x(t)}_{\mathcal{F}}\,.\,e^{-j\omega t}\,.\,dt$$

$$\boxed{-jt\,x(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{d\,X(j\omega)}{d\omega}}$$

Desplazamiento en frecuencia

$$e^{j\omega_0 t}\,.\,x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j(\omega-\omega_0))$$

Ejemplo: "modulación" $f_0 = 1\,\text{MHz}$

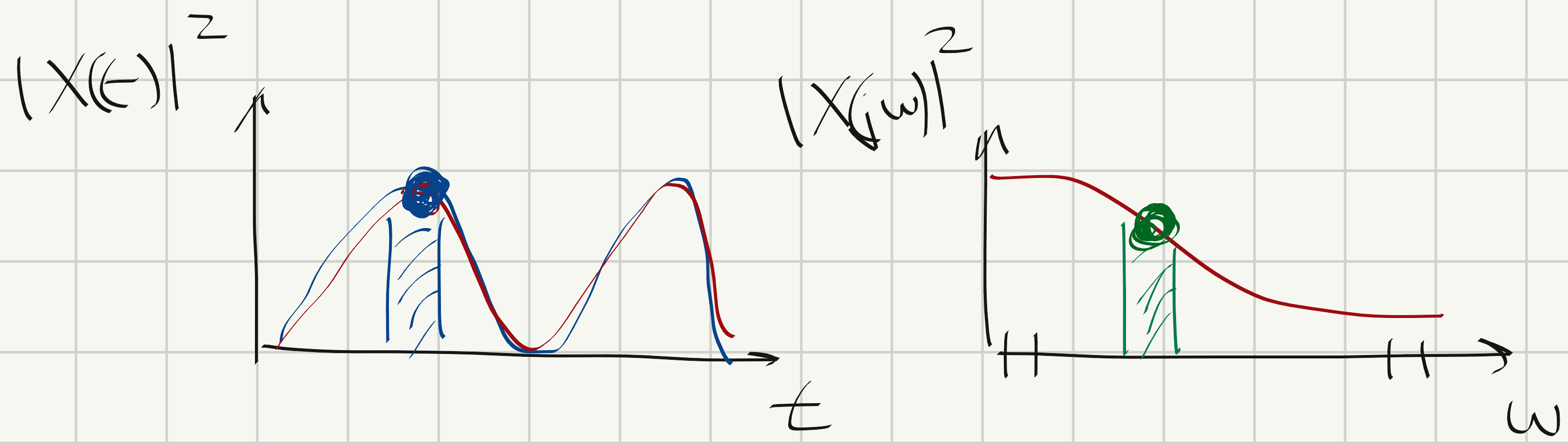


Relación de Parseval

$$E = \int_{-\infty}^{\infty} \underbrace{|x(t)|^2}_{\text{potencia}} dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{|X(j\omega)|^2}_{\text{energía por unidad de frecuencia}} d\omega$$

potencia → Energía por unidad de tiempo

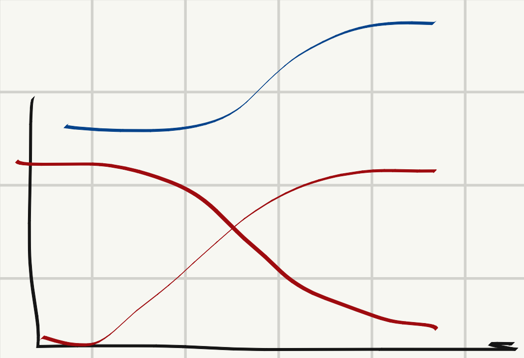
energía por unidad de frecuencia → densidad espectral de potencia



Obs: * $|x(t)|^2$: Potencia (densidad temporal de potencia)

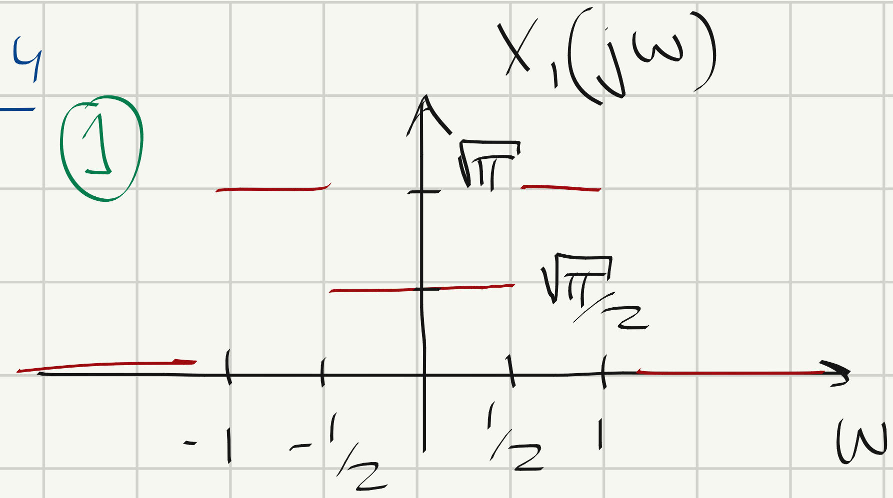
* $|X(j\omega)|^2$: Densidad espectral de potencia

* En el práctico 10 hicimos reconstrucción parcial usando esto

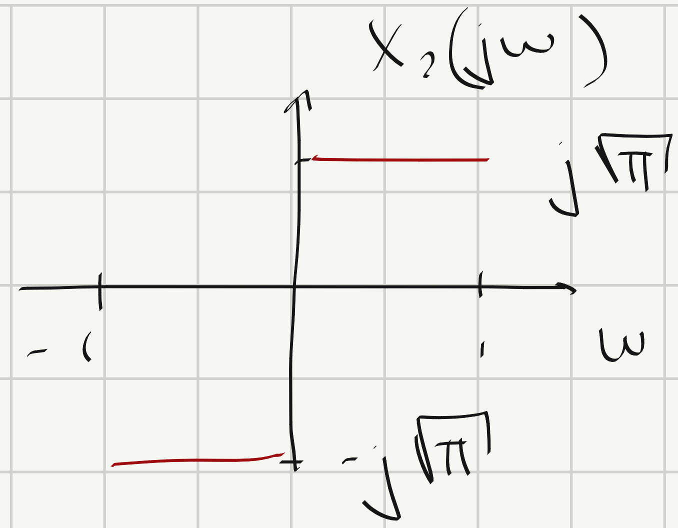


Ejemplo 4.14

1



2



$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt \qquad D = \left. \frac{d}{dt} x(t) \right|_{t=0}$$

a) energías

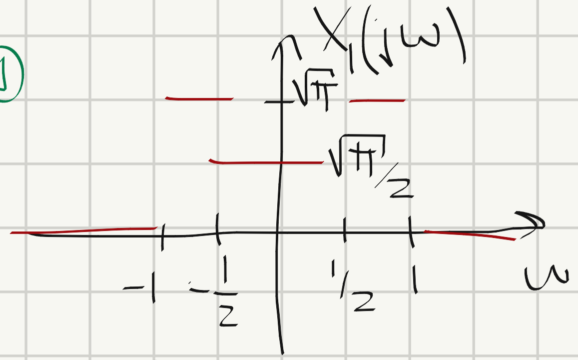
$$E_1 = \frac{1}{2\pi} \left[\frac{1}{2} \cdot \pi + 1 \cdot \frac{\pi}{4} + \frac{1}{2} \cdot \pi \right] = \frac{1}{2\cancel{\pi}} \cdot \frac{5}{4} \cancel{\pi} = \boxed{\frac{5}{8}}$$

$$E_2 = \frac{1}{2\pi} 2 \cdot 1 \cdot \pi = \boxed{1}$$

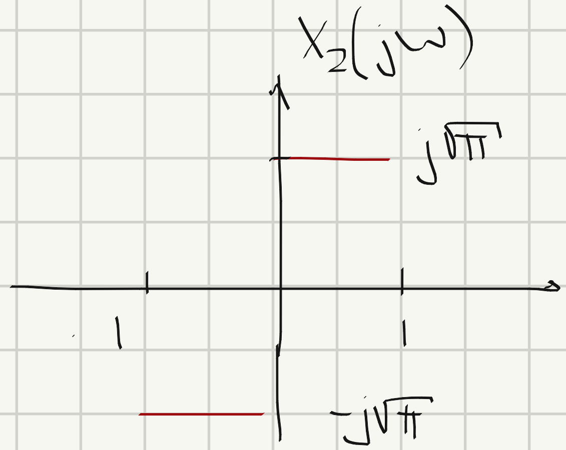
Obs: * Puedo calcular cantidades en el tiempo sin la formula explicita (sin reconstruir)

Ejemplo 4.14

1



2



$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt \qquad D = \frac{d}{dt} x(t) \Big|_{t=0}$$

d) energías

$$E_1 = \frac{1}{2\pi}\left[\frac{1}{2}(\sqrt{\pi})^2 + 1\left(\frac{\sqrt{\pi}}{2}\right)^2 + \frac{1}{2}(\sqrt{\pi})^2\right] = \frac{\pi + \frac{\pi}{4}}{2\pi} = \boxed{\frac{5}{8}}$$

$$E_2 = \frac{2 \cdot 1 \cdot \pi}{2\pi} = \boxed{1}$$

b) derivadas

$$g(t) = \frac{d\,x(t)}{dt} \longrightarrow D = g(0)$$

$$g(t) \xleftrightarrow{\mathcal{F}} jw\,X(jw) = G(jw)$$

ec. síntesis de $g(t)$

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} jw\,X(jw)\,e^{jwt}\,dw \qquad t \to 0$$

$t=0$

$$\boxed{g(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} jw\,X(jw)\,dw}$$

$D_1 = 0$ (impar)

$$D_2 = 2\,\frac{1}{\cancel{2}\pi}\int_0^1 \cancel{j}w\,\cancel{j}^{-1}\sqrt{\pi}\,dw = -\frac{\sqrt{\pi}}{\pi}\,\frac{w^2}{2}\Big|_0^1 = \boxed{-\frac{1}{2\sqrt{\pi}}}$$

par

obs: * repaso del concepto de continua.

$$X(j0) = \int_{-\infty}^{\infty} x(t)\,dt$$

* valores puntuales en un dominio se transforman en integrales en el otro.

Clase 21

Transformada de Fourier

Propiedades (cont)

Propiedades aplicadas a sistemas

Convolución

$$y(t) = h(t) * x(t) \qquad x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$\mathcal{F}\{y(t)\} = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \right) e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(\tau) \underbrace{\int_{-\infty}^{\infty} h(t-\tau)\, e^{-j\omega t}\, dt}_{A}\, d\tau$$

C.V.: $r = t - \tau$

$$A = \int_{-\infty}^{\infty} h(r)\, e^{-j(r+\tau)\omega}\, dr = e^{-j\omega\tau} \overbrace{\int_{-\infty}^{\infty} h(r)\, e^{-j\omega r}\, dr}^{H(j\omega)}$$

$$= \boxed{H(j\omega)\, e^{-j\omega\tau}}$$

$$Y(j\omega) \overset{A}{=} \int_{-\infty}^{\infty} x(\tau)\, \underline{H(j\omega)}\, e^{-j\omega\tau}\, d\tau = H(j\omega) \underbrace{\int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau}\, d\tau}_{X(j\omega)}$$

$$\boxed{Y(j\omega) = H(j\omega)\, X(j\omega)}$$

$$\boxed{y(t) = h(t) * x(t) \xleftrightarrow{\mathcal{F}} Y(\omega) = H(\omega) \cdot X(j\omega)}$$

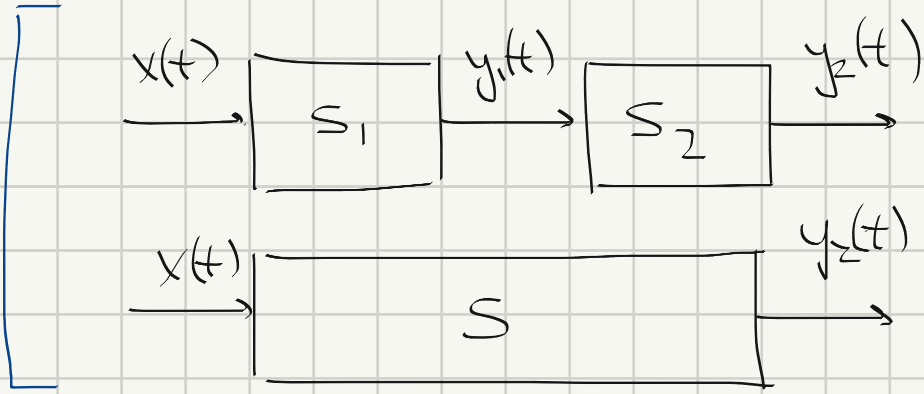
obs: * nos permite interpretar los sistemas como filtros y ver como modifican el espectro de la entrada

* Como $h(t)$ caracteriza completamente el sistema, entonces $H(j\omega)$ también

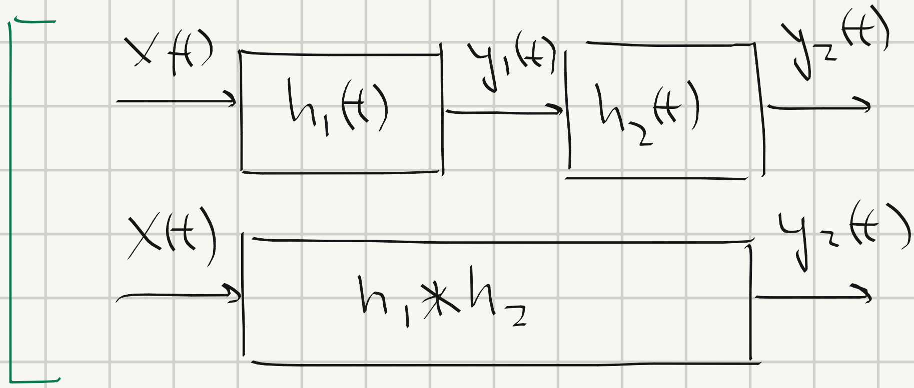
* puedo ver propiedades del sistema en $H(j\omega)$

Ejemplo:

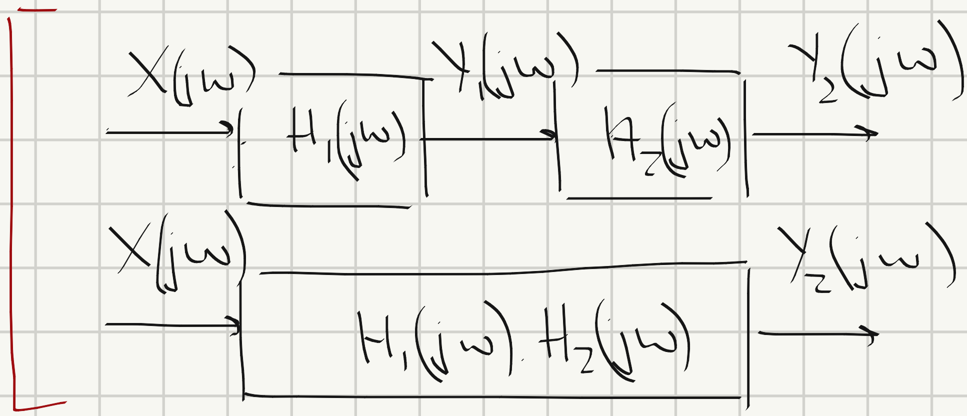
SLIT
Def. tiempo
Ec. transf.



SLIT
tiempo
convolución



SLIT
Espectral
T. Fourier

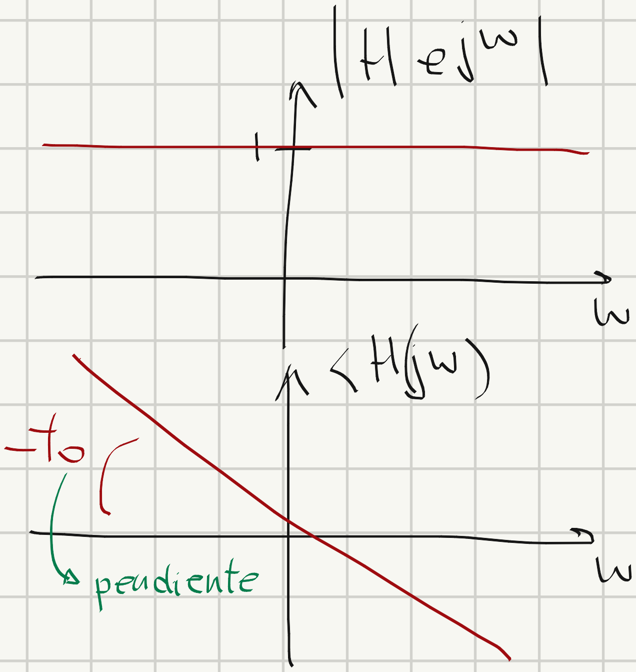


obs: * todas las propiedades se vuelven más evidentes
* por ejemplo la asociativa y conmutativa.

Ejemplo 4.15 (p. 317)

$h(t) = \delta(t - t_0) \quad H(j\omega) = e^{-j\omega t_0}$

$Y(j\omega) = e^{-j\omega t_0} X(j\omega)$



$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$

$\downarrow S$

$x(t-t_0) \overset{?}{\xleftrightarrow{\mathcal{F}}} e^{-j\omega t_0} X(j\omega)$

$\boxed{y(t) = x(t - t_0)}$

$\tau(\omega) = -\frac{d\,\theta(j\omega)}{d\omega} = \boxed{t_0}$

Ejemplo 4.16 (Diferenciador)

$y(t) = \frac{d\,x(t)}{dt} \rightarrow Y(j\omega) = j\omega \cdot X(j\omega)$

$\boxed{H(j\omega) = j\omega}$

obs: * uso la propiedad al revés
* es difícil obtener $h(t) = \delta'(t)$

Ejemplo 4.19 (p. 320)

$$h(t) = u(t) \cdot e^{-at} \qquad x(t) = u(t) \cdot e^{-bt}$$

$$H(jw) = \frac{1}{b+jw} \qquad X(jw) = \frac{1}{a+jw}$$

a) salida en frecuencia b) salida en el tiempo

1 $$Y(jw) = H(jw) \cdot X(jw) = \frac{1}{b+jw} \cdot \frac{1}{a+jw} \overset{b \neq a}{=} \frac{A}{a+jw} + \frac{B}{b+jw}$$

$$\boxed{Y(jw) = \frac{1}{b-a} \left(\frac{1}{a+jw} - \frac{1}{b+jw} \right)}$$

$$\boxed{y(t) = \frac{1}{b-a} u(t) \cdot \left(e^{-at} - e^{-bt} \right)}$$

2 $b = a$ $$Y(jw) = \frac{1}{(a+jw)^2} = \frac{1}{-j} \cdot \frac{d}{dw} \left(\frac{1}{a+jw} \right)$$

$$\boxed{Y(jw) = j \cdot \frac{d}{dw} \left(\frac{1}{a+jw} \right)}$$

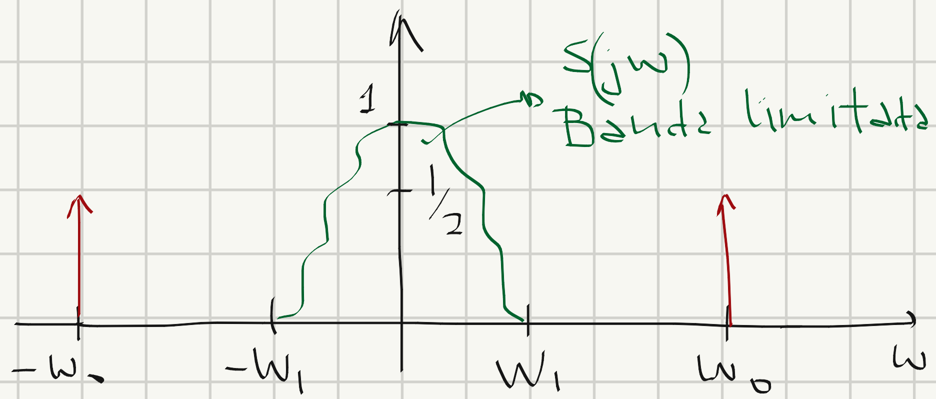
$$y(t) = j(-j \cdot t) u(t) \cdot e^{-at} \Rightarrow \boxed{y(t) = u(t) \cdot t \cdot e^{-at}}$$

Multiplicación (modulación)

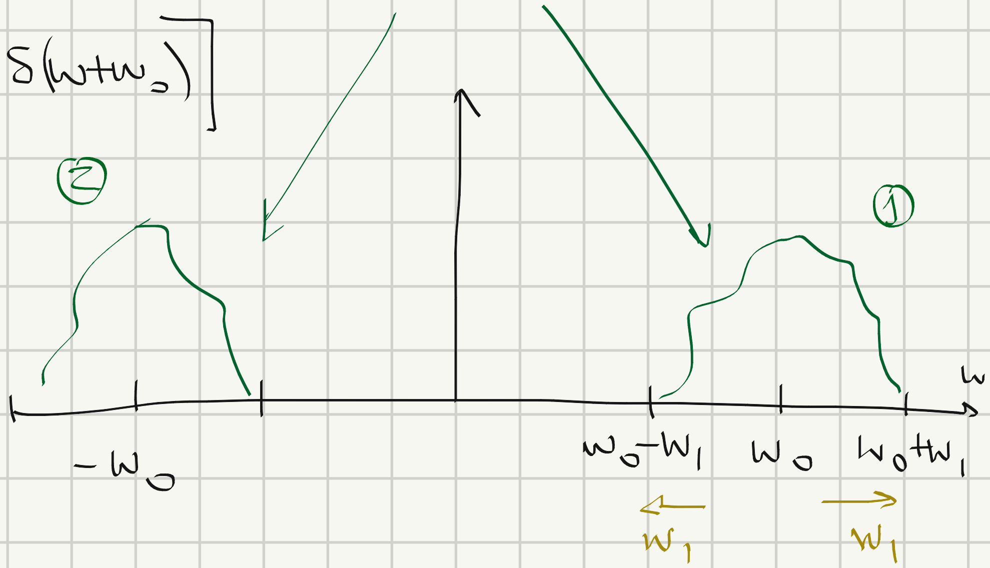
$$m(t) = \underbrace{s(t)}_{\text{señal}} \cdot \underbrace{p(t)}_{\text{portadora}} \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi}\left[S(j\omega) * P(j\omega)\right] = M(j\omega)$$

Ejemplo 4.21

$$p(t) = \cos(\omega_0 t)$$

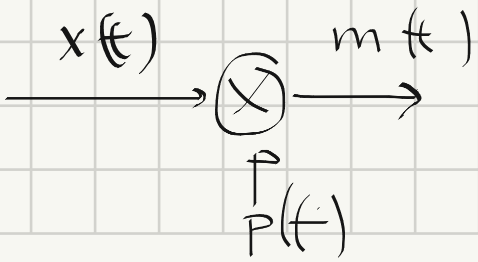


$$P(j\omega) = \frac{1}{2}\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$$



$$Y(j\omega) = S(j\omega) * \frac{1}{2}\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$$

$$\boxed{Y(j\omega) = \frac{1}{2} \underbrace{S(j(\omega-\omega_0))}_{1} + \frac{1}{2} \underbrace{S(j(\omega+\omega_0))}_{2}}$$



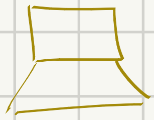
Ejemplo: "modulación en amplitud"

→ voz: 300 Hz – 4 kHz.

S: $x_1(t) = \text{sen}(2\pi f_1 \cdot t)$, $f_1 = 400\,\text{Hz}$ $\quad f_1 \ll f_2$

P: $x_2(t) = \text{sen}(2\pi f_2 \cdot t)$, $f_2 = 5000\,\text{Hz}$

Obs: * ver en simulación



* $x_1(t) = \text{sen}(\omega_1 t) + k$

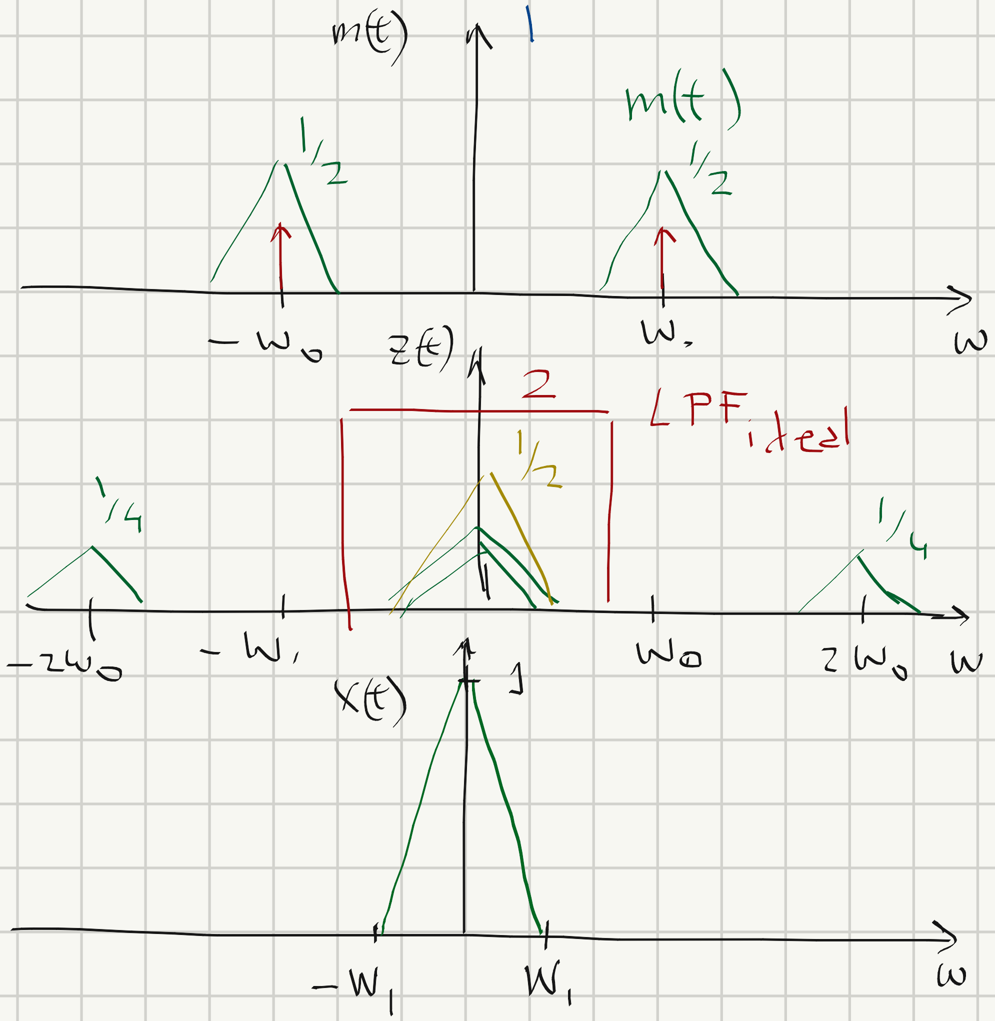
Ejemplo: "demodulador"

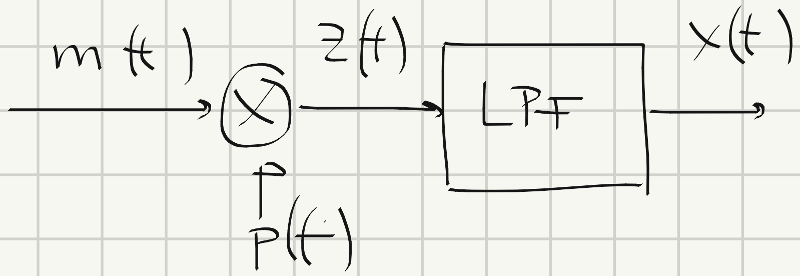
$$z(t) = [\overbrace{s(t)\,p(t)}^{m(t)}]\,\cos(w_0 t)$$

$$Z(jw) = \frac{1}{2}\left[S(j(w-w_0)) + S(j(w+w_0))\right] * \frac{1}{2}\left[S(w-w_0) + S(w+w_0)\right]$$

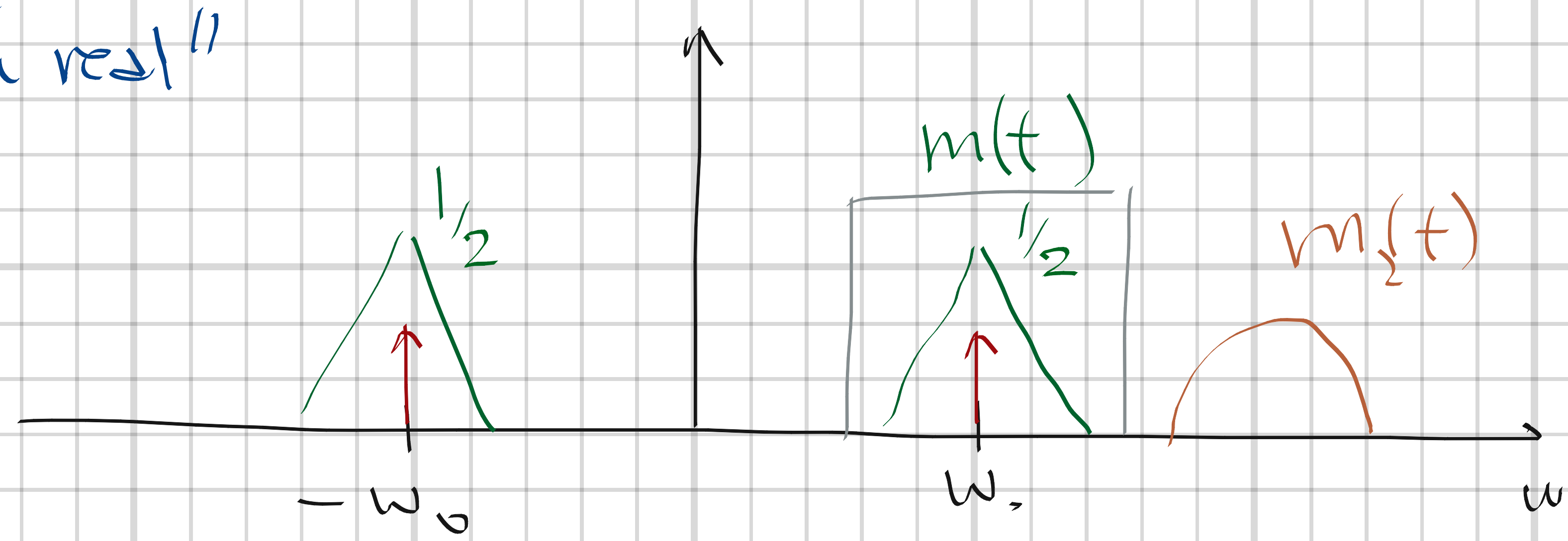
$$Z(jw) = \frac{1}{4} S(j(w-2w_0)) + \underbrace{\frac{1}{4} S(jw) + \frac{1}{4} S(jw)}_{} + \frac{1}{4} S(j(w+2w_0))$$

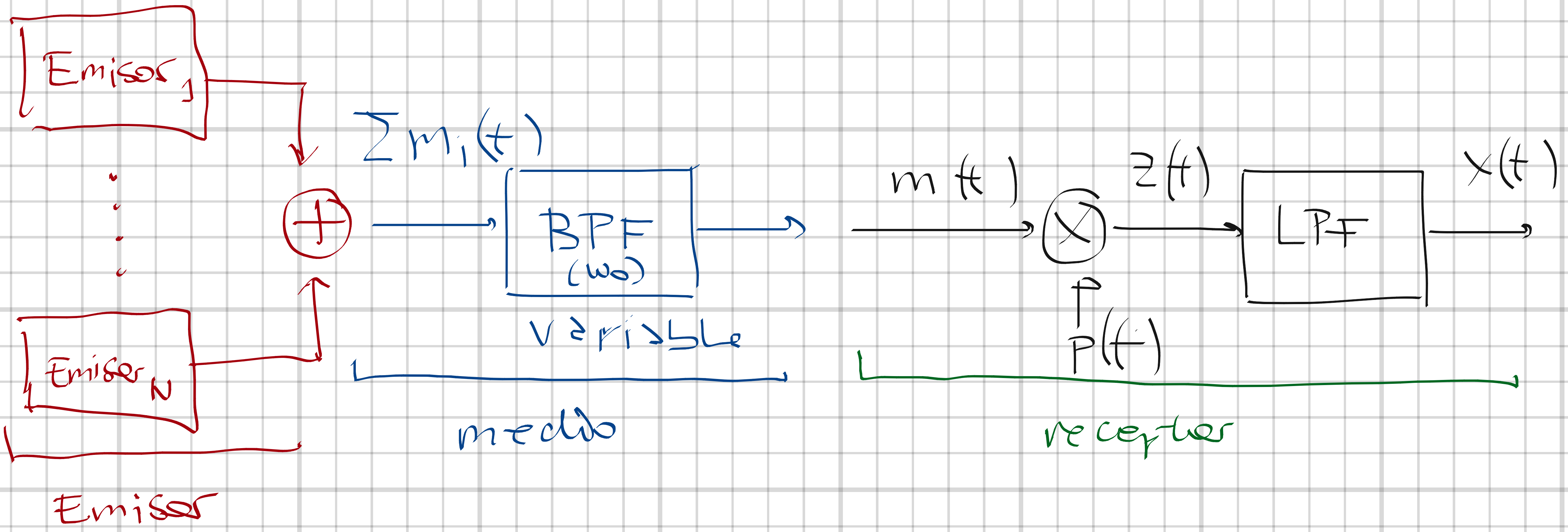
$$\boxed{Z(jw) = \frac{1}{2} S(jw) + \frac{1}{4} S(j(w-2w_0)) + \frac{1}{4} S(j(w+2w_0))}$$





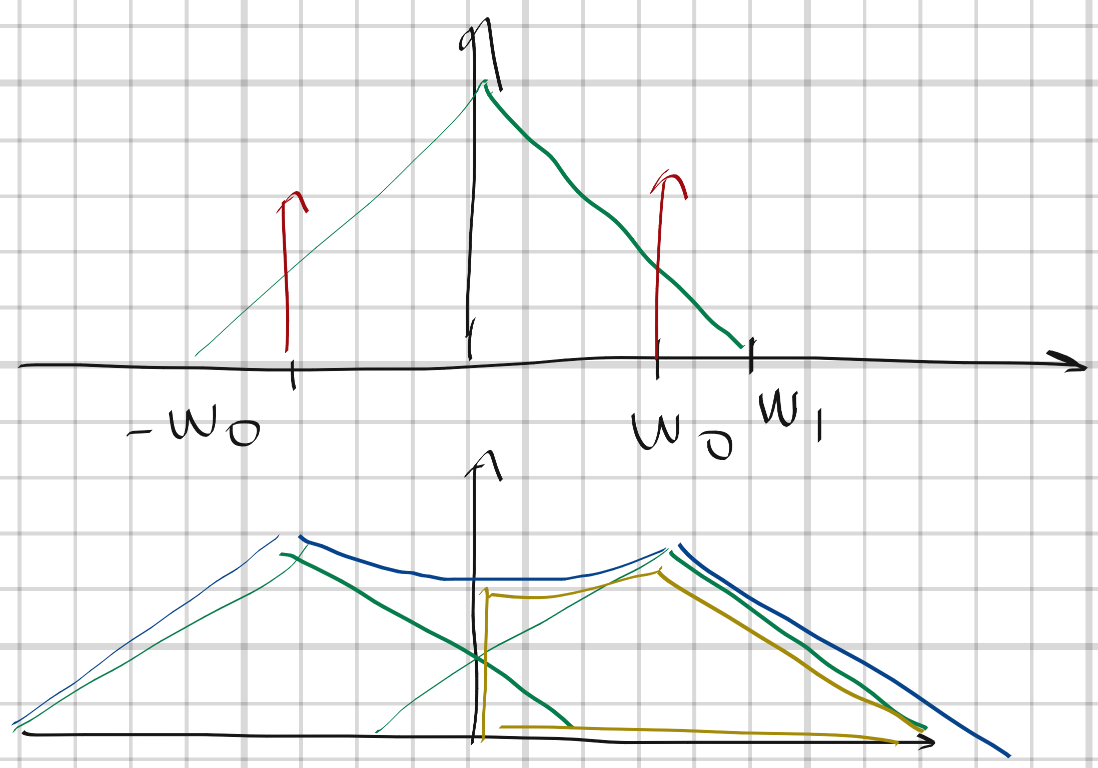
Ejemplo: "reconstrucción real"

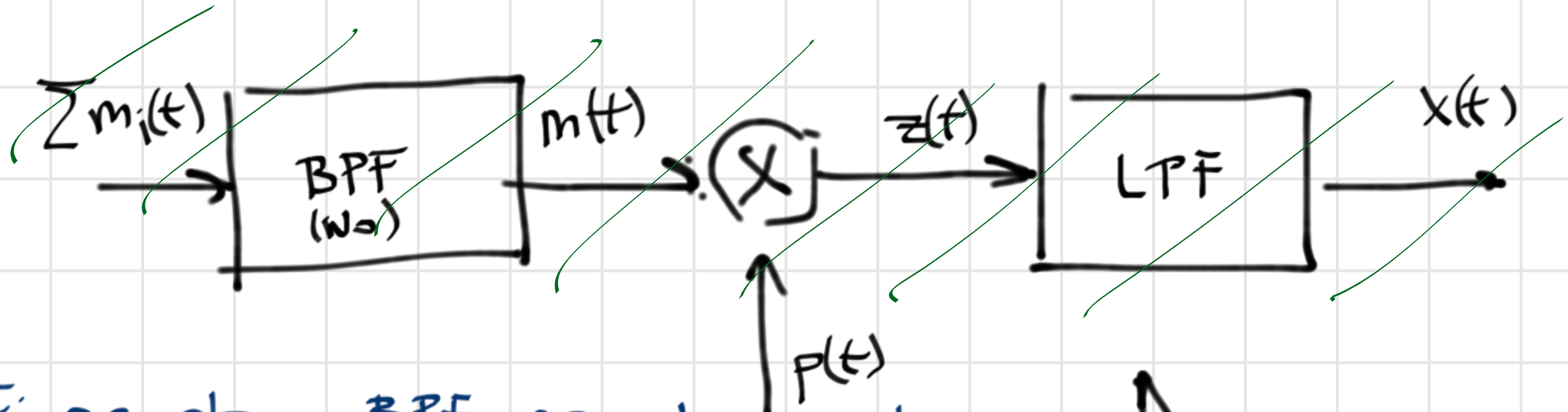




obs: "banda limitada"

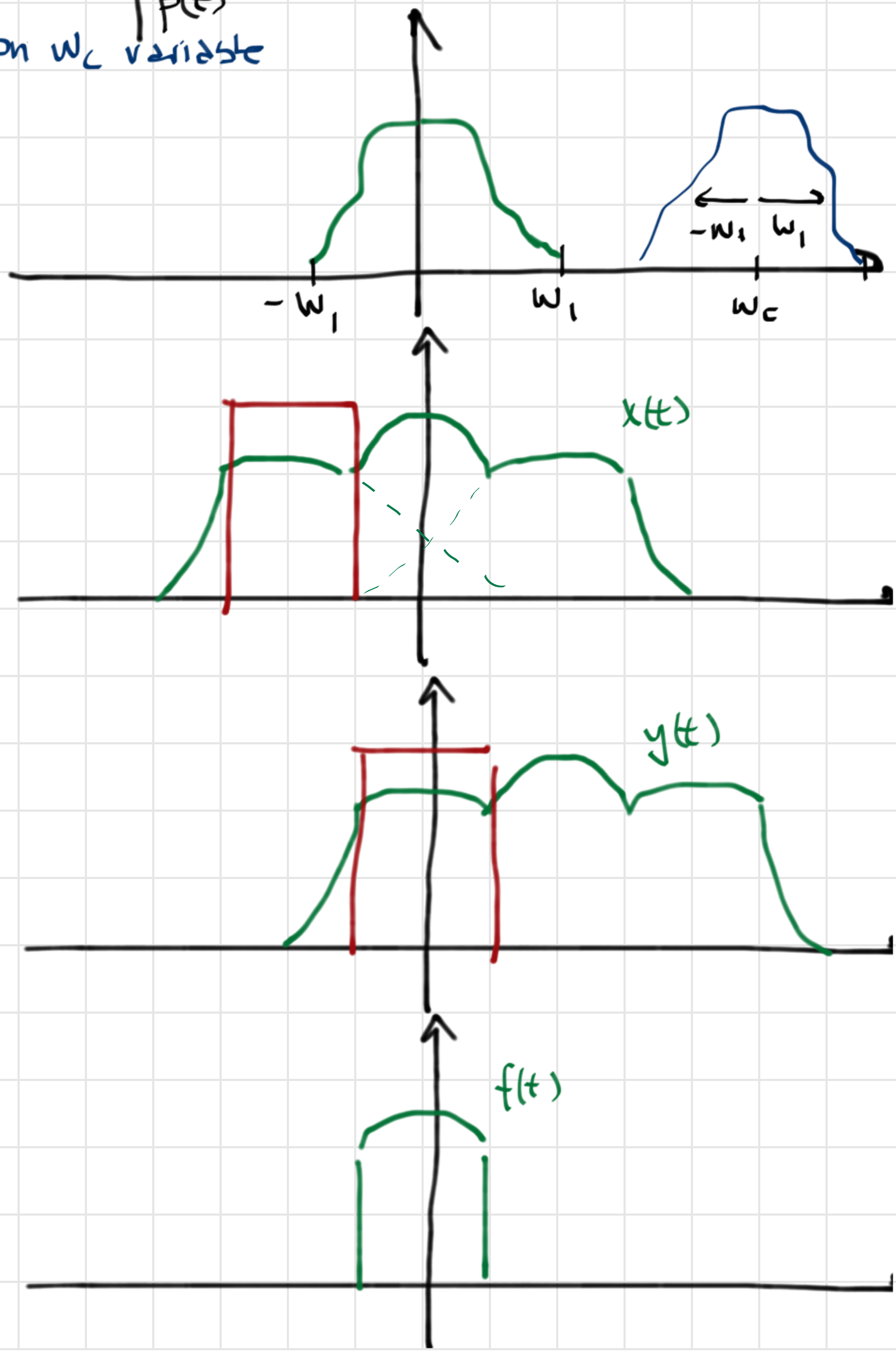
$$W_1 < \frac{w_0}{2}$$

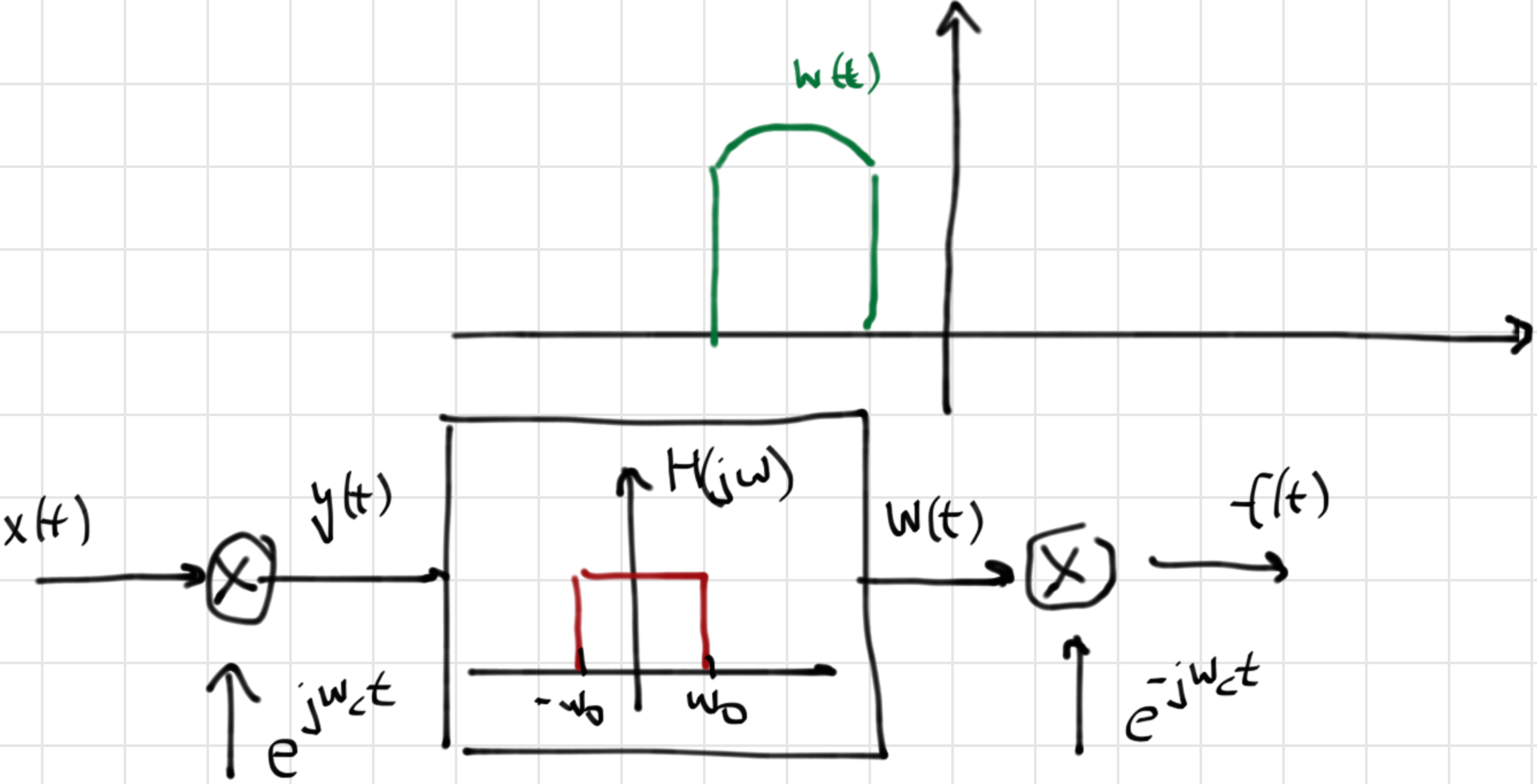




Ejemplo: BPF con w_c variable

Hay que ver que hacer con este.




w(t)
x(t)
y(t)
H(jw)
W(t)
f(t)
$e^{jw_c t}$
$-w_0$
w_0
$e^{-jw_c t}$

Practico 11

Transformada de Fourier

Ejemplos integradores

Ejemplo integrador

* limitaciones del análisis: osciloprama vs batido

Clase 22

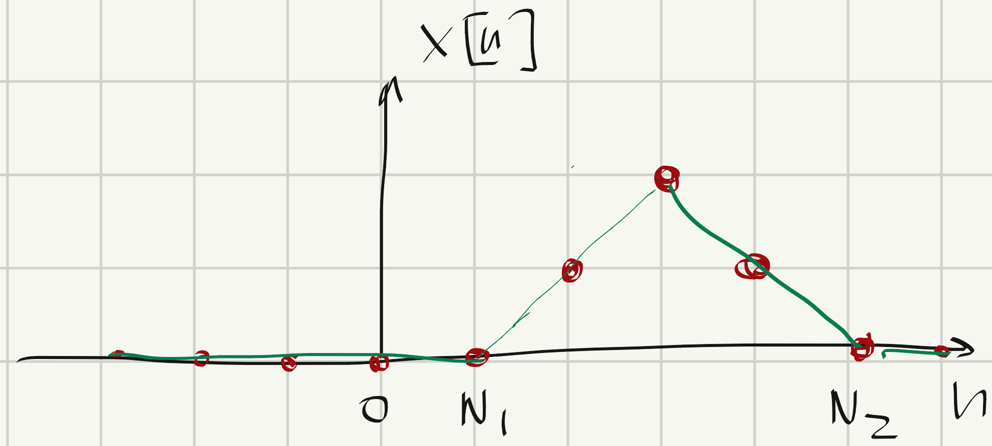
Transformada de Fourier

T. Fourier en tiempo discreto

T. FOURIER EN TIEMPO DISCRETO (DTFT)

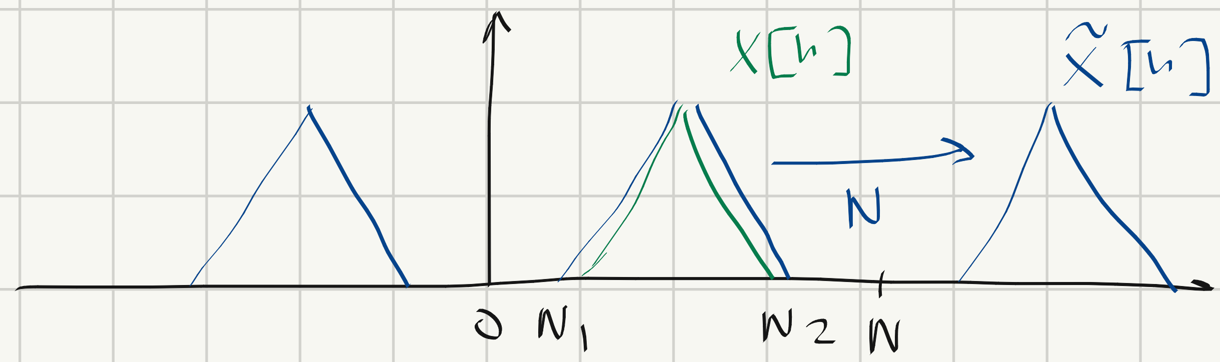
Aproximación periódica

$x[n]$ de soporte acotado $[N_1, N_2]$



$\tilde{x}[n]$ periódica N tal que:

$$\tilde{x}[n] = x[n] \quad \text{en } [N_1, N_2]$$



$$\lim_{N\to\infty} \tilde{x}[n] = x[n]$$

Serie de Fourier de $\tilde{x}[n]$ (DTFS)

$$\tilde{x}[n] = \sum_{k=\langle N\rangle} a_k \, e^{jk\omega_0 n}, \quad \omega_0 = \frac{2\pi}{N}$$

Análisis: $a_k = \frac{1}{N} \sum_{n=\langle N\rangle} \tilde{x}[n] \, e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=N_1}^{N_2} x[n] \, e^{-jk\omega_0 n} \Rightarrow$ ($x[n]$ sop. acotado)

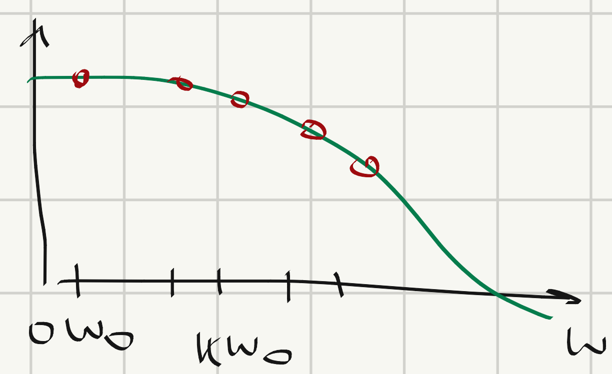
$$a_k = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] \, e^{-jk\omega_0 n} \Rightarrow N a_k = \underbrace{\sum_{n=-\infty}^{\infty} x[n] \, e^{-jk\omega_0 n}}_{f(k\omega_0) = f(\omega)\big|_{\omega = k\omega_0}}$$

$k\omega_0 \to \omega$

$$\lim_{N\to\infty} N a_k = \sum_{n=-\infty}^{\infty} x[n] \, e^{-j\omega n} \qquad \begin{matrix} N\to\infty \\ \omega_0 \to 0 \end{matrix}$$

$$\boxed{X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] \, e^{-j\omega n}}$$

ec. análisis

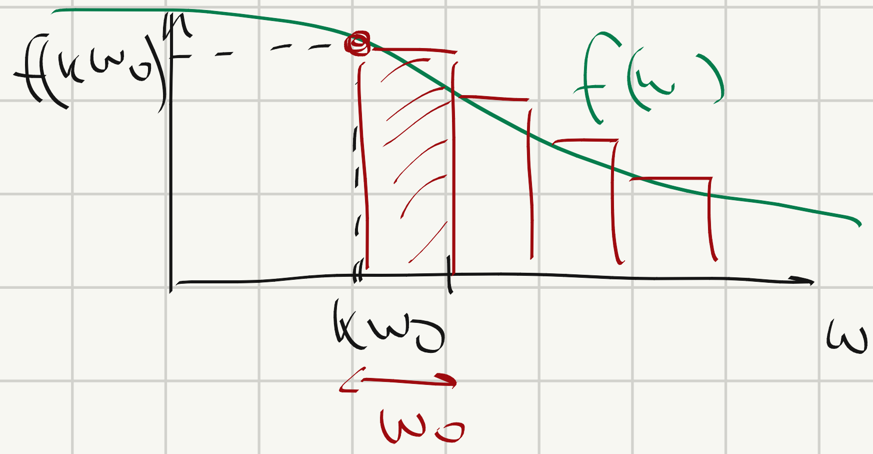


$$\boxed{a_k = \frac{1}{N} X(e^{j\omega})\Big|_{\omega = k\omega_0}}$$

Síntesis $\quad \tilde{x}[n] = \sum_{k=\langle N\rangle} a_k \, e^{jk\omega_0 n} \quad , \quad \omega_0 = \frac{2\pi}{N} \; , \quad \frac{1}{N} = \frac{\omega_0}{2\pi}$

$$\tilde{x}[n] = \sum_{k=\langle N\rangle} \overset{\frac{\omega_0}{2\pi} \,''}{\frac{1}{N}} \, X(e^{j\omega})\Big|_{\omega=k\omega_0} e^{jk\omega_0 n} = \frac{1}{2\pi} \sum_{k=\langle N\rangle} \underbrace{X(e^{j\omega})\Big|_{\omega=k\omega_0} e^{jk\omega_0 n}}_{f(k\omega_0) = f(\omega)|_{\omega=k\omega_0}} \cdot \omega_0$$

$$x[n] = \lim_{N\to\infty} \tilde{x}[n] \Rightarrow$$



$$\boxed{x[n] = \frac{1}{2\pi} \int_{\langle 2\pi\rangle} X(e^{j\omega}) \, e^{j\omega n} \, d\omega}$$

ec. síntesis

Transformada de Fourier de Tiempo Discreto

Análisis:

$$\boxed{X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}}$$

$$\boxed{a_k = \frac{1}{N} \cdot X(e^{j\omega})\Big|_{\omega = k\omega_0}}$$ Coefs. Fourier

Síntesis:

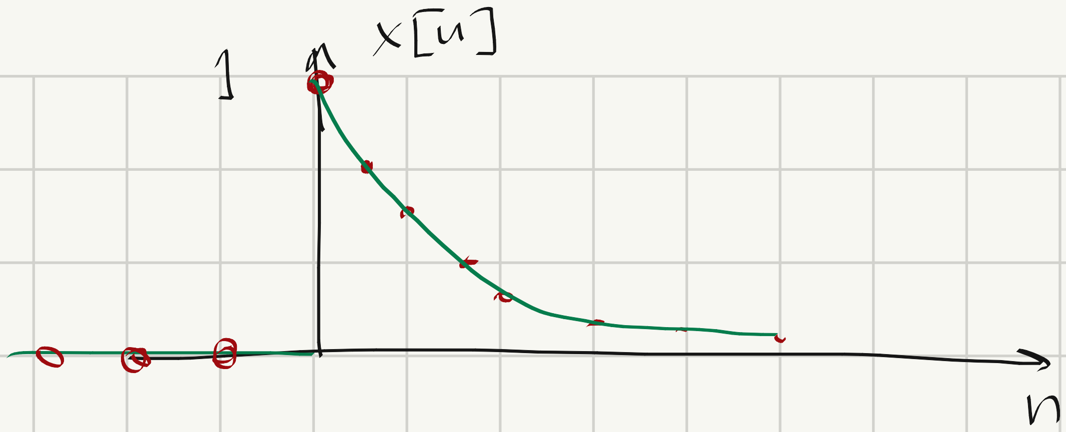
$$\boxed{x[n] = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(e^{j\omega}) \cdot e^{j\omega n} \cdot d\omega}$$

Obs: * Terminamos con Fourier (continuaciones)

* Descomposición como C.L de exponenciales
* DTFT son muestras de la DTFT
* DIFERENCIA con CTFT: → $X(e^{j\omega})$ periódica
  → Ec. síntesis finita.

Ejemplo 5.1 (p. 362)

$$x[n] = u[n] \cdot a^n, \quad |a|<1$$



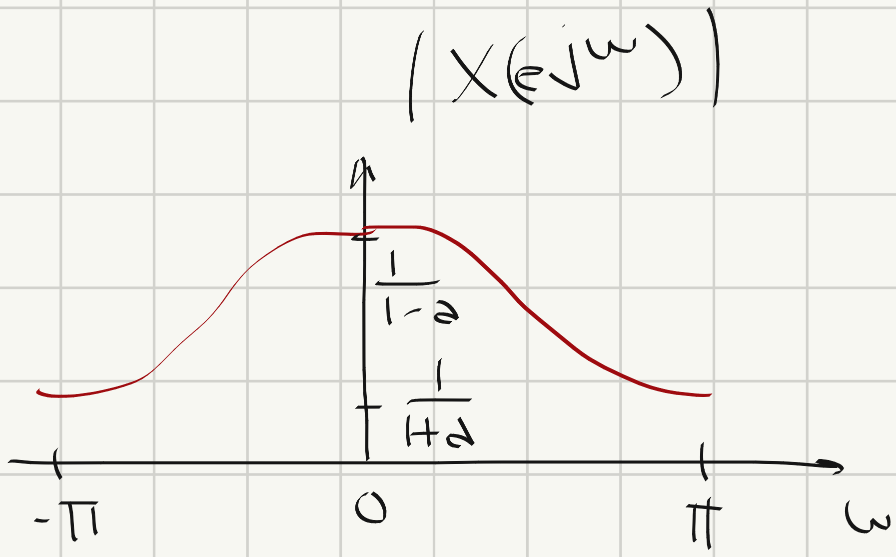
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} u[n] \cdot a^n \cdot e^{-j\omega n} = \sum_{k=0}^{\infty} \left(ae^{-j\omega}\right)^n \overset{a\neq 1}{=} \frac{1-\left(ae^{-j\omega}\right)^{\cancel{N+1}\to 0}}{1-ae^{-j\omega}}$$

$$\boxed{X(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}}$$

$$|X(e^{j\omega})| = \frac{1}{\sqrt{(1-a\cos\omega)^2 + a^2 \operatorname{sen}^2\omega}}$$

$$|X(e^{j0})| = \frac{1}{\sqrt{(1-a)^2}} = \frac{1}{1-a}$$

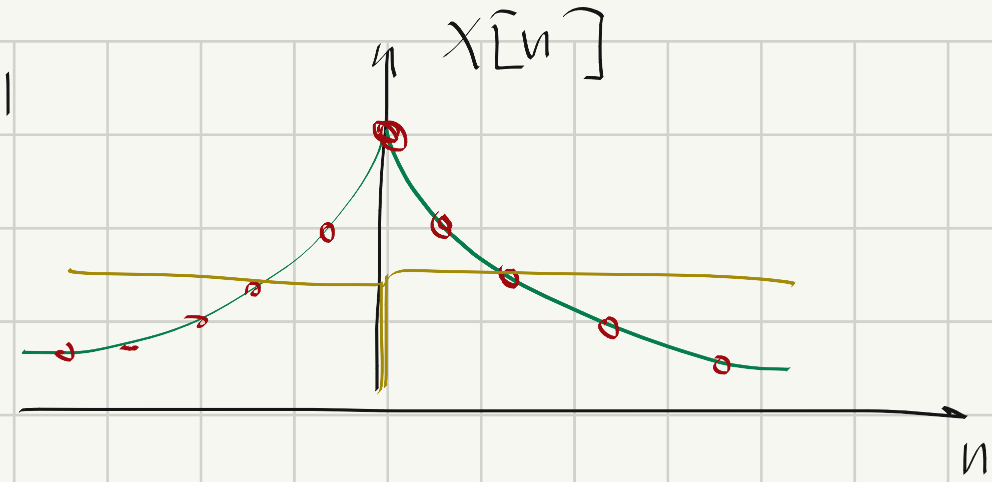
$$|X(e^{j\pi})| = \frac{1}{\sqrt{(1+a)^2}} = \frac{1}{1+a}$$



b) Código

Ejemplo 5.2 (p. 364)

$|a| < 1$



$$x[n] = u[n] \cdot a^{n} + u[-n] \cdot a^{-n}$$

$$x[n] = a^{|n|}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\omega n} = \underbrace{\sum_{n=0}^{\infty} a^{|n|} \cdot e^{-j\omega n}}_{S.1} + \underbrace{\sum_{n=-\infty}^{-1} a^{|n|} \cdot e^{-j\omega n}}_{2}$$

C.V. $m=n$

$$2 = \sum_{n=-\infty}^{-1} a^{-n} \cdot e^{-jn\omega} = \sum_{n=1}^{\infty} a^{m} \cdot e^{j\omega m} = \sum_{m=0}^{\infty} a^{m} e^{jm\omega} - 1$$

$m = -n$

$$= \frac{1 - \underbrace{(a e^{j\omega})^{\cancel{M+1} \to \infty}}_{\to 0}}{1 - (a e^{j\omega})}$$

Tamara

$$\frac{2}{1-e^{-j\omega}} - a$$

Manuel

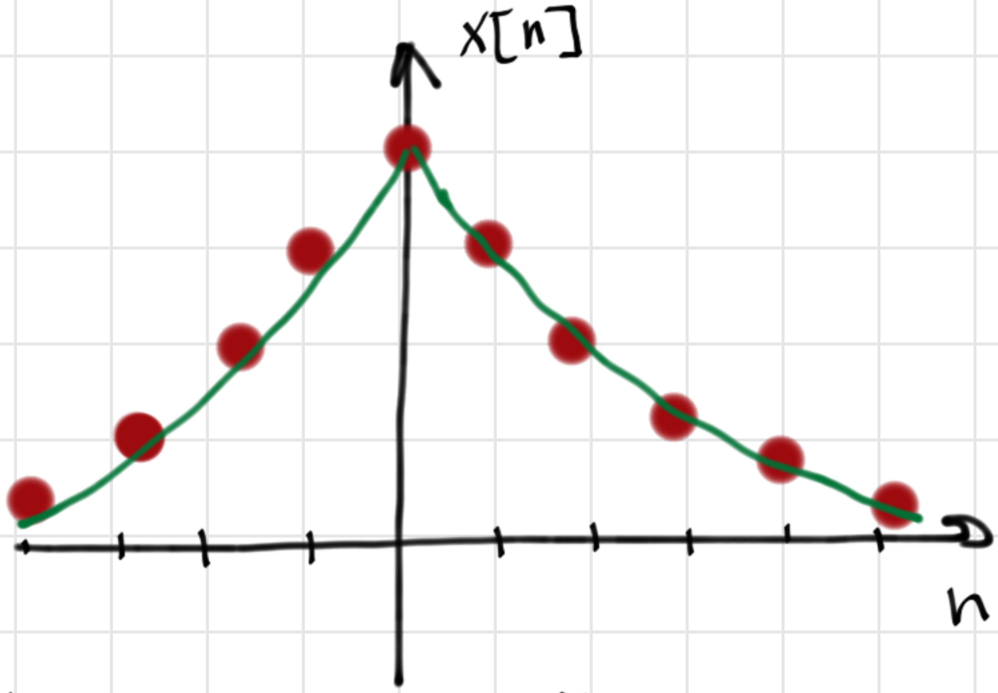
$$\frac{1}{1-ae^{-j\omega}} + \frac{1}{1-\frac{e^{-j\omega}}{a}} - 1$$

Lucía

$$\boxed{\frac{1}{1-ae^{-j\omega}} + \frac{1}{1-ae^{j\omega}} - 1}$$

Ejemplo 5.2 (p. 364)

$$x[n] = a^{|n|} \quad , |a|<1$$



$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-jnw}$$

$$= \sum_{n=-\infty}^{\infty} a^{|n|} \cdot e^{-jnw}$$

$$= \underbrace{\sum_{n=0}^{\infty} a^{n} \cdot e^{-jnw}} + \sum_{n=-\infty}^{-1} a^{-n} \cdot e^{-jnw} = \frac{1}{1-ae^{-jw}} + \sum_{n=1}^{\infty} a^{n} \cdot e^{jnw}$$

$$= \frac{1}{1-ae^{-jw}} + \sum_{n=0}^{\infty} a^{n} \cdot e^{jnw} - 1 = \frac{1}{1-ae^{-jw}} + \frac{1}{1-ae^{+jw}} - 1$$

$$= \frac{1 - \cancel{ae^{jw}} + \cancel{1} - \cancel{ae^{-jw}} - \overbrace{(\cancel{1} - \cancel{ae^{-jw}} - \cancel{ae^{jw}} + a^2)}^{*_1}}{*_1}$$

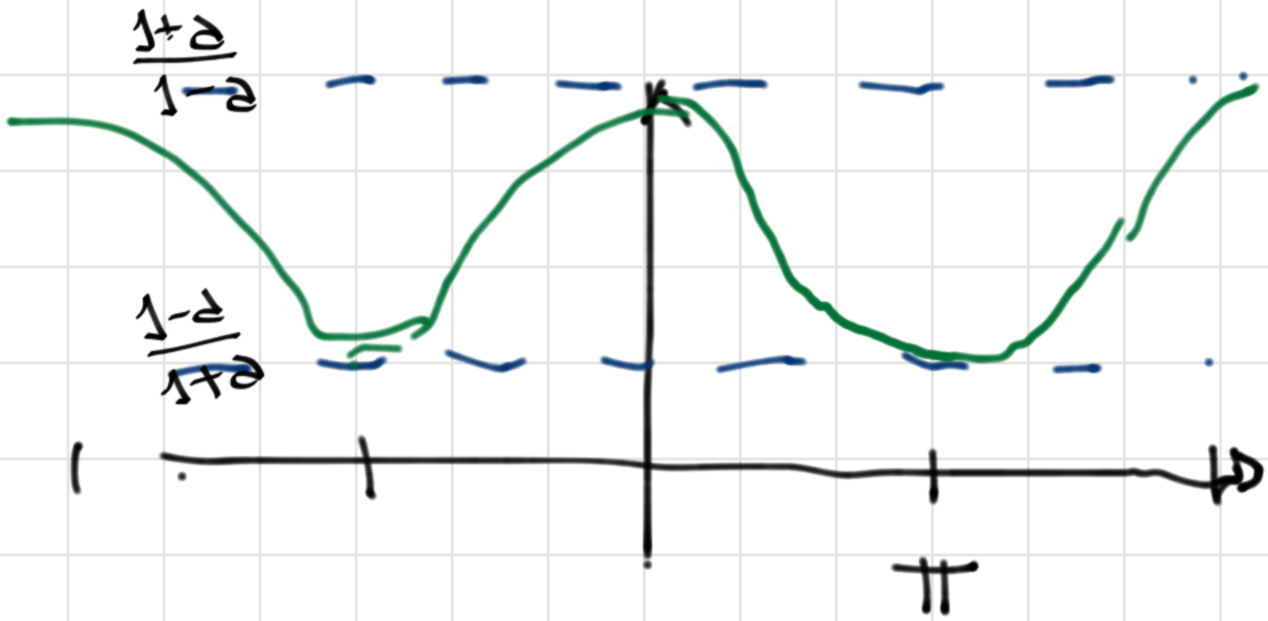
$$= \boxed{\frac{1-a^2}{1-2a\cos w + a^2}}$$

obs: $*$ $X(e^{jw}) \in \mathbb{R}$

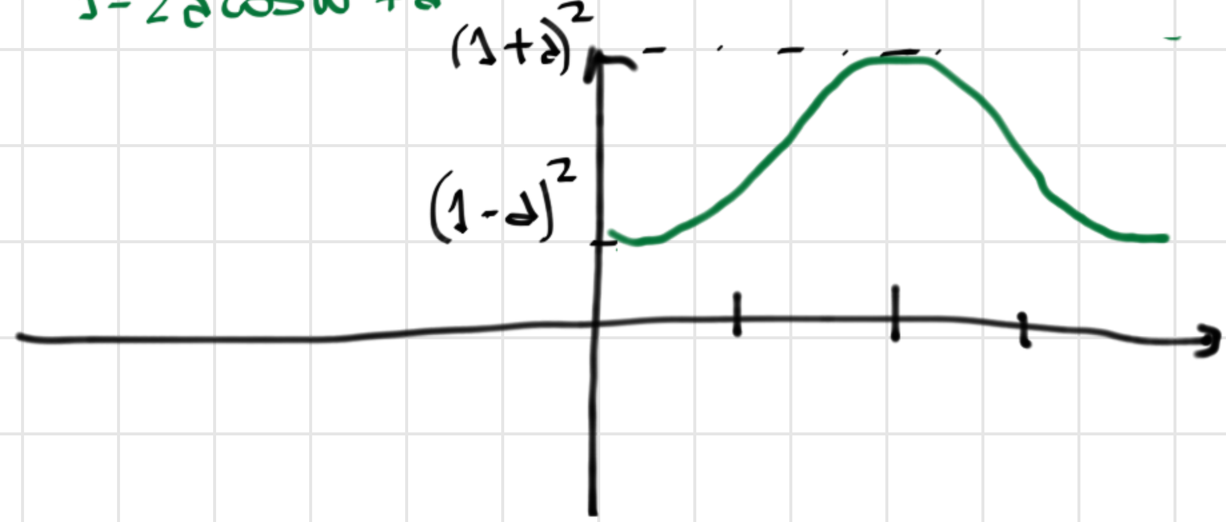
$$X(e^{j0}) = \frac{1-a^2}{1-2a+a^2} = \frac{\cancel{(1-a)}(1+a)}{(1-a)^{\cancel{2}}} = \boxed{\frac{1+a}{1-a}}$$

$$X(e^{j\pi}) = \frac{1-a^2}{1+2a+a^2} = \frac{\cancel{(1+a)}(1-a)}{(1+a)^{\cancel{2}}} = \boxed{\frac{1-a}{1+a}}$$

corregir



$1 - 2a\cos w + a^2$



Practico 12

Transformada de Fourier

Ejemplos tiempo discreto

Ejemplo 5.3 (p. 365)

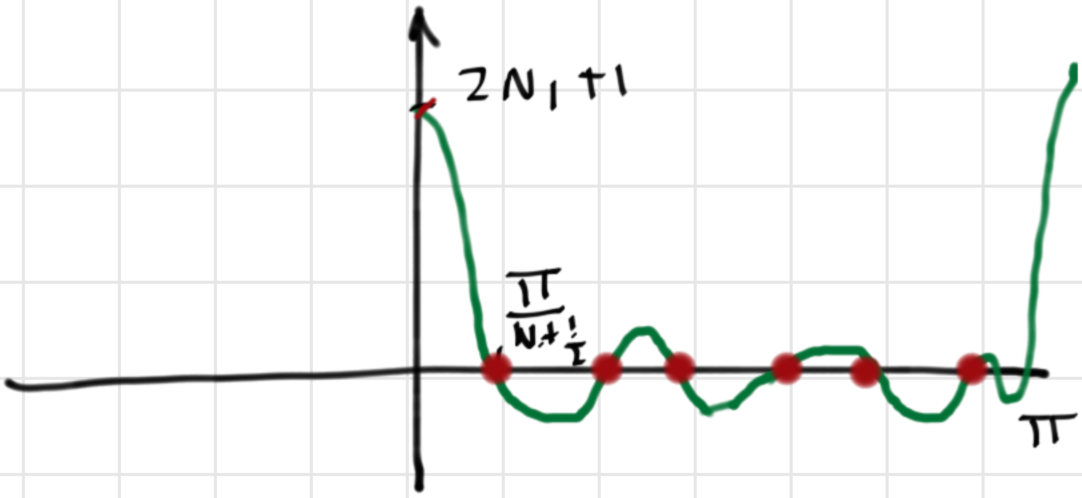
$$x[n] = \begin{cases} 1 & |n| \leq N_1 \\ 0 & |n| > N_1 \end{cases}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jn\omega} = \sum_{n=-N_1}^{N_1} e^{-jn\omega} \underset{\substack{\uparrow \\ \text{C.V.} \\ m = n+N_1}}{=} \sum_{n=0}^{2N_1} e^{-j(m-N_1)\omega}$$

$$= e^{+jN_1\omega} \sum_{n=0}^{2N_1} e^{-jm\omega} = e^{+jN_1\omega} \cdot \frac{1 - e^{-j(2N_1+1)\omega}}{1 - e^{-j\omega}} =$$

$$= \frac{e^{+jN_1\omega} - e^{-j(N_1+1)\omega}}{1 - e^{-j\omega}} \cdot \frac{e^{j\frac{\omega}{2}}}{e^{j\frac{\omega}{2}}} = \frac{e^{j(N_1+\frac{1}{2})\omega} - e^{-j(N_1+\frac{1}{2})\omega}}{e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}}$$

$$\boxed{X(e^{j\omega}) = \frac{\operatorname{sen}[(N_1+\frac{1}{2})\omega]}{\operatorname{sen}(\frac{\omega}{2})}}$$



obs: * senoc periódico

$$\lim_{\omega \to 0} X(e^{j\omega}) = \frac{\omega(N_1+\frac{1}{2})}{\omega/2} = \boxed{2N_1+1}$$

ceros: $\boxed{\omega_k = \frac{\pi}{N_1+\frac{1}{2}}}$

Clase 23

Transformada de Fourier

T. Fourier en tiempo discreto

CONVERGENCIA

Obs:

* análisis: $\boxed{X(e^{j\omega}) = \frac{1}{N}\sum_{n=-\infty}^{\infty} x[n] \cdot e^{-jn\omega}}$ serie X

* síntesis: $\boxed{x[n] = \frac{1}{2\pi}\int_{\langle 2\pi \rangle} X(e^{j\omega}) \cdot e^{j\omega n} \, d\omega}$ integral finita ✓

Condiciones

1) Energía: $\sum_{k=-\infty}^{\infty} |x[n]|^2 < \infty$

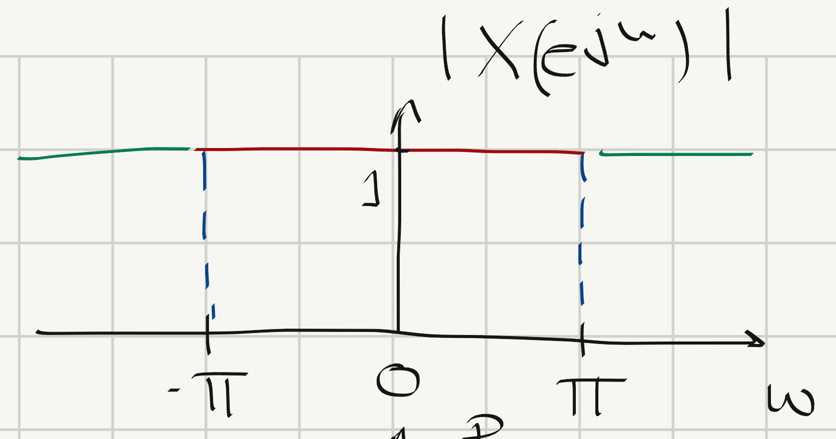
2) Dirichlet: * $\sum_{k=-\infty}^{\infty} |x[n]| < \infty$

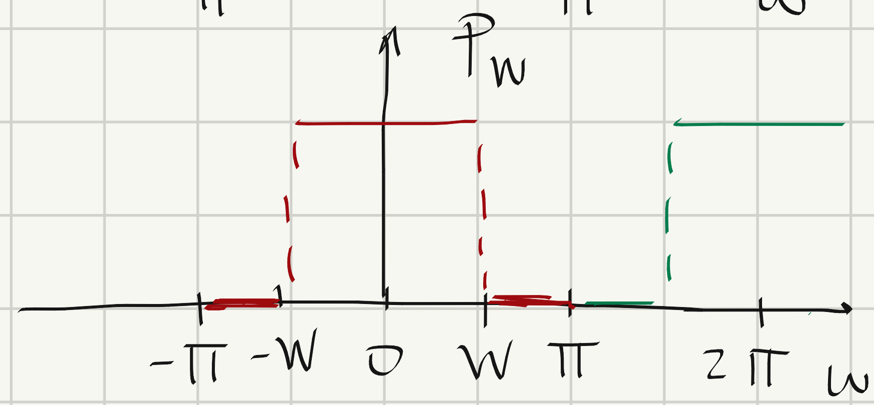
* otras: máximos y discontinuidades

Ejemplo 5.4

$$x[n] = \delta[n] \overset{\mathcal{F}}{\longleftrightarrow} 1 = X(e^{j\omega})$$

$$\tilde{x}[n] = \,? \longleftarrow \overset{\mathcal{F}}{\longleftrightarrow} P_W(e^{j\omega})$$

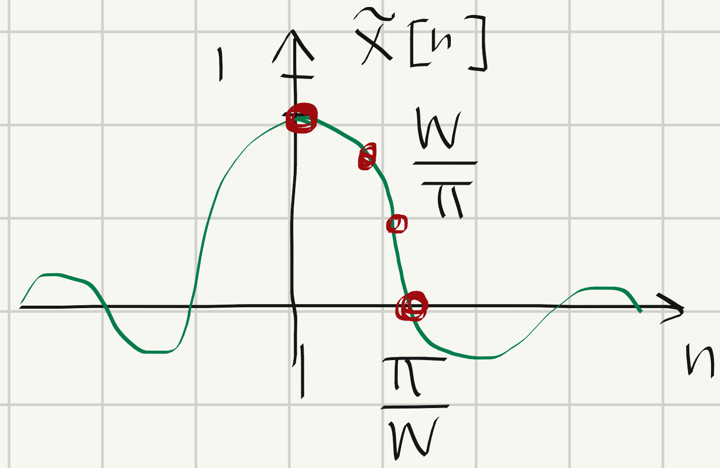




$$\tilde{x}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_W(e^{j\omega})\, e^{jn\omega}\, d\omega$$

$$\tilde{x}[n] = \frac{1}{2\pi}\int_{-W}^{W} 1 \cdot e^{jn\omega}\, d\omega \overset{n \neq 0}{=} \frac{1}{2\pi}\, \frac{e^{jWn} - e^{-jWn}}{jn} = \boxed{\frac{\operatorname{sen}[Wn]}{\pi n}} \quad n \neq 0$$

$$\tilde{x}[0] = \frac{2W}{2\pi} = \boxed{\frac{W}{\pi}}$$



$$x[n] = \lim_{n \to 0} \frac{\operatorname{sen}[Wn]}{\pi n} = \frac{W\cancel{n}}{\pi \cancel{n}} = \boxed{\frac{W}{\pi}}$$

obs: $\tilde{x}[n] \to x[n]$

$W \to \pi$

$$x[n] = \lim_{W \to \pi} \tilde{x}[n] = \lim_{W \to \pi} \frac{\operatorname{sen}[Wn]}{\pi n} = \frac{\operatorname{sen}[\pi n]}{\pi n} = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

$$\boxed{x[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}} \qquad x[n] = \delta[n]$$

PROPIEDADES

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}) \quad , \quad y[n] \overset{\mathcal{F}}{\longleftrightarrow} Y(e^{j\omega})$$

Periodicidad

$$X\left(e^{j(\omega+2\pi)}\right) = X(e^{j\omega})$$

Linealidad

$$a\, x[n] + b\, y[n] \overset{\mathcal{F}}{\longleftrightarrow} a\, X(e^{j\omega}) + b\, Y(e^{j\omega})$$

Desplazamiento en el tiempo

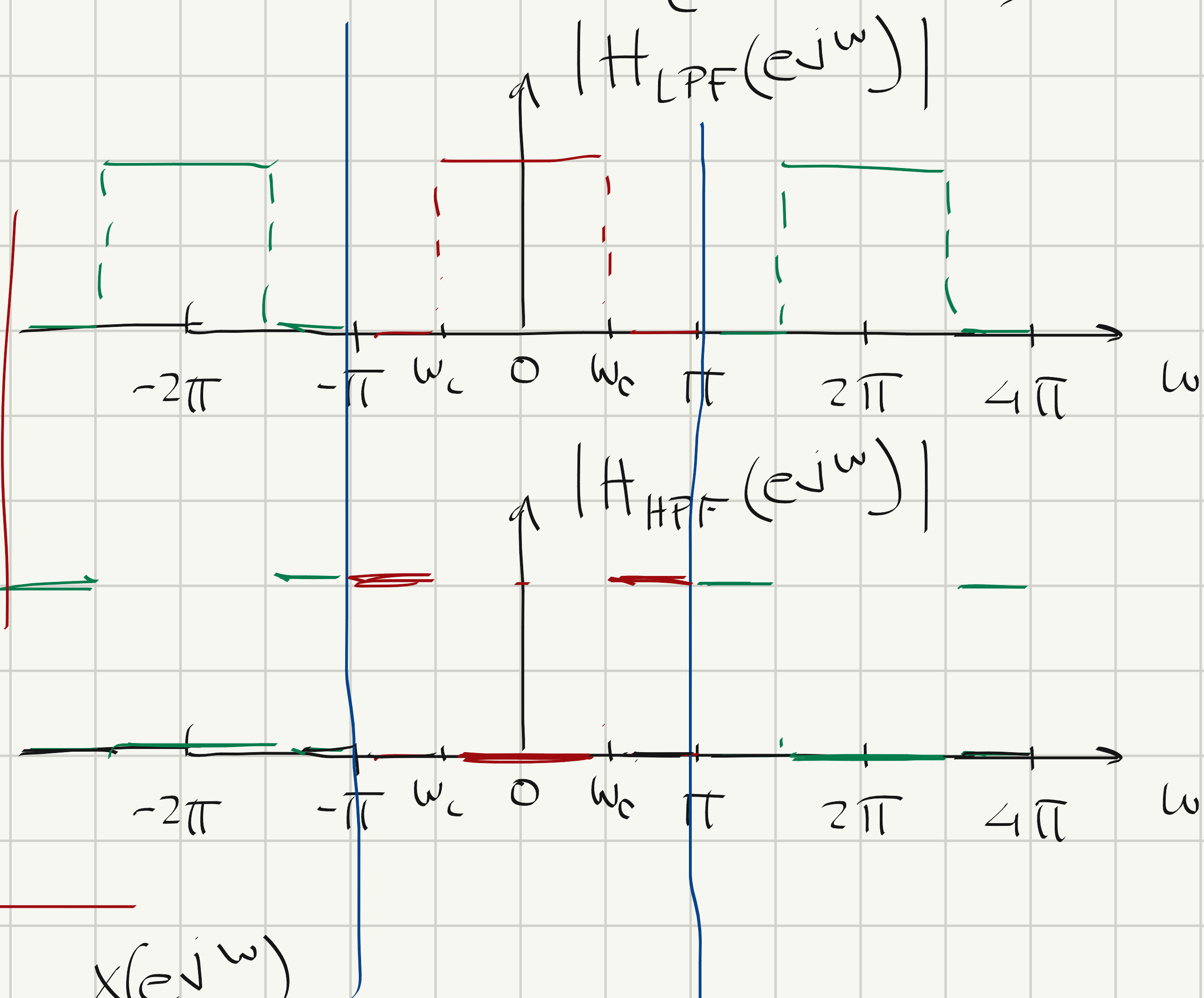
$$x[n-n_0] \overset{\mathcal{F}}{\longleftrightarrow} e^{-jn_0\omega} \cdot X(e^{j\omega})$$

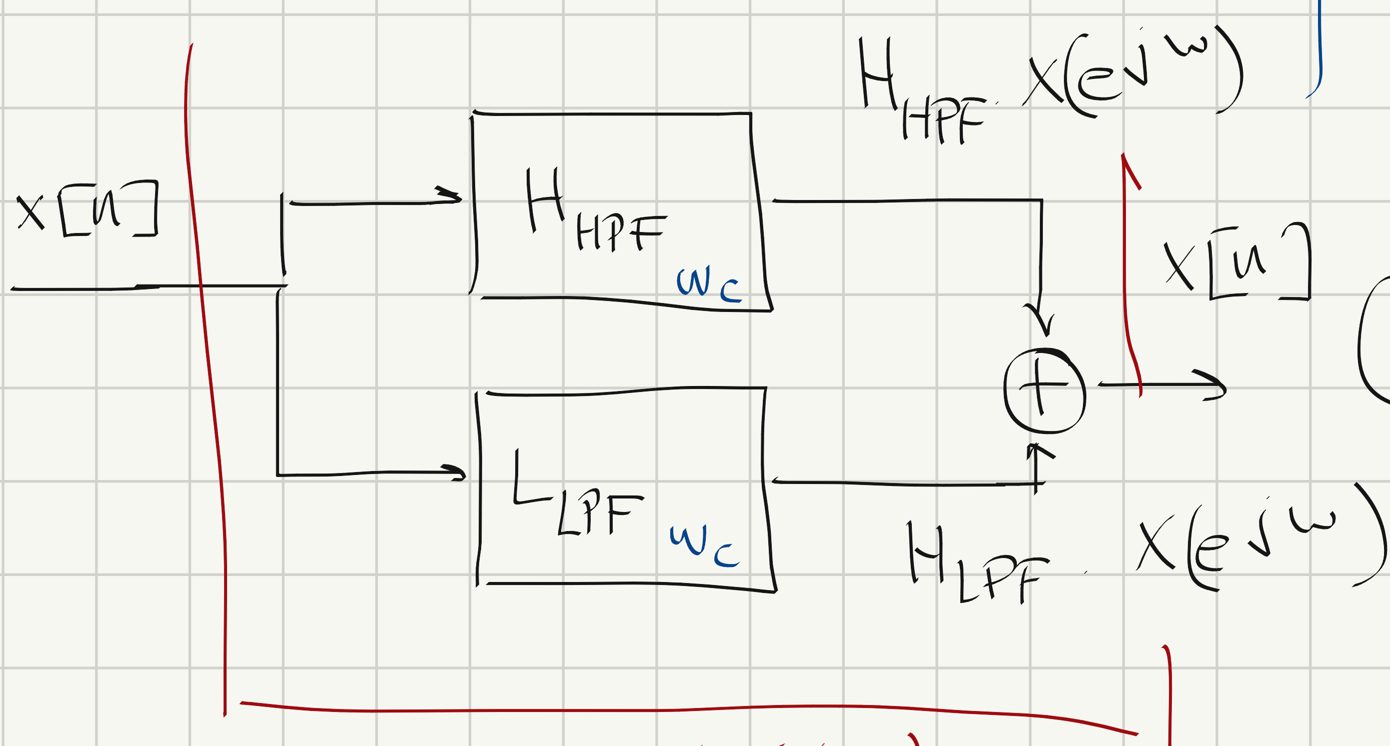
Desplazamiento en frecuencia

$$e^{j\omega_0 n} \cdot x[n] \overset{\mathcal{F}}{\longleftrightarrow} X\left(e^{j(\omega-\omega_0)}\right)$$

Ejemplo 5.7 "filtros ideales"

$$\boxed{\begin{aligned} H_{HPF}(e^{j\omega}) &= H_{LPF}\left(e^{j(\omega-\omega_c)}\right) \\ h_{HPF}[n] &= e^{j\omega_c n} \cdot h_{LPF}[n] \end{aligned}}$$





$$\underbrace{\left(H_{LPF} + H_{HPF}\right)}_{H=1} X(e^{j\omega})$$

$$H(e^{j\omega}) = 1$$

Conjugación y simetría conjugada

$$x^*[n] \xleftrightarrow{\mathcal{F}} X^*(e^{-j\omega})$$

$*$ $x[n] \in \mathbb{R}$, $X(e^{j\omega}) = X^*(e^{-j\omega})$

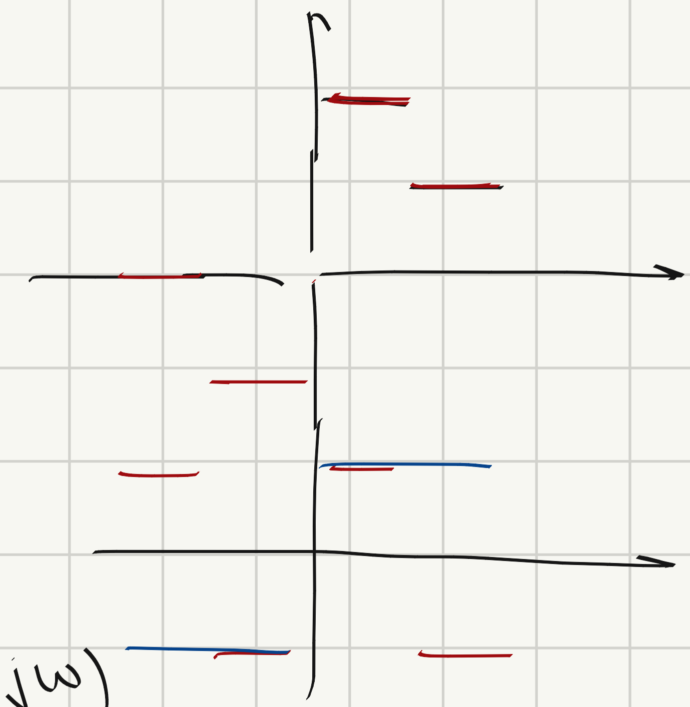
$\rightarrow$ $x[n]$ par $\xleftrightarrow{\mathcal{F}}$ $X(e^{j\omega})$ real y par

$\rightarrow$ $x[n]$ impar $\xleftrightarrow{\mathcal{F}}$ $X(e^{j\omega})$ imaginaria pura e impar

$*$ $x[n] \in \mathbb{R}$ $\rightarrow$ $x[n] = x_{par}[n] + x_{impar}[n]$

$\rightarrow$ $x_{par}[n] \xleftrightarrow{\mathcal{F}} \mathrm{Re}\{X(e^{j\omega})\}$

$\rightarrow$ $x_{impar}[n] \xleftrightarrow{\mathcal{F}} j\,\mathrm{Im}\{X(e^{j\omega})\}$



Diferenciación

$$y[n] = x[n] - x[n-1] \xleftrightarrow{\mathcal{F}} Y(e^{j\omega}) = (1 - e^{-j\omega})\, X(e^{j\omega})$$

obs: descomposición par $f(x)$

$*$ $f_{par}(x) = \dfrac{f(x) + f(-x)}{2}$

$f_{par}(-x) = \dfrac{f(-x) + f(x)}{2}$

$*$ $f_{impar}(x) = \dfrac{f(x) - f(-x)}{2}$

$*$ $f_{par}(x) + f_{impar}(x) = f(x)$

Acumulación

$$y[n] = \sum_{k=-\infty}^{n} x[k] \xleftrightarrow{\mathcal{F}} \ ?$$

C.V. $\ell = n - k$, $k = n - \ell$ $\Rightarrow$ $y[n] = \sum_{\ell=\infty}^{0} x[n-\ell]$

C.V. $m = n - \ell$, $n = m + \ell$

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n] \cdot e^{-jn\omega} = \sum_{n=-\infty}^{\infty} \sum_{\ell=0}^{\infty} x[\underbrace{n-\ell}_{m}] \cdot e^{-jn\omega} \Longrightarrow$$

$$Y(e^{j\omega}) = \sum_{m=-\infty}^{\infty} \sum_{\ell=0}^{\infty} x[m] \cdot e^{-j(m+\ell)\omega} = \sum_{\ell=0}^{\infty} e^{-j\ell\omega} \underbrace{\sum_{m=-\infty}^{\infty} x[m] \cdot e^{-jm\omega}}_{X(e^{j\omega})}$$

$$Y(e^{j\omega}) = X(e^{j\omega}) \cdot \sum_{\ell=0}^{\infty} e^{-j\ell\omega}$$

$$Y(e^{j\omega}) = X(e^{j\omega}) \sum_{\ell=0}^{\infty} (\underbrace{e^{-j\omega}}_{r})^{\ell}$$

$$\boxed{Y(e^{j\omega}) = \frac{1}{1 - e^{-j\omega}} \cdot X(e^{j\omega})}$$

$$\boxed{y[n] = \sum_{k=-\infty}^{n} x[k] \xleftrightarrow{\mathcal{F}} Y(e^{j\omega}) = \frac{1}{1 - e^{-j\omega}} \cdot X(e^{j\omega}) + \pi \cdot X(e^{j0}) \cdot \delta(\omega)}$$

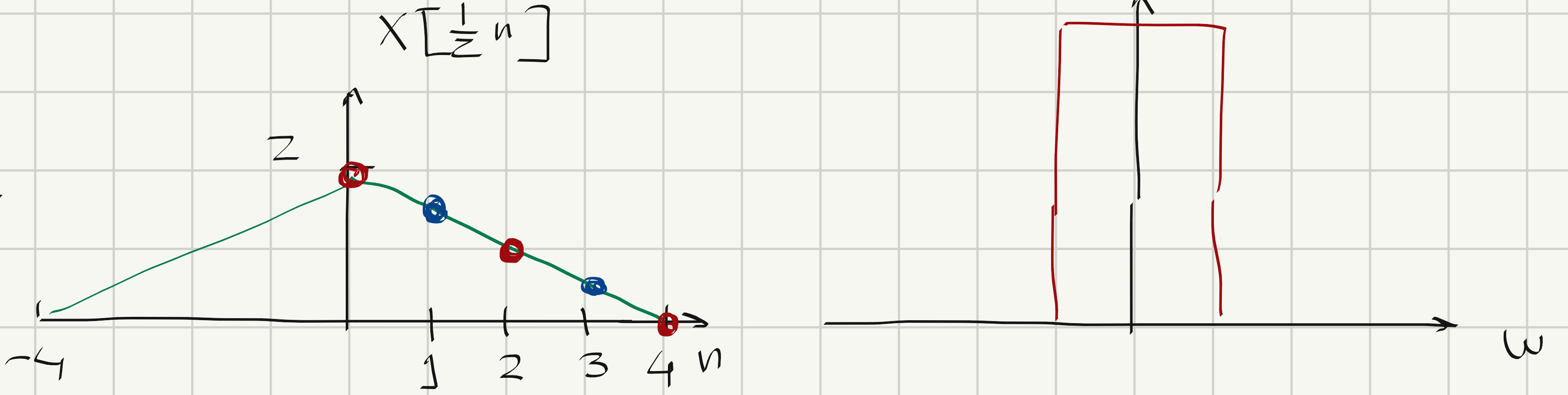
Inversión temporal

$$x[-n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{-j\omega})$$

Expansión temporal

$$x[an] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} \cdot X\left(e^{j\frac{\omega}{a}}\right)$$

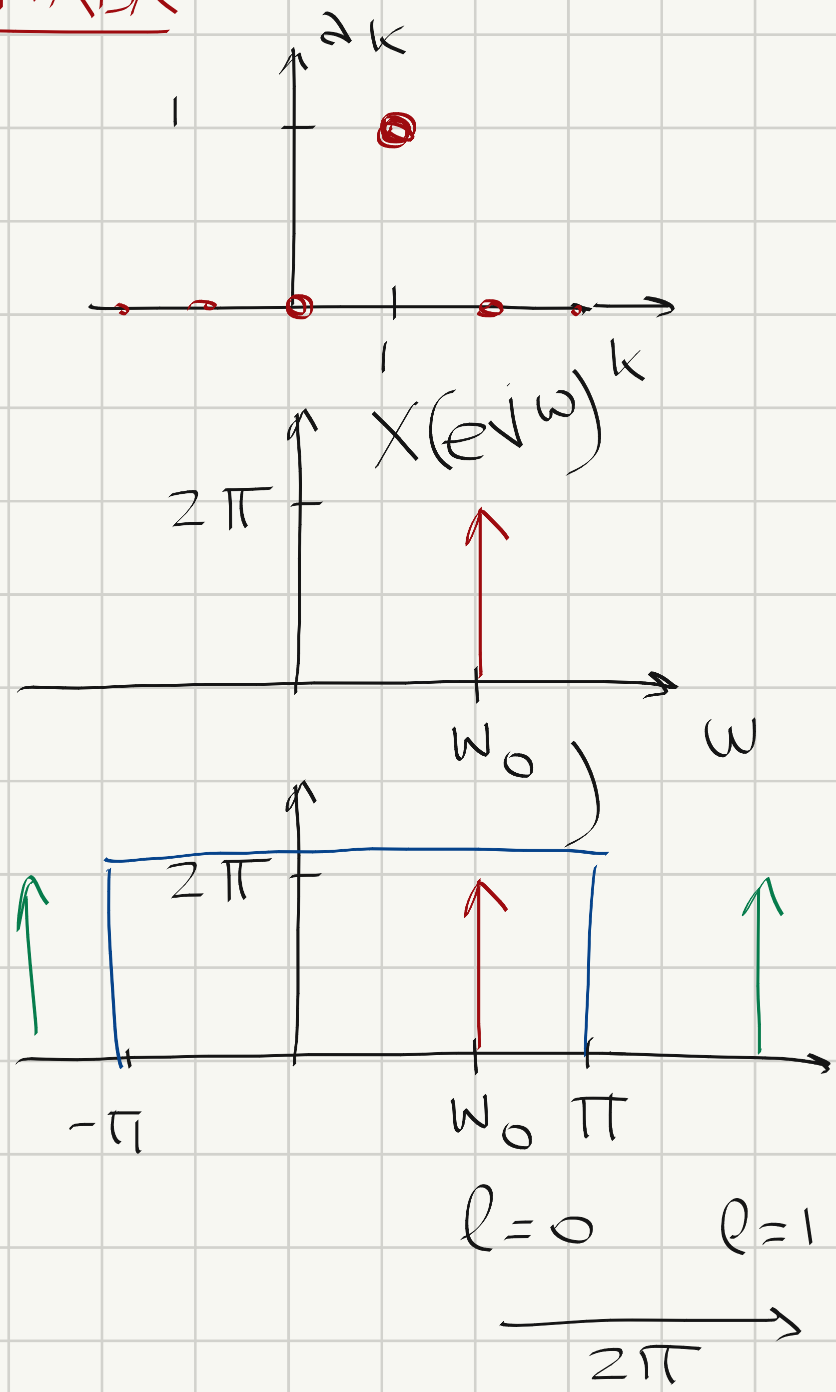
$a = \frac{1}{2}$



$a = 2$

Relación serie-transformada

a) $x[n] = e^{j\omega_0 n} \overset{DTFS}{\longleftrightarrow} a_k = \begin{cases} 1 & n=1 \\ 0 & n\neq 1 \end{cases}$



DTFS	$\longleftrightarrow$	DTFT
a_k		$X(e^{j\omega})$
$\delta[k]$		$2\pi\,\delta(\omega-\omega_0)$ periódica 2π

$$\boxed{X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi\,\delta(\omega-\omega_0-2\pi l)}$$

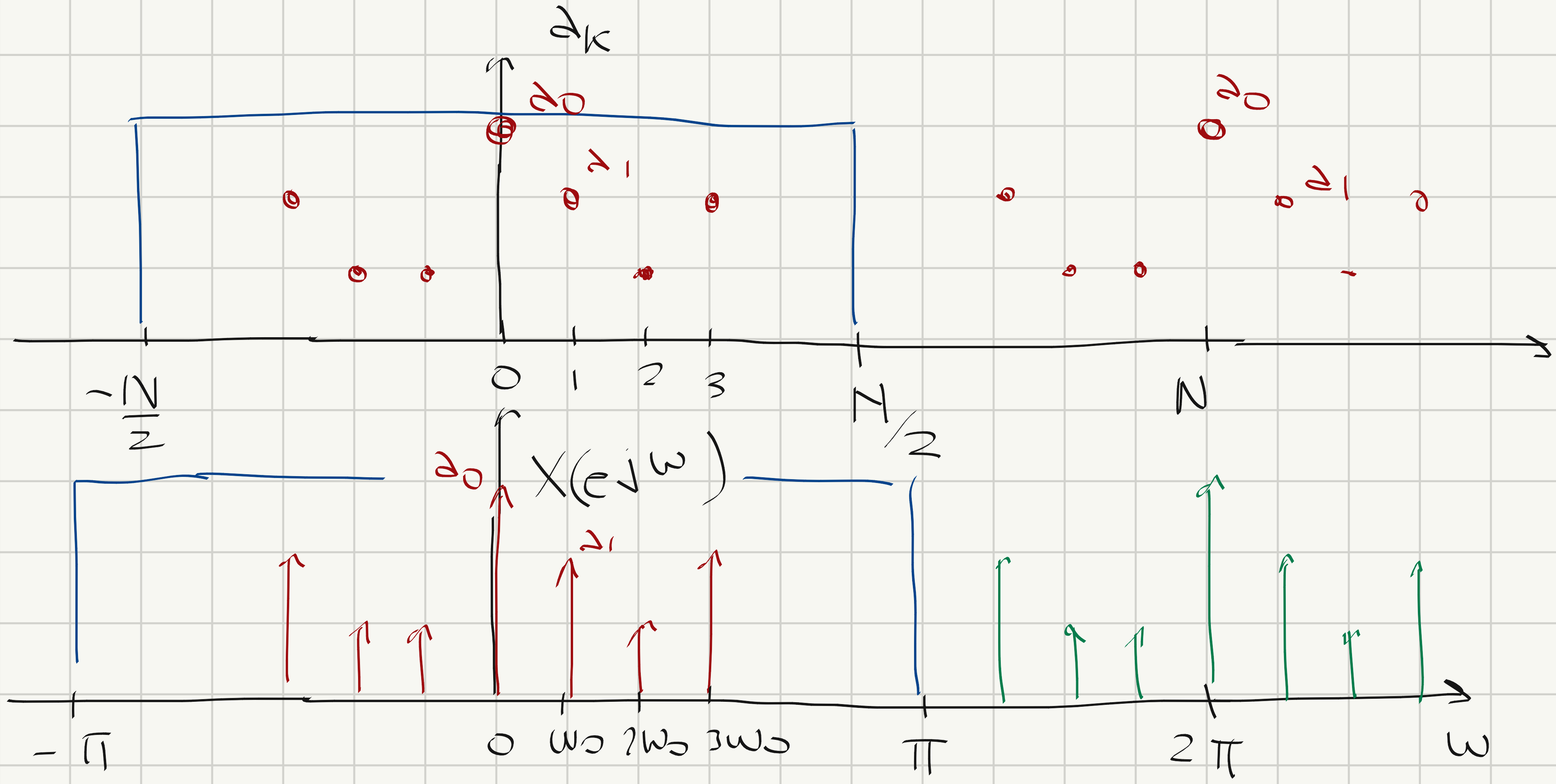
verif

$$\tilde{x}[n] \overset{DTFT}{\longleftrightarrow} X(e^{j\omega}) = 2\pi\,\delta(\omega-\omega_0)$$

$$\tilde{x}[n] = \frac{1}{\cancel{2\pi}}\int_{-\pi}^{\pi} \cancel{2\pi}\,\delta(\omega-\omega_0)\,e^{jn\omega}\,d\omega = \boxed{e^{j\omega_0 n}} \quad \checkmark$$

$$a_k = \frac{1}{2\pi}\,X(e^{j\omega})\Big|_{\omega=k\omega_0} = \frac{1}{\cancel{2\pi}}\,\cancel{2\pi}\,\delta(\underbrace{k\omega_0-\omega_0}_{=0}) = \begin{cases} 1 & k=1 \\ 0 & k\neq 0 \end{cases}$$

b) $x[n] = \sum_{k=\langle N \rangle} a_k \cdot e^{jk\omega_0 n}$



$$\boxed{X(e^{j\omega}) = \sum_{k=\langle N \rangle} 2\pi \cdot a_k \cdot \delta(\omega - k\omega_0)}$$ periódica π

$$\boxed{X(e^{j\omega}) = \sum_{\ell=-\infty}^{\infty} \sum_{k=\langle N \rangle} 2\pi \cdot a_k \, \delta(\omega - k\omega_0 - 2\pi\ell)}$$

escrito con repetición explícita

Clase 24

T. de Fourier tiempo discreto

Propiedades

Ejemplo 5.9

$z[n] = y_1[n] + y_2[n]$

$x[n] = p_2[n-2]$ $*_1$

$$y_1[n] = \begin{cases} x\left[\frac{n}{2}\right] & n \text{ par} \\ 0 & n \text{ impar} \end{cases} \quad *_2$$

$y_2[n] = \frac{1}{2}\, y_1[n-1]$ $*_3$

$$P_{N_1}(e^{jw}) = \frac{\operatorname{sen}\left(\frac{N_1+1}{2}\, w\right)}{\operatorname{sen}\left(\frac{w}{2}\right)}$$

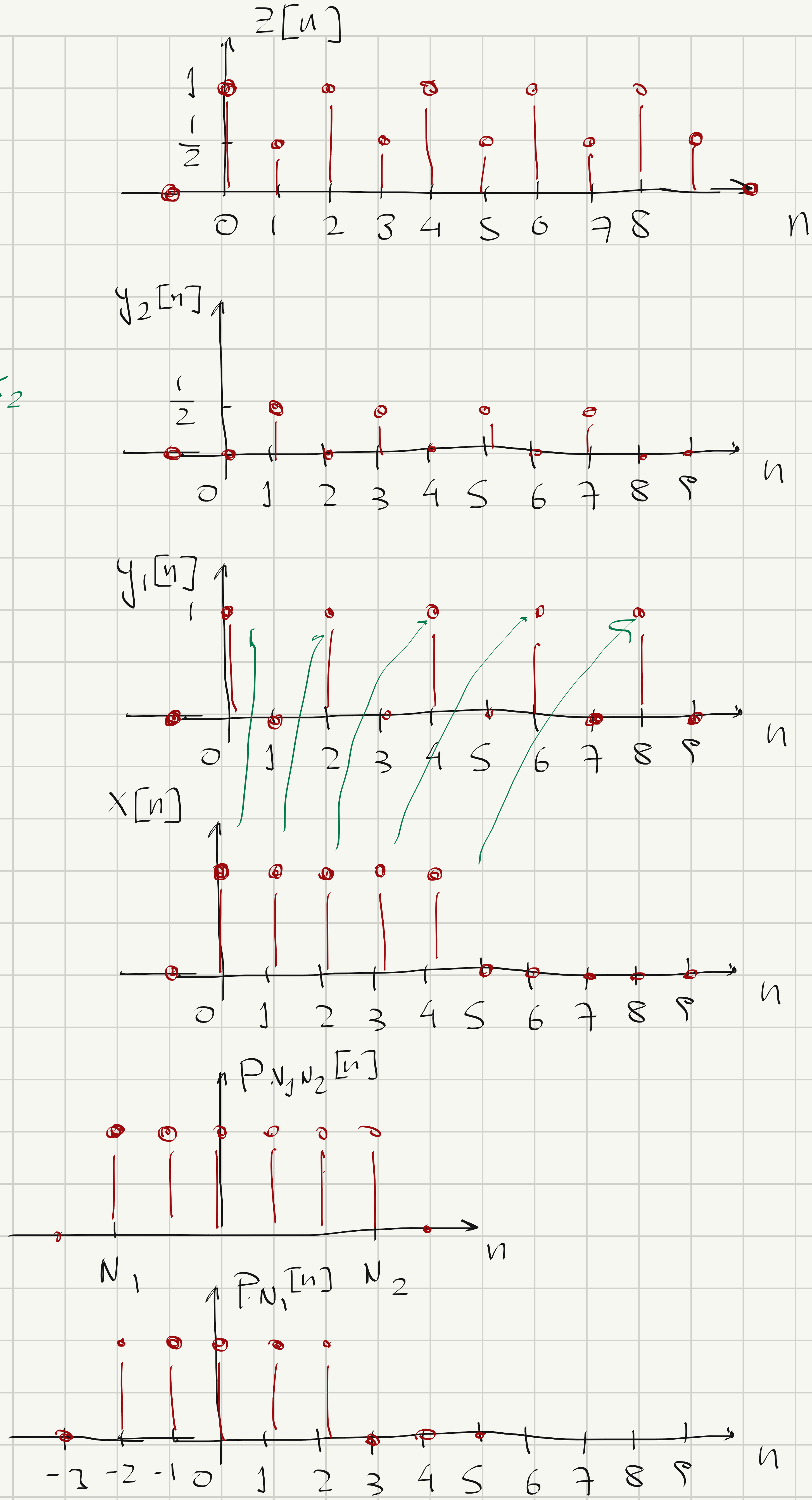
$N_1 = 2$

$$P_2(e^{jw}) = \frac{\operatorname{sen}\left(\frac{3}{2}w\right)}{\operatorname{sen}\left(\frac{w}{2}\right)}$$

$*_1$ $X(e^{jw}) = e^{-j2w}\, P_2(e^{jw})$

$*_2$ $Y(e^{jw}) = 2\, X(e^{j2w})$

$*_3$ $Y_2(e^{jw}) = \frac{1}{2}\, e^{-jw}\, Y(e^{jw})$



$Z(e^{jw}) = Y_1(e^{jw}) + Y_2(e^{jw})$

$Z(e^{jw}) = 2\, X(e^{j2w}) + e^{-jw}\, X(e^{j2w})$

$$\boxed{Z(e^{jw}) = (2 + e^{-jw})\, e^{-j2w}\, \frac{\operatorname{sen}(3w)}{\operatorname{sen}(w)}}$$

Diferenciación en frecuencia

$$\boxed{-jn\, x[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{d\, X(e^{j\omega})}{d\omega}}$$

Relación de Parseval

$$\boxed{\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{\langle 2\pi \rangle} |X(e^{j\omega})|^2\, d\omega}$$

Convolución

$$\boxed{y[n] = x[n] * h[n] \overset{\mathcal{F}}{\longleftrightarrow} Y(e^{j\omega}) = X(e^{j\omega})\, H(e^{j\omega})}$$

Multiplicación (modulación)

$$\boxed{x[n]\, y[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}) * Y(e^{j\omega})}$$

$$= \frac{1}{2\pi}\int_{\langle 2\pi \rangle} X(e^{j\theta})\, Y\left(e^{j(\omega-\theta)}\right) d\theta$$

conv. periódica

DUALIDAD

Obs: * no hay dualidad en el sentido estricto

Dualidad de la DTFS

$$x[n] \overset{DTFS}{\longleftrightarrow} a_k = a[k]$$

periódica, t. discreto — periódica, f. discreta

$$x[n] = \sum_{k=\langle N\rangle} a[k] \cdot e^{jk\omega_0 n} \Rightarrow x[-n] = \sum_{k=\langle N\rangle} a[k] \cdot e^{-jk\omega_0 n}$$

$$\frac{1}{N} \cdot x[-n] = \frac{1}{N} \cdot \sum_{k=\langle N\rangle} \underbrace{a[k]}_{\text{f. periódica}} \underbrace{e^{-jk\omega_0 n}}_{\text{exp. comp. conj.}}$$

→ es una ec. de análisis DTFS

$$\boxed{a[k] \overset{DTFS}{\longleftrightarrow} \frac{1}{N} \cdot x[-n]}$$

$$\boxed{\begin{array}{ccc} x[n] & \overset{\mathcal{F}}{\longleftrightarrow} & a[k] \\ a[n] & \overset{\mathcal{F}}{\longleftrightarrow} & \frac{1}{N} \cdot x[-k] \end{array}}$$

Obs: * cálculos
* propiedades (deducir)

Dualidad cruzada

* DTFT: → análisis: $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\omega n}$

→ síntesis: $x[n] = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(e^{j\omega}) \cdot e^{j\omega n} \cdot d\omega$

$\updownarrow$ dual

* CTFS: → análisis: $a[k] = \frac{1}{T} \int_{\langle T \rangle} x(t) \cdot e^{-jk\omega_0 t} \cdot dt$

→ síntesis: $x(t) = \sum_{k=-\infty}^{\infty} a[k] \cdot e^{jk\omega_0 t}$

obs: * ver resumen en slides

SIST. BASADOS EN EC. EN DIFERENCIAS

Ec. dif.

$N \rightarrow$ orden: cant. retardos de la salida

$$\sum_{k=0}^{N} a_k \cdot y[n-k] = \sum_{\ell=0}^{M} b_k \cdot x[n-\ell]$$

$$y[n] = h[n] * x[n] \overset{\mathcal{F}}{\longleftrightarrow} Y(e^{j\omega}) = X(e^{j\omega}) \cdot H(e^{j\omega})$$ $\leftarrow$ prop. conv.

$$\sum_{k=0}^{N} a_k \cdot e^{-jk\omega} \cdot Y(e^{j\omega}) = \sum_{\ell=0}^{M} b_k \cdot e^{-j\ell\omega} \cdot X(e^{j\omega})$$ $\leftarrow$ prop. linealidad y desp. temporal

$$\sum_{k=0}^{\infty} a_k \cdot e^{-jk\omega} \cdot \underbrace{H(e^{j\omega})}_{\text{no depende de } k} \cancel{X(e^{j\omega})} = \sum_{\ell=0}^{M} b_k \cdot e^{-j\ell\omega} \cdot \cancel{X(e^{j\omega})}$$

$$\boxed{H(e^{j\omega}) = \frac{\sum_{\ell=0}^{N} b_\ell \cdot e^{-j\ell\omega}}{\sum_{k=0}^{N} a_k \cdot e^{-jk\omega}}} \longrightarrow$$

C.V. $z = e^{-j\omega}$

$$\boxed{H(e^{j\omega}) = \frac{\sum_{\ell=0}^{N} b_\ell \cdot z^{\ell}}{\sum_{k=0}^{M} a_k \cdot z^{k}}}$$

obs:

* función racional en $e^{-j\omega}$
* expresión general para cualquier sistema LTI discreto
* Relación directa con la implementación $\rightarrow$ software
  $\rightarrow$ electrónica
* tema 2: sistemas en el tiempo (implementación)

Implementación en CPU

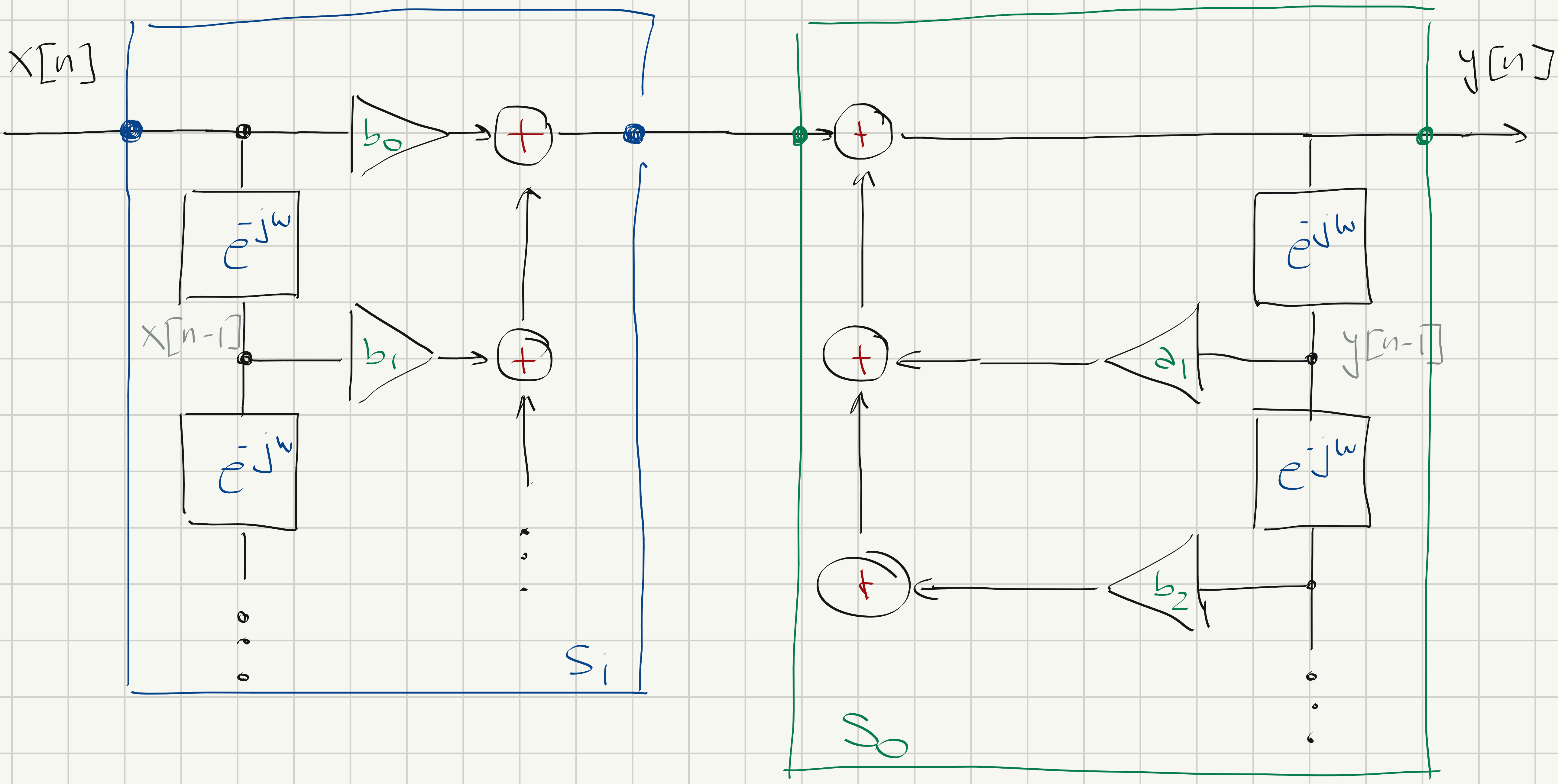
$$y[n] = \underbrace{\sum_{\ell=0}^{M} b_\ell \, x[n-\ell]}_{S_i} - \underbrace{\sum_{k=1}^{N} a_k \, y[n-k]}_{S_o}$$

```
for (int n=0; n<L; n++) {
    float Si=0;                                  // Si
    for (int l=0; l<M; l++)
        Si += b[l] * x[n-l];
    float So=0;                                  // So
    for (int k=0; k<N; k++)
        So += a[k] * y[n-k];
    y[n] = Si - So;
}
```

$n > \ell$, $n > k$: cond. de borde

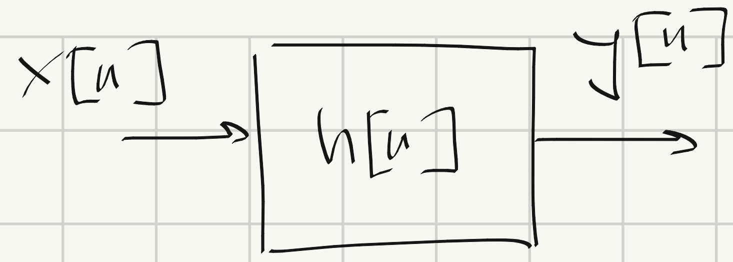
Obs: $*$ es lo que está adentro del lfilter de Python

Implementación en electrónica



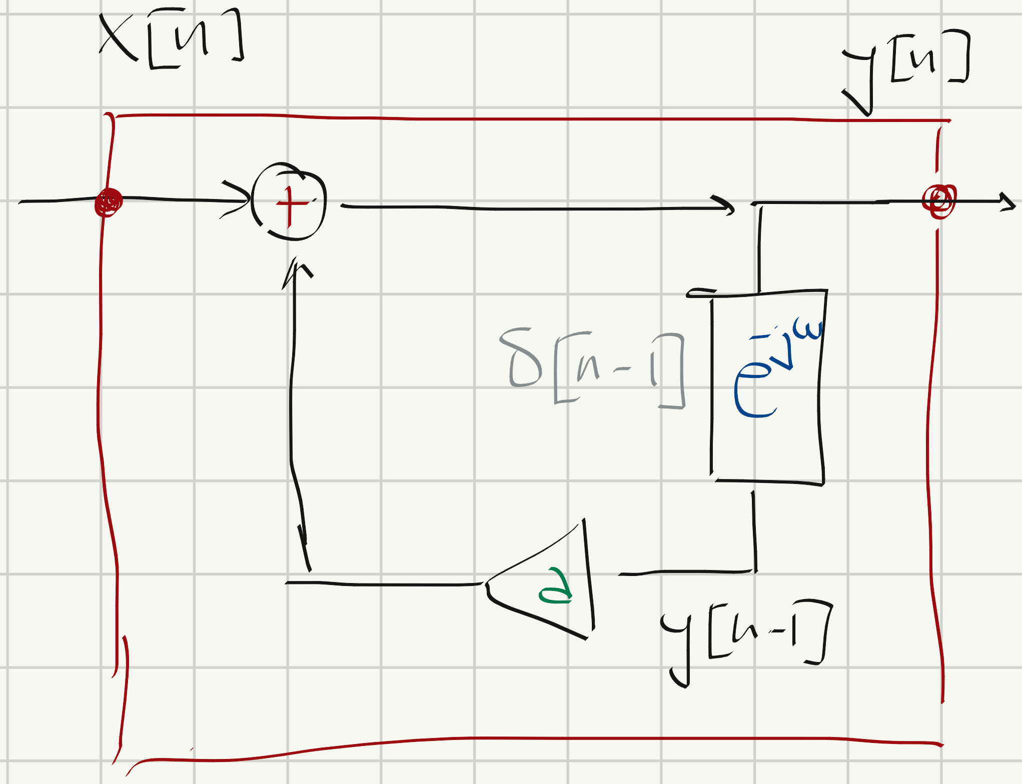
obs: * muy cerca de la implementación electrónica

Ejemplo 5.18 (p. 397)



$$y[n] - a\,y[n-1] = x[n] \quad , |a|<1 \leftarrow \text{ ec. diferencias}$$

a) Análisis en el tiempo (ejemplo IIR):



$$y[n] - a\,y[n-1] = x[n] \rightarrow h[n] = u[n]\,a^n$$

b) DTFS (ejemplo 3.17):

$$h[n] = u[n]\,a^n \xrightarrow[\text{indirecta}]{} H(e^{j\omega}) = \frac{1}{1 - a e^{-j\omega}}$$

c) DTFT (ej. 5.1):

$$x[n] = u[n]\,a^n \rightarrow X(e^{j\omega}) = \frac{1}{1 - a e^{-j\omega}}$$

d) $$y[n] - a\,y[n-1] = x[n]$$

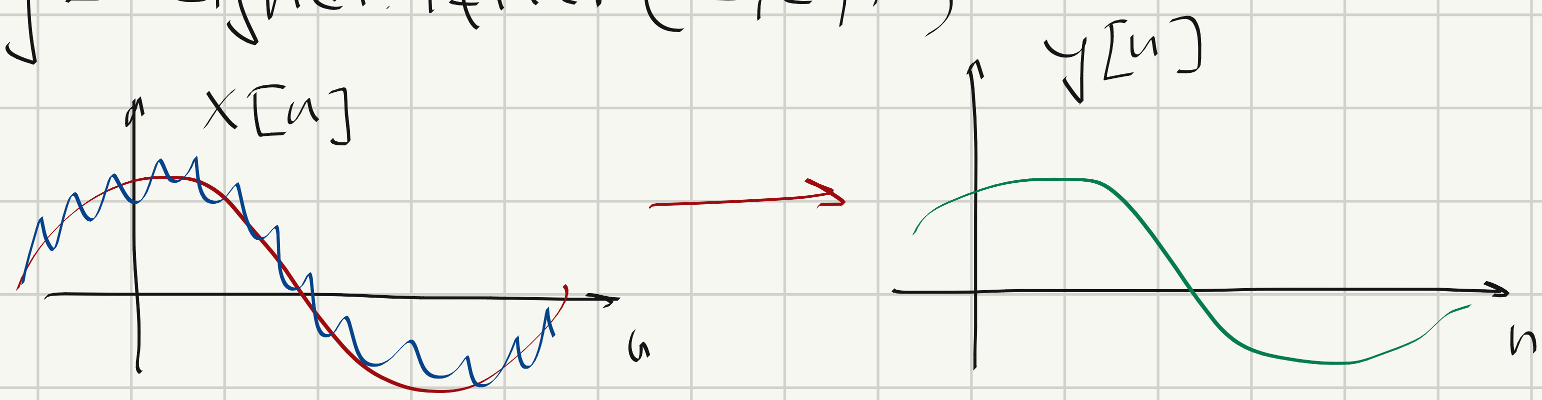
$a_0 = 1 \quad a_1 = -a \quad b_0 = 1$

$$\boxed{H(e^{j\omega}) = \frac{1}{1 - a\,e^{-j\omega}}} \xrightarrow{\mathcal{F}^{-1}} h[n] = u[n]\,a^n$$

obs: * se implementa fácil en computadora
* tanto en tiempo como en frecuencia.

Implementación (s.18)

a) $a = [1 \ \ -a]$, $b = [1]$ $\Rightarrow$ y = signal.lfilter(b, a, x)



b) $a = [1 \ \ -a]$, $b = [1]$ $\Rightarrow$ [H, θ] = freqz(b, a)

Clase 25

Muestreo

Definición y representación en frecuencia

MUESTREO

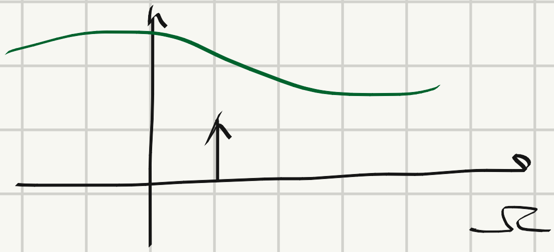
Introducción: * hemos visto T.C. y T.D por separado

* ¿Cómo se juntan?

[slides hasta pág. 9]

Representación del muestreo en el dominio de la frecuencia

a) modulación



$$x_s(t) = x_c(t) \cdot s(t)$$

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

modulación

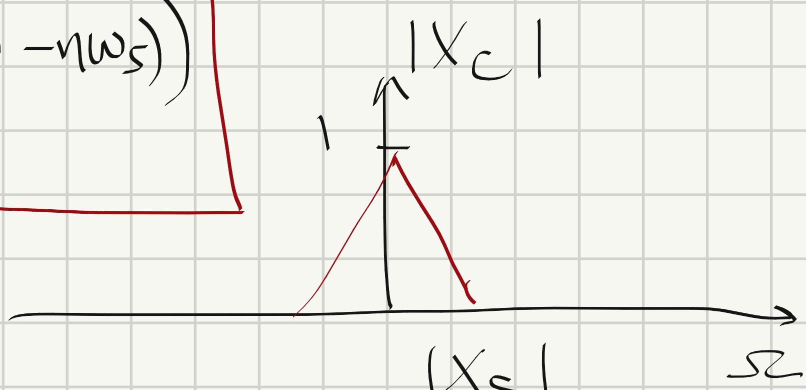
$$X_s(j\omega) = \frac{1}{2\pi} X_c(j\omega) * S(j\omega)$$

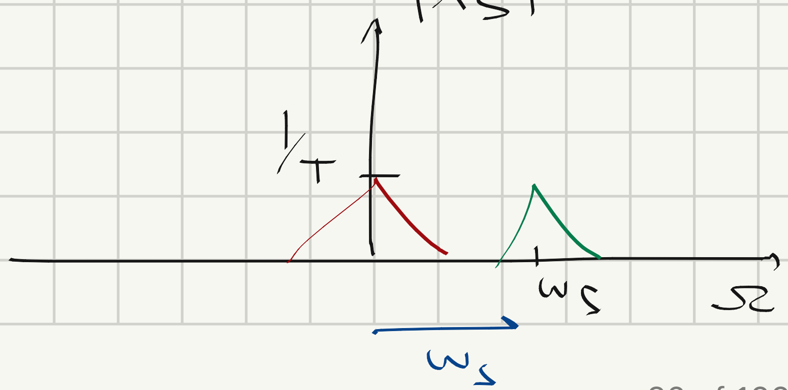
$$S(j\omega) = \overbrace{\frac{2\pi}{T}}^{\omega_s} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)$$

$$X_s(j\omega) = \frac{1}{2\pi} X_c(j\omega) * \left[\omega_s \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)\right]$$

$$X_s(j\omega) = \overbrace{\frac{\omega_s}{2\pi}}^{1/T} \sum_{n=-\infty}^{\infty} \underbrace{X_c(j\omega) * \delta(\omega - n\omega_s)}_{\text{desplazo } X_c \text{ en frecuencia}}$$

$$\boxed{X_s(j\omega) = \frac{1}{T} \cdot \sum_{n=-\infty}^{n=\infty} X_c\left(j(\omega - n\omega_s)\right)}$$



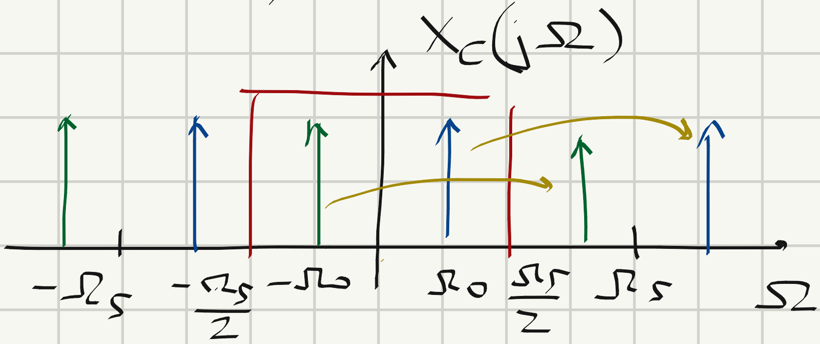


[volver a slide 12 a 18]

ver conclusiones en 12

Ejemplo: $x_c(t) = \cos(\Omega_0 t)$, muestreo Ω_s

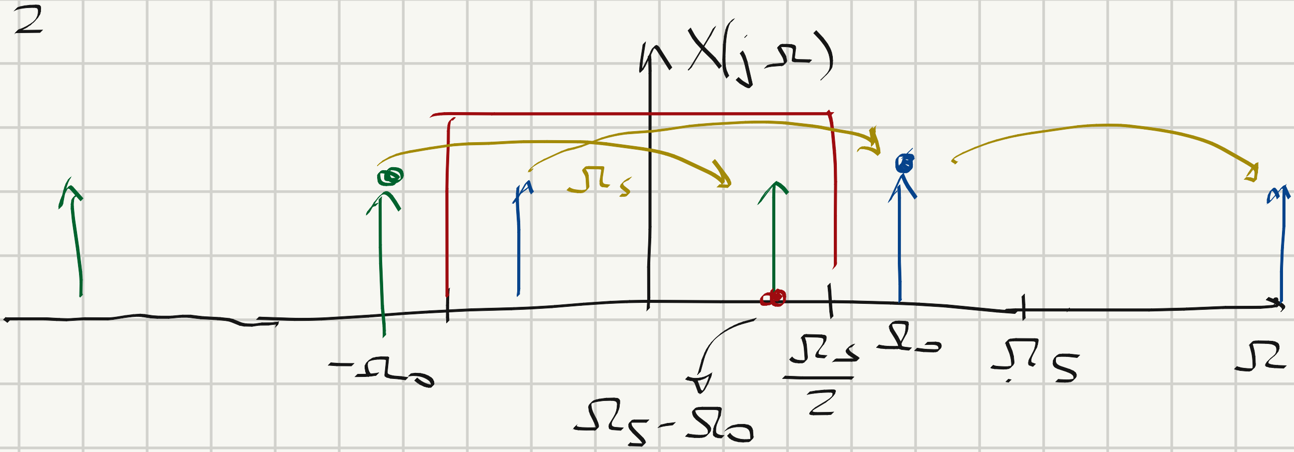
a) $\frac{\Omega_s}{2} > \Omega_0$



$\Omega_0 < \Omega_c < \frac{\Omega_s}{2}$

$x_r(t) = \cos(\Omega_0 t)$

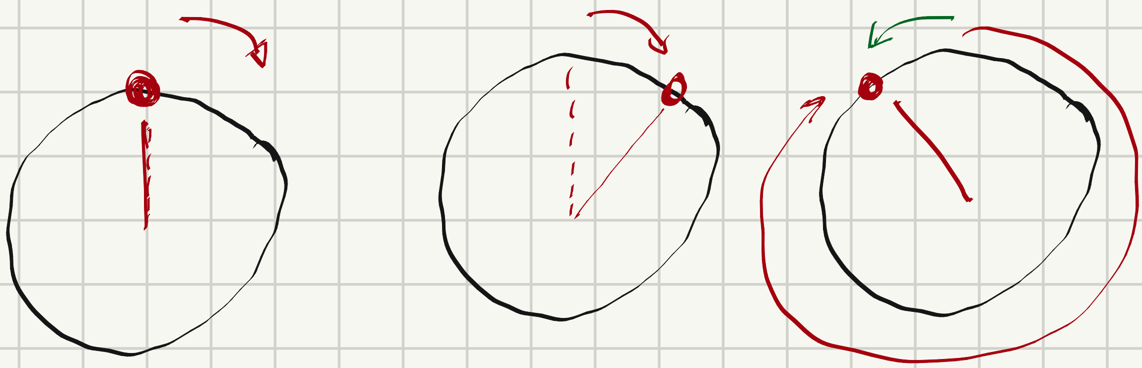
b) $\frac{\Omega_s}{2} < \Omega_0$



$$\boxed{x_r(t) = \cos[(\Omega_s - \Omega_0)t]}$$

[volver al slide 18 al 22]

Obs:



$t = 0s$ $\quad$ $t = 1s$ $\quad$ $t = 7s$

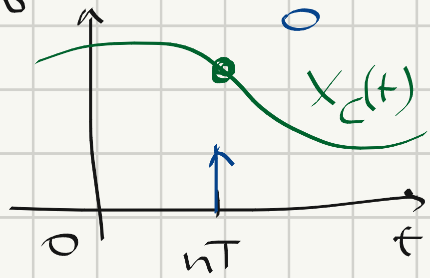
$\Omega_0 = \frac{\pi}{4}$

Representación en frecuencia

b) expansión

$$x_s(t) = x_c(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t-nT) = \sum_{n=-\infty}^{\infty} x_c(t) \cdot \delta(\underbrace{t-nT}_{=0})$$

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT) \cdot \delta(t-nT) \quad *$$



$$X_s(jw) = \int_{-\infty}^{\infty} x_s(t) \cdot e^{-jwt} \cdot dt = \int_{-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} x_c(nT) \cdot \delta(t-nT) \right) e^{-jwt} \cdot dt$$

$$X_s(jw) = \sum_{n=-\infty}^{\infty} x_c(nT) \left[\int_{-\infty}^{\infty} \delta(t-nT) \cdot e^{-jwt} dt \right] = \sum_{n=-\infty}^{\infty} x_c(nT) \cdot e^{-jnTw}$$

C.V ⇒ Ω

$$X_s(j\Omega) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-jnT\Omega}$$

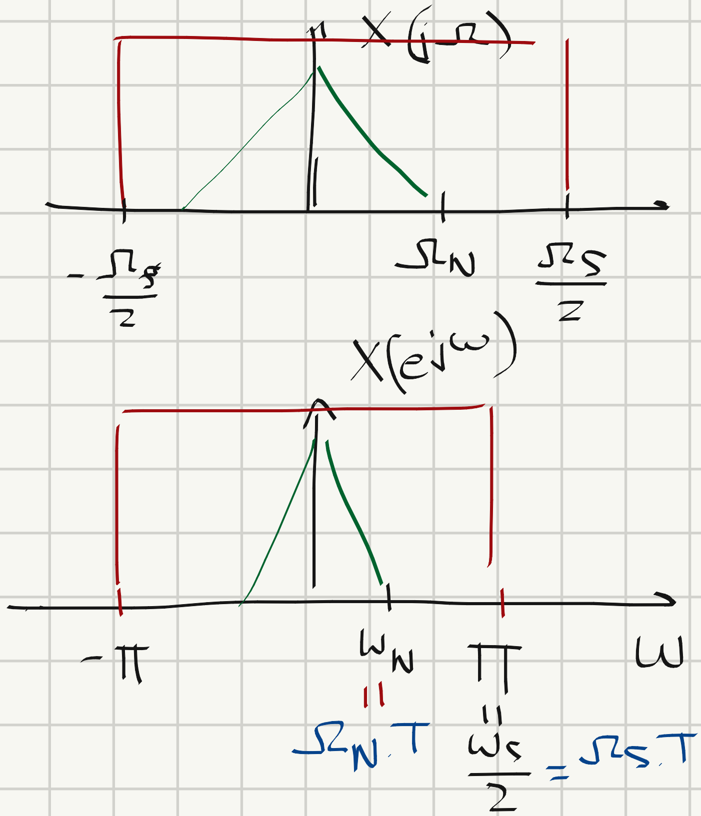
$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-jnw}$$

$$\Rightarrow \boxed{X(e^{jw}) = X_s(j\Omega)} \qquad \boxed{\omega = \Omega T}$$

$$\boxed{X(e^{jw}) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X_c\left(j\left(\frac{w}{T} - k \cdot w_s\right)\right)}$$

[cubre del slide 22 al 25]

[ver conclusiones en el 25]



X(jΩ)
-Ωs/2
ΩN
Ωs/2
X(e^jω)
-π
ωN
π
ω
ΩN.T
ωs/2
= Ωs.T

Ejemplo $f_0 = 2kHz$, $f_s = 6kHz \Rightarrow f_s > 2f_0$ ✓

$$T_s = \frac{1}{6000} s = 166 \mu s \qquad \Omega_s = 2\pi \cdot 6k \cong 36k \ rad/s$$

a) $$x_c(t) = \cos(2\pi 2k t) = \cos(4000\pi t)$$

$$x[n] = \cos(4000\pi nT) = \boxed{\cos\left[\frac{2}{3}\pi n\right]}$$

b) $$X_c(j\Omega) = \frac{1}{2}\left[\delta(\Omega - 4000\pi) + \delta(\Omega + 4000\pi)\right]$$ *

$$X_s(j\Omega) = \frac{1}{T} X_c(j\Omega)$$ en $\left[-\frac{\Omega_s}{2}, \frac{\Omega_s}{2}\right]$

periódica Ω_s

$$X(e^{j\omega}) = X_s\left(j\frac{\omega}{T}\right) = \frac{1}{T} X_c\left(j\frac{\omega}{T}\right)$$

$$\boxed{X(e^{j\omega}) = \frac{1}{2T}\left[\delta\left(\underbrace{\frac{\omega}{T} - 4000\pi}_{=0}\right) + \delta\left(\frac{\omega}{T} + 4000\pi\right)\right]}$$

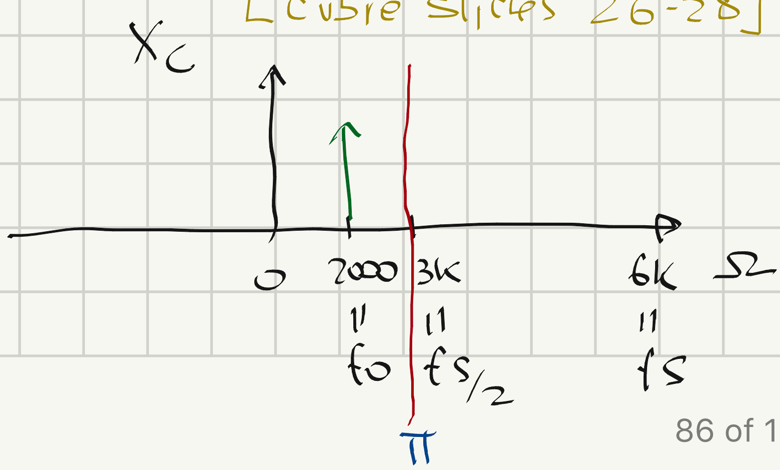
para saber la frec. del coseno

$$\frac{\omega_0}{T} - 4000\pi = 0 \Rightarrow \omega_0 = 4000\pi \cdot T = \frac{4000\pi}{6000} = \boxed{\frac{2\pi}{3}}$$

$$\boxed{x[n] = \cos\left[\frac{2\pi}{3} n\right]}$$

[cubre slides 26-28]

$f_0 = 2000 \longrightarrow \omega_0 = \frac{2\pi}{3}$



Clase 26

Muestreo

reconstrucción y procesamiento

Reconstrucción

[Empezar con slide 30, seguir con estas cuentas]

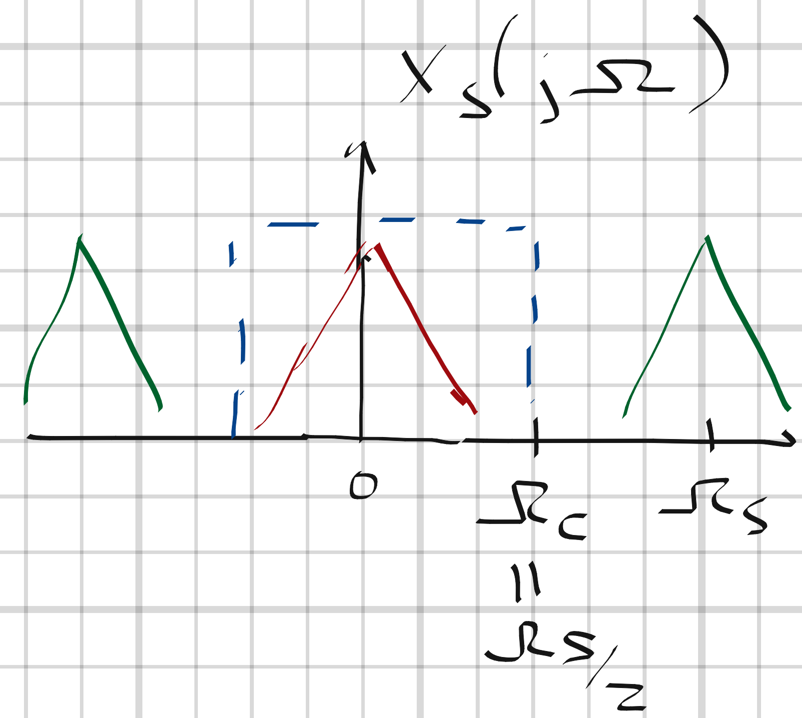
$$x_r(t) = x_s(t) * h_r(t)$$

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\,\delta(t-nT)$$

$$x_r(t) = \left(\sum_{n=-\infty}^{\infty} x_c(nT)\,\delta(t-nT) \right) * h_r(t)$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]\,\left(\underline{\delta(t-nT) * h_r(t)}\right)$$

$$\boxed{x_r(t) = \sum_{n=-\infty}^{\infty} x[n]\,h_r(t-nT)} \quad (5)$$



[repasar diagrama en slide 31]

[slide 32, empezar con figura]

$$H_r(j\Omega) = \begin{cases} T & |\Omega| < \Omega_c = \frac{\Omega_s}{2} = \frac{\pi}{T} \\ 0 & \text{e.o.c.} \end{cases}$$

$$\boxed{h_r(t) = \frac{\operatorname{sen}(\Omega_c t)}{\Omega_c t}} \qquad \boxed{h_r(t) = \operatorname{senc}(\Omega_c t)} \quad (6)$$

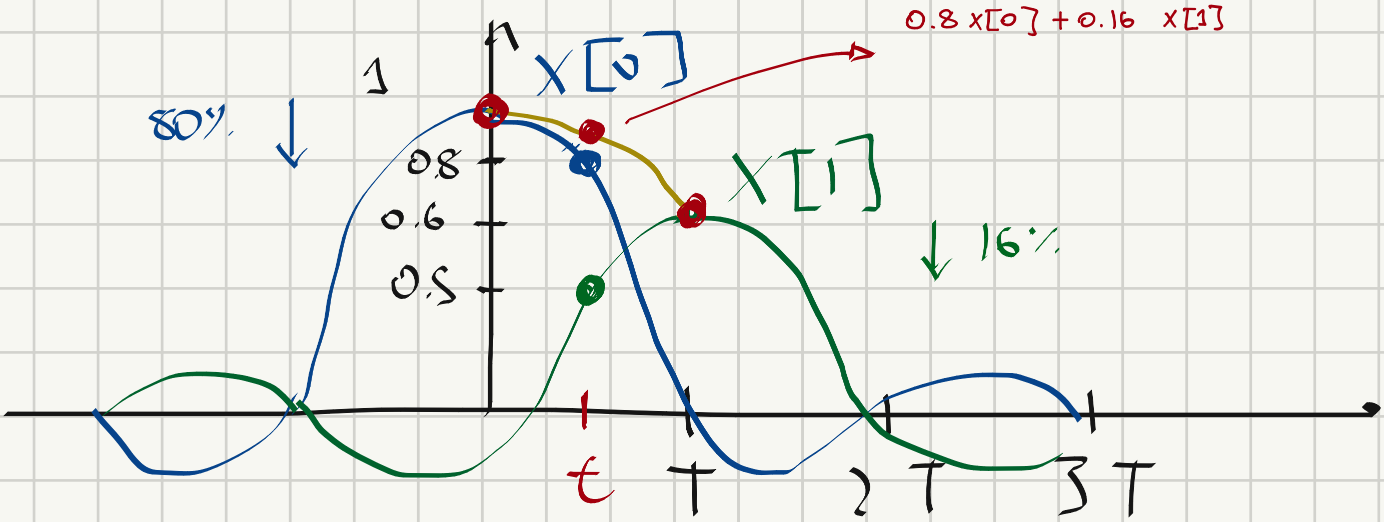
$$(5) \text{ y } (6) \Rightarrow \boxed{x_r(t) = \sum_{n=-\infty}^{\infty} x[n]\,\operatorname{sen}(\Omega_c (t-nT))} \quad *_1$$

reconstruccion ideal

[slide 33 - dibujo de $*_1$]

[slide 34 - conclusiones]

$$\boxed{x_1(t) = \sum_{n=-\infty}^{\infty} x[n] \cdot \frac{\operatorname{sen}\left(\pi(t-nT)/T\right)}{\pi(t-nT)/T}}$$



Obs: función interpolante

[slide 35 - retomando lo visto en la figura]

Conversor DC ideal

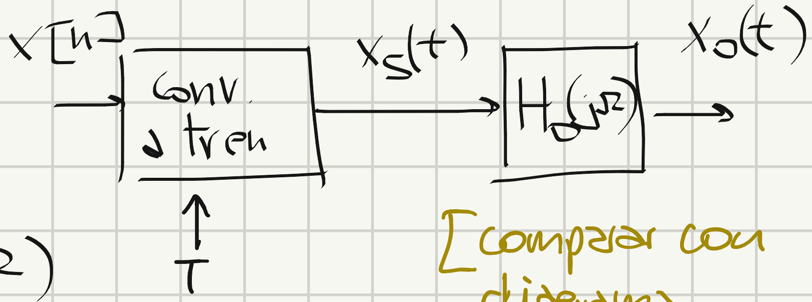
$\boxed{\text{slide 36}}$

Procesamiento en tiempo discreto

slides 37 - 41

RECONSTRUCCIÓN REAL

Mantenedor de orden 0
(interpolador)

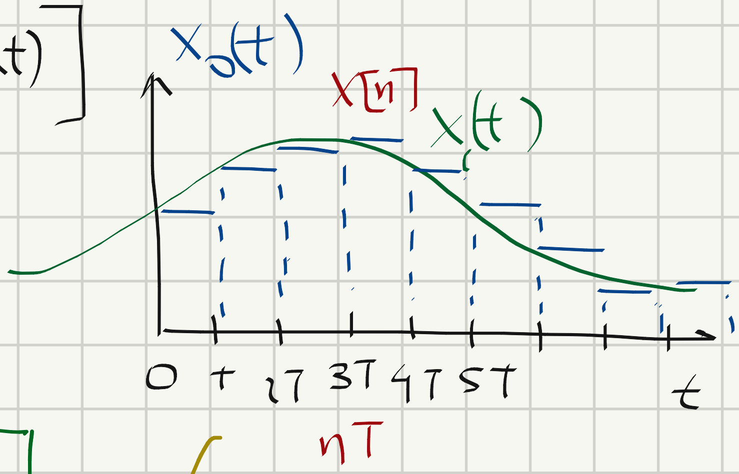


[comparar con diagrama en slide 36]

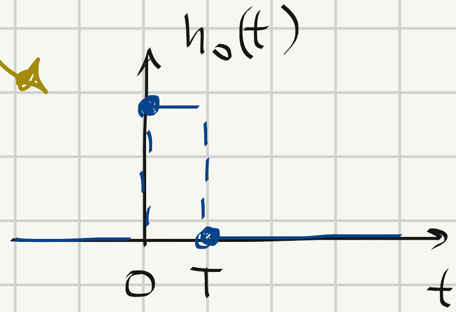
$$X_0(j\Omega) = X_s(j\Omega) \cdot H_0(j\Omega)$$

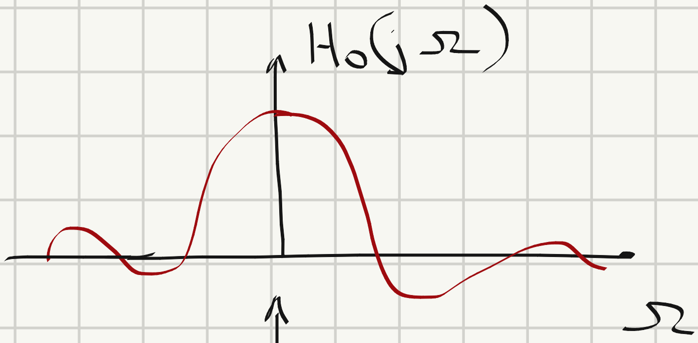
$$x_0(t) = \sum_{n=-\infty}^{\infty} x[n] \cdot \left[\delta(t-nT) * h_0(t) \right]$$

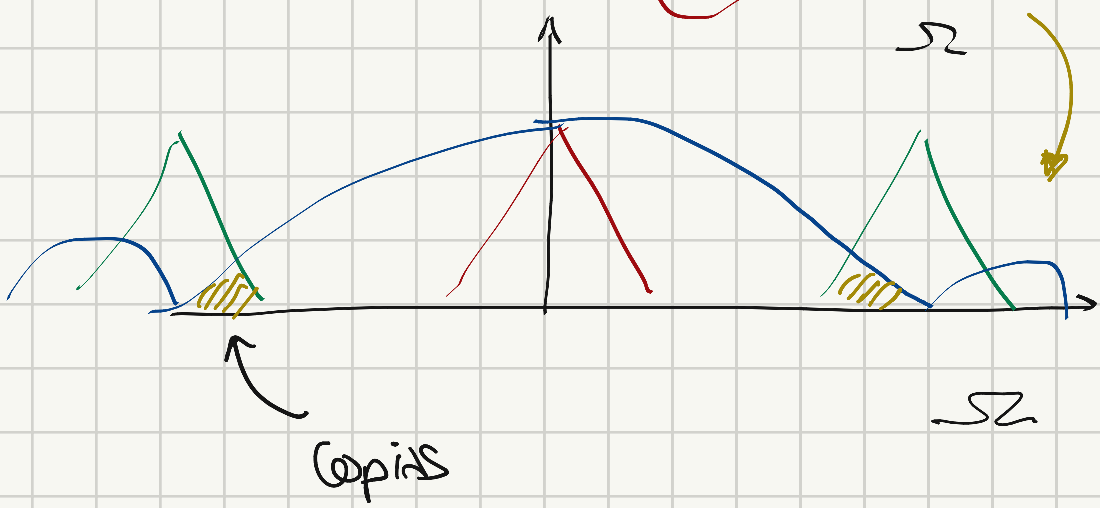
$$\boxed{x_0(t) = \sum_{n=-\infty}^{\infty} x[n] \cdot h_0(t-nT)}$$



$$\boxed{\left| H_0(j\Omega) \right| = T \cdot \operatorname{senc}\left(\frac{T}{2}\Omega\right)}$$

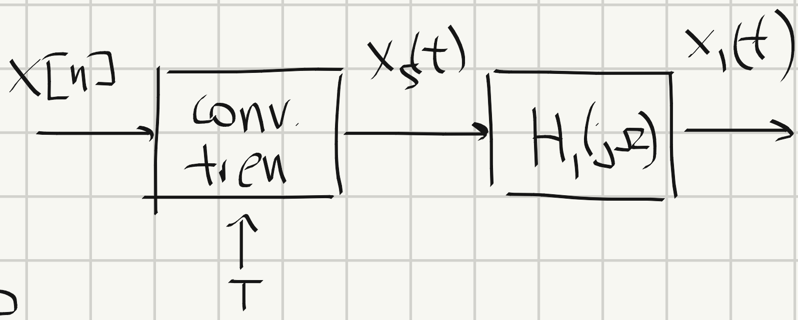


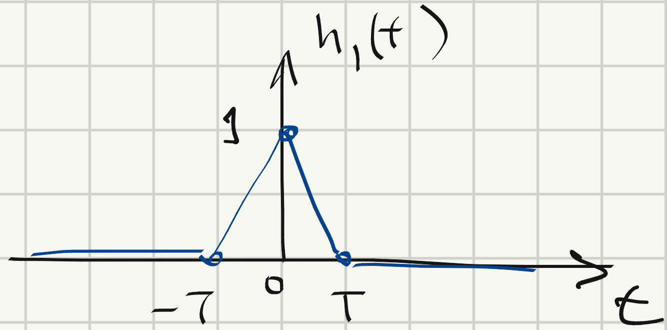




Interpolación lineal (orden 1)

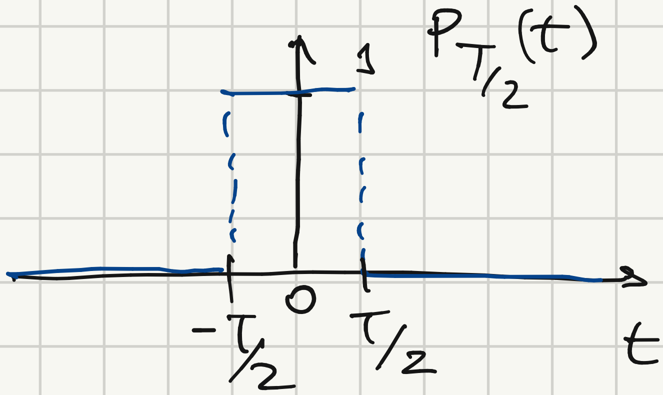
$$h_1(t) = \begin{cases} 0 & T<t \\ \dfrac{T-t}{T} & 0<t<T \\ \dfrac{t+T}{T} & -T<t\leq 0 \\ 0 & t\leq -T \end{cases}$$





$$h_1(t) = p_{T/2}(t) * p_{T/2}(t)$$

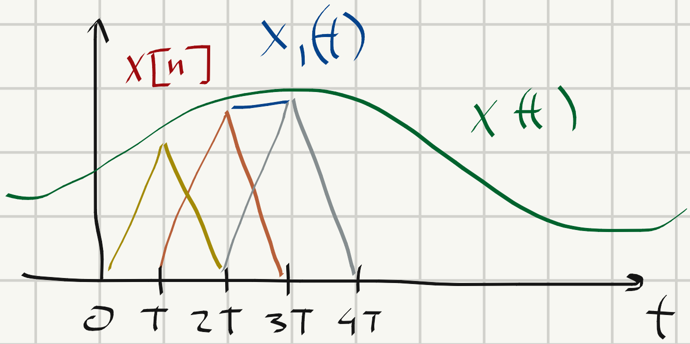
$$\Rightarrow H_1(j\Omega) = P_{T/2}^2(j\Omega)$$

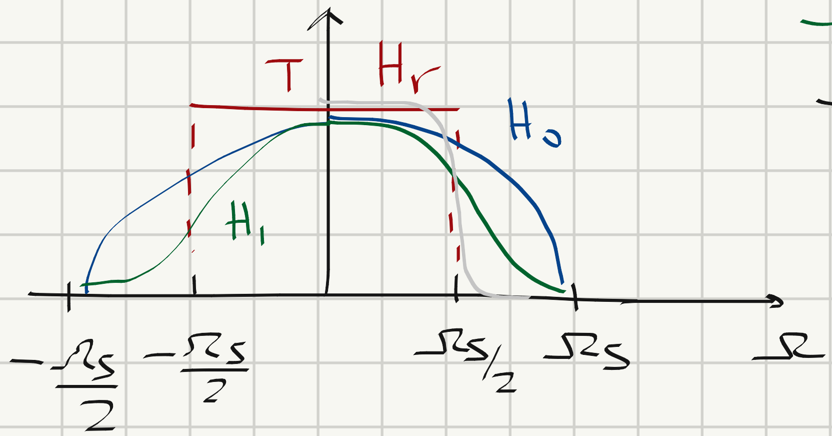


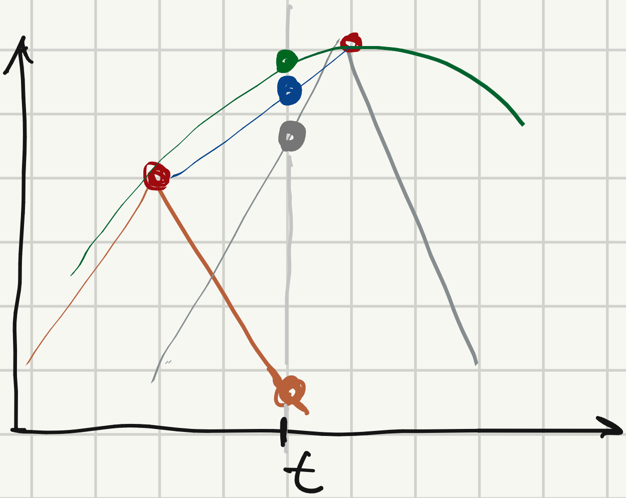
$$P_{T/2}(j\Omega) = \mathrm{senc}\left(\frac{T}{2}\Omega\right)$$

$$\boxed{H_1(j\Omega) = T\cdot \mathrm{senc}^2\left(\frac{T}{2}\Omega\right)}$$

la agrego después







Clase 27

Muestreo

Reconstrucción y cambio de frecuencia

Nota: Avanza en slides 42-45
Sigue en ppt de Aplicaciones

Nota: ver slides 46-47 junto con esta.

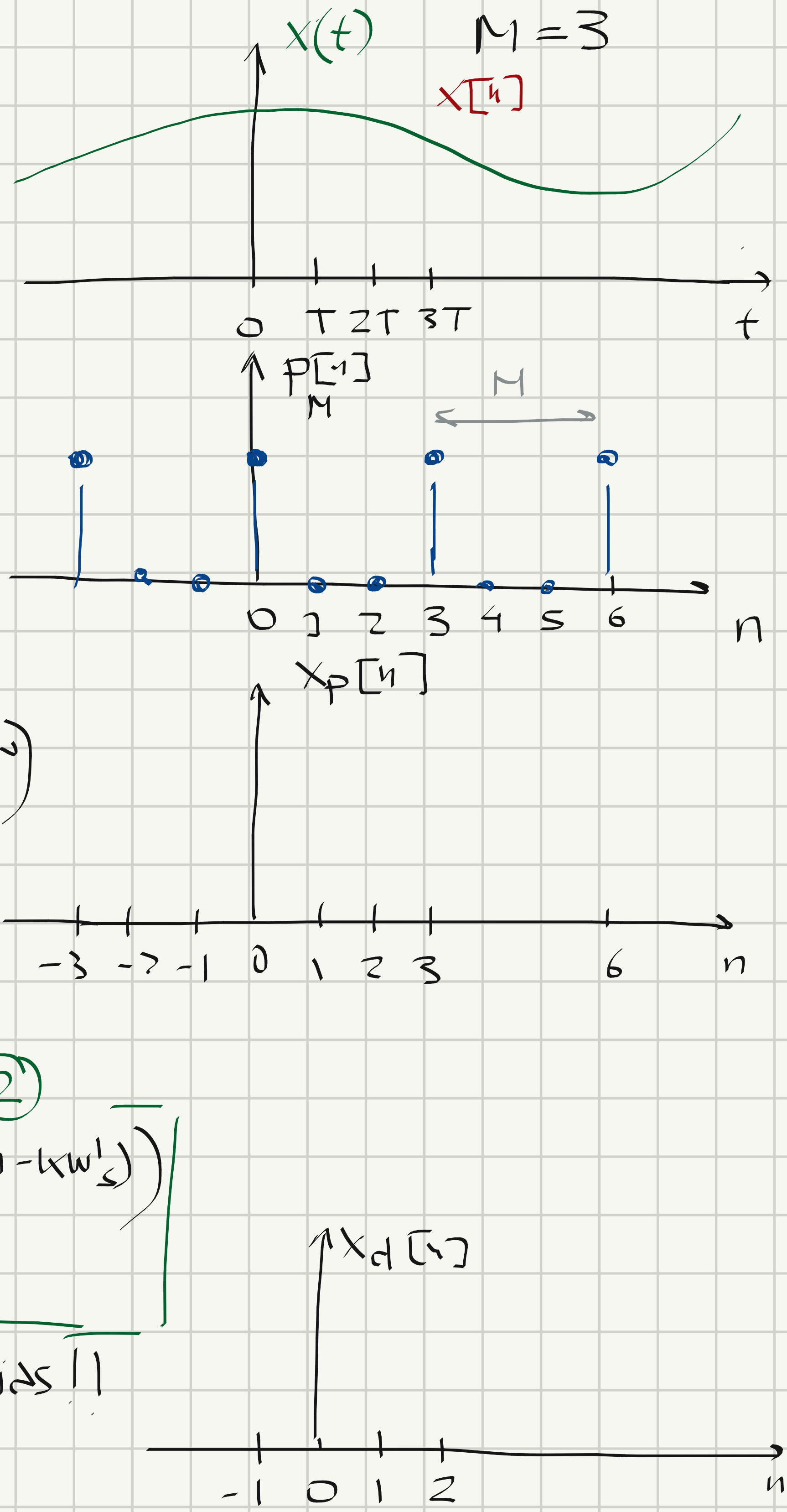
SUBMUESTREO (ANÁLISIS EN FRECUENCIA)

$$x_d[n] = x[Mn]$$

(1) $x_p[n] = x[n] \cdot p_M[n]$

$$p_M[n] = \sum_{k=-\infty}^{\infty} \delta[n - k \cdot M]$$

$$x_p[n] = \sum_{k=-\infty}^{\infty} x[kM] \cdot \delta[n - kM]$$



de (1)

$$X_p(e^{j\omega}) = X(e^{j\omega}) * P_M(e^{j\omega})$$

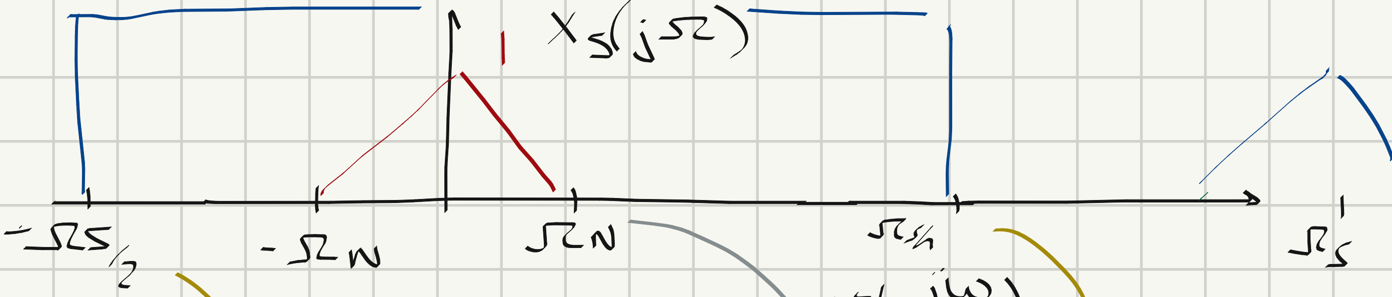
$$P_M(e^{j\omega}) = \frac{2\pi}{M} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$

$\omega'_s = \frac{2\pi}{M}$

(2)

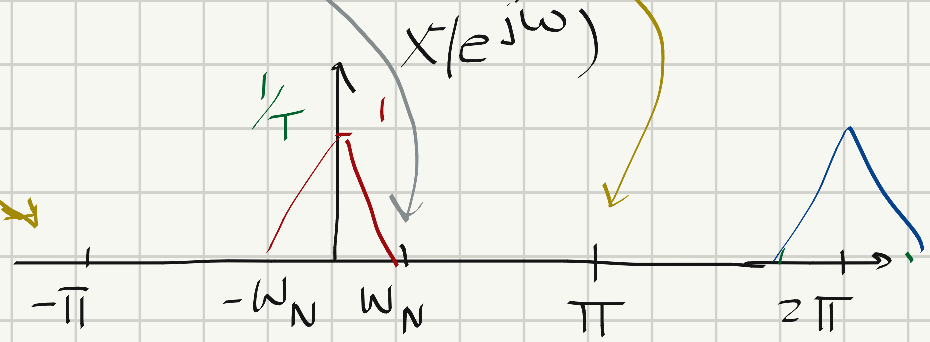
$$\boxed{X_p(e^{j\omega}) = \frac{1}{M} \sum_{k=-\infty}^{\infty} X(e^{j(\omega - k\omega'_s)})}$$

Obs: * aparecen copias!!



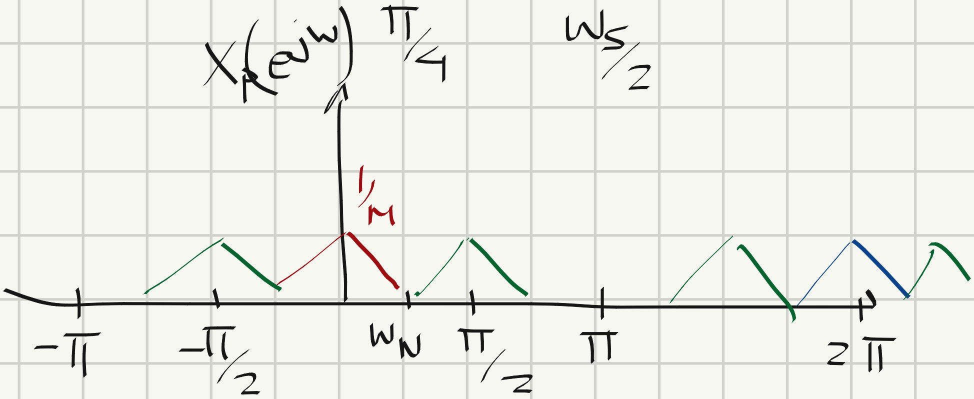
$\Omega_s = 8\,\Omega_N$

$M = 4$



$X\left(e^{j(\omega - k\omega_s')}\right)$

$\omega_s' = \dfrac{2\pi}{M} = \dfrac{\pi}{2}$



② $x_d[n] = x[M.n]$

$$X_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_d[n]\,.e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \underbrace{x[M.n]}_{x_p}\,.e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x_p[M n]\,.\,e^{-j\omega n} \underset{\uparrow}{=} \underbrace{\sum_{k=-\infty}^{\infty} x_p[k]\,.\,e^{-j\omega \frac{k}{M}}}$$

C.V. $k = Mn$

$n = \dfrac{k}{M}$

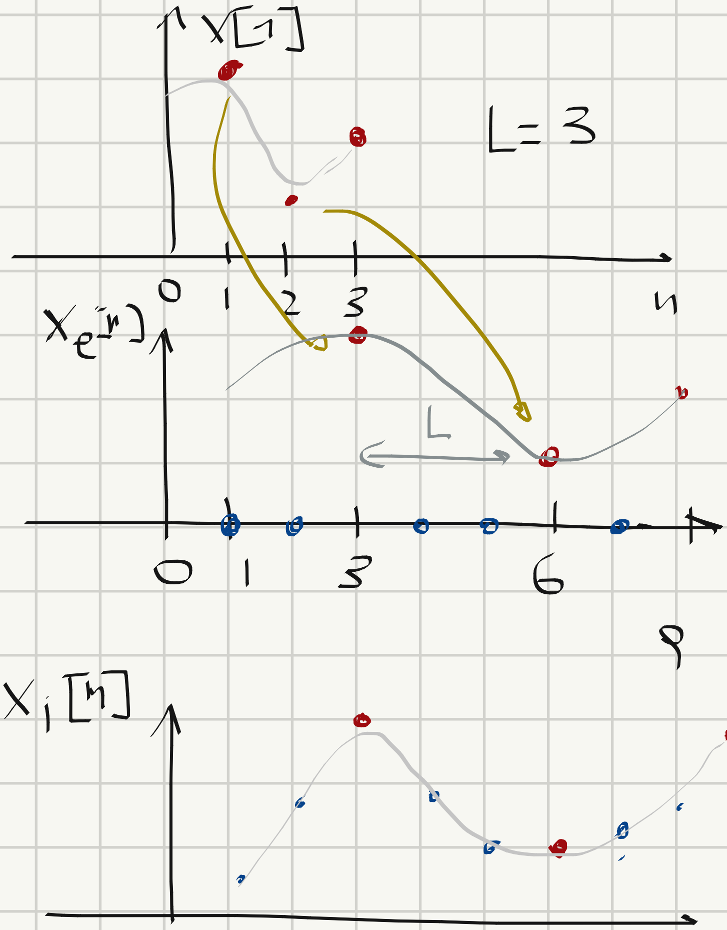
↳ es una DTFT

$$\boxed{X_d(e^{j\omega}) = X_p(e^{j\frac{\omega}{M}})}$$

Nota: sigue en slides 56-58

AUMENTO DE LA FRECUENCIA DE MUESTREO

[Slides 60-62]



$$x_e[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-kL]$$

$$X_e(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_e[n]\, e^{-j\omega n}$$

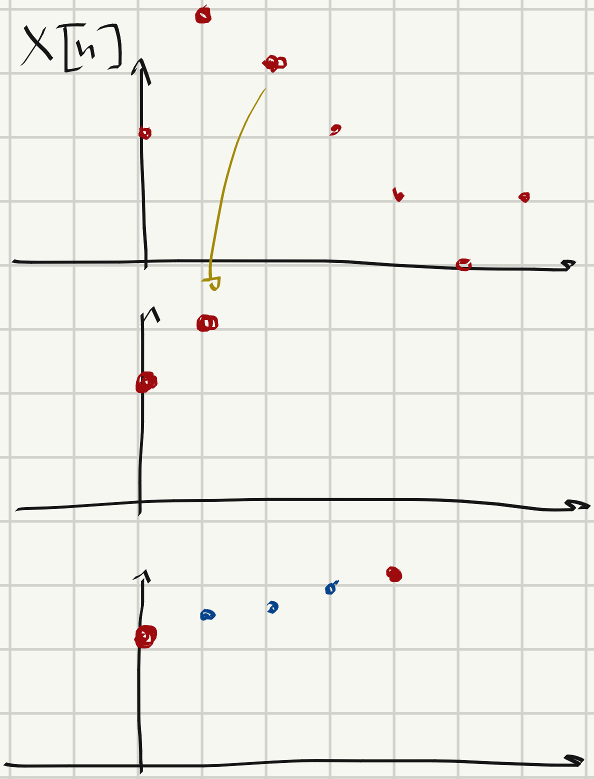
$$= \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-kL] \right) e^{-j\omega n}$$

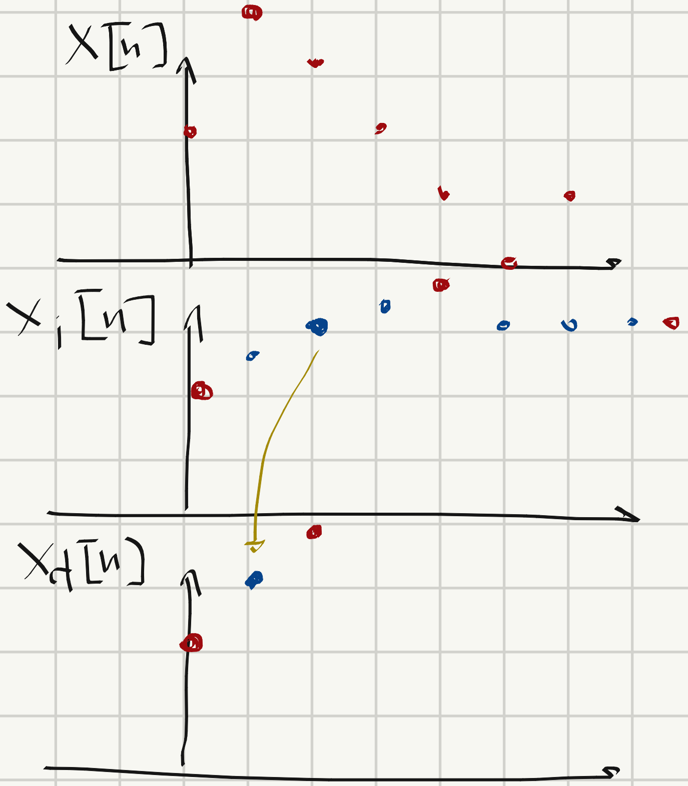
$$= \sum_{k=-\infty}^{\infty} x[k] \sum_{n=-\infty}^{\infty} \underbrace{\delta[n-kL]}_{n=kL}\, e^{-j\omega n}$$

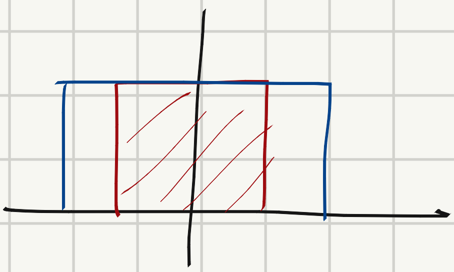
$$= \sum_{k=-\infty}^{\infty} x[k]\, e^{-j\omega kL}$$

$$\boxed{X_e(e^{j\omega}) = X(e^{j\omega L})}$$

CAMBIO ARBITRARIO DE FS







Clase 28

Transformada de Laplace

Definición

RESUMEN DEL CURSO

1) Introducción / Motivación T.C. ⟷ T.D.

2) Señales: def., propiedades, clasificación → "lindas"

3) Sistemas: def., propiedades, clasificación → sist. LIT

$\boxed{\text{Sistemas LIT}}$

4) Análisis temporal: → modelado
→ resp. impulso
→ convolución

5) Análisis espectral: → periódicas (armónico)
→ CTFS, DTFS

6) Análisis espectral: → generales
→ CTFT, DTFS

7) Muestreo: relación T.C. ⟷ T.D.

8) Análisis gral.: → T. Laplace y ~~T. Z~~
→ Análisis "completo" de un sistema
→ "Diseño" de sistemas

T. LAPLACE

Introducción

* Fourier es útil para analizar SLITs:
  - base de funciones sencillas: exp. imaginarios puros
  - autovectores de los SLIT
  - descomposición y superposición
  - funciones de energía finita

$s = j\omega$

* Si generalizo a exp. complejos generales e^{st}
  - muchas de estas propiedades se mantienen.
  - Análisis de sistemas "inestables"
  - Herramienta de cálculo

Idea: $x(t) = e^{st}$, $s \in \mathbb{C}$; sist. LIT; $y(t) = T\{x(t)\}$ $\Rightarrow$ $y(t) = \underbrace{H(s)}_{\text{valor propio}} \cdot \underbrace{e^{st}}_{\text{vector propio}}$

$$H(s) = \int_{-\infty}^{\infty} h(t) \cdot e^{-st} \cdot dt \xRightarrow{s = j\omega} H(j\omega) = \int_{-\infty}^{\infty} h(t) \cdot e^{-j\omega t} \cdot dt$$

$H(s)$: es una T.L.

$H(j\omega)$: no es una CTFS; si es una CTFT

Def:

$$\boxed{X(s) = \int_{-\infty \;(0)}^{\infty} x(t) \; e^{-st} \; dt}$$

ec. análisis

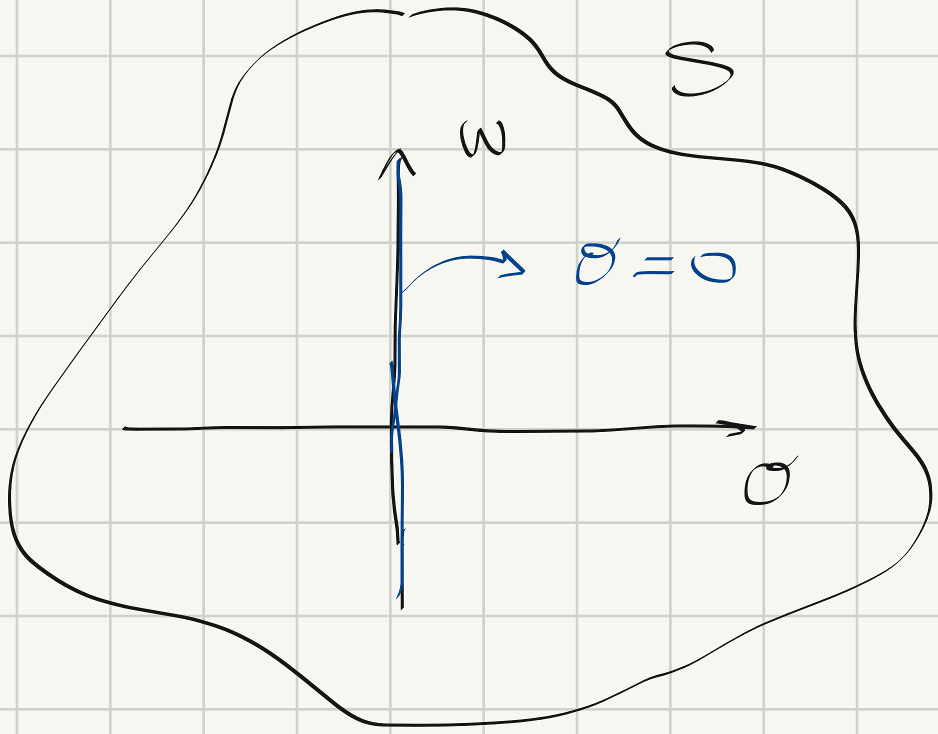
T. bilateral de Laplace (unilateral)

Obs: $X(s) = \langle x(t), e^{st} \rangle$

Notación:

$X(j\omega): \mathbb{R} \rightarrow \mathbb{C}(\mathbb{R}^2)$

* $s = \sigma + j\omega$
* $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$
* $X(s) = \mathcal{L}\{x(t)\}$
* $X(s): \mathbb{C} \rightarrow \mathbb{C}$



obs: * $X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \, dt \Rightarrow \boxed{X(j\omega) = X(s)\Big|_{s=j\omega}}$

* $X(s) = X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-(\sigma+j\omega)t} \, dt = \underbrace{\int_{-\infty}^{\infty} (\underbrace{x(t) \cdot e^{-\sigma t}}_{f(t)}) \, e^{-j\omega t} \, dt}_{\text{CTFT}}$

$$\boxed{X(s) = \mathcal{F}\{x(t) \cdot e^{-\sigma t}\}}$$

* Hay relación estrecha entre $\mathcal{F}$ y $\mathcal{L}$
  ↳ en particular en la convergencia.

Ejemplo 9.1: $x(t) = u(t)\, e^{-at}, \quad a \in \mathbb{R}$

del ej. 4.1 $\rightarrow$ $X(j\omega) = \int_{-\infty}^{\infty} u(t)\, e^{-at}\, e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-(\overbrace{a}^{>0}+j\omega)t}\, dt$

$$\boxed{X(j\omega) = \frac{1}{a+j\omega}}$$, converge si $a>0$

$X(s) = \int_{-\infty}^{\infty} u(t)\, e^{-at}\, e^{-st}\, dt = \int_{0}^{\infty} e^{-(a+s)t}\, dt = \int_{0}^{\infty} e^{-(\overbrace{a+\sigma}^{>0})t}\, e^{-j\omega t}\, dt$

$$\boxed{X(s) = \frac{1}{a+s}}$$, converge si $\sigma > -a$ o $\mathrm{Re}\{s\} > -a$

$$\boxed{u(t)\, e^{-at} \xleftrightarrow{\mathcal{L}} \frac{1}{a+s}, \quad \mathrm{Re}\{s\} > -a}$$

obs: * si $s = j\omega \Rightarrow$ coinciden!!

* si $a = 0$, $u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s}$ $\rightarrow$ integrador!!

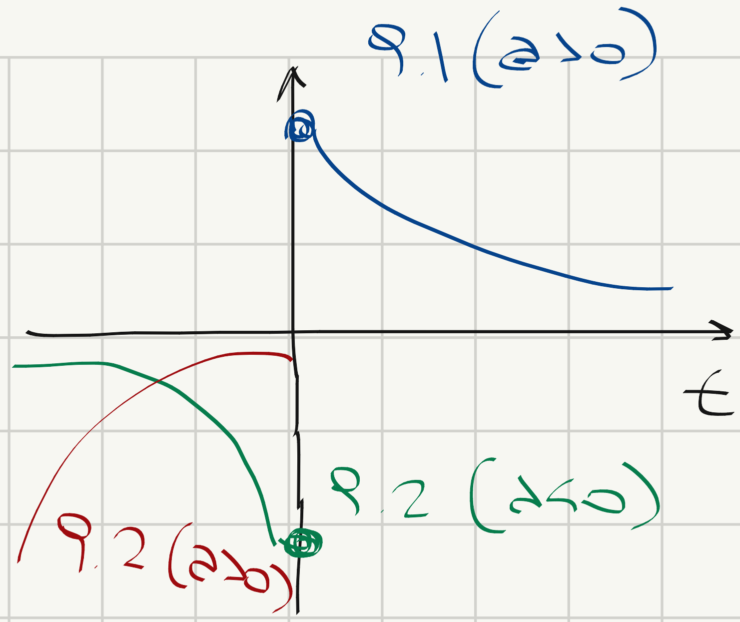
obs: * si $a>0$, converge en $\sigma = 0 \Rightarrow \exists\, \mathcal{F}\{x(t)\}$

* y se calcula como $\boxed{X(0+j\omega) = \frac{1}{a+j\omega}}$

* si $a<0$, $\exists\, \mathcal{L}$ pero $\nexists\, \mathcal{F}$

↳ caso nuevo que antes no podíamos estudiar!!

Ejemplo 9.2: $x(t) = -u(-t)\, e^{-at}$



$$X(s) = \int_{-\infty}^{\infty} -u(-t)\, e^{-at}\, e^{-st}\, dt$$

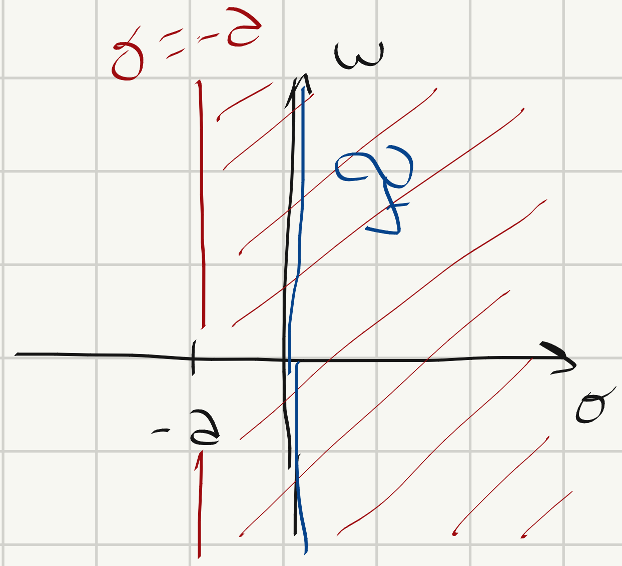
$$X(s) = -\int_{-\infty}^{0} e^{-(a+s)t}\, dt = \cancel{-} \left. \frac{e^{-(a+s)t}}{\cancel{-}(a+s)} \right|_{-\infty}^{0}$$

$$\boxed{X(s) = \frac{1}{a+s}} \qquad -(a+\sigma) > 0 \Rightarrow \boxed{\sigma < -a \Rightarrow \mathrm{Re}\{s\} < -a}$$

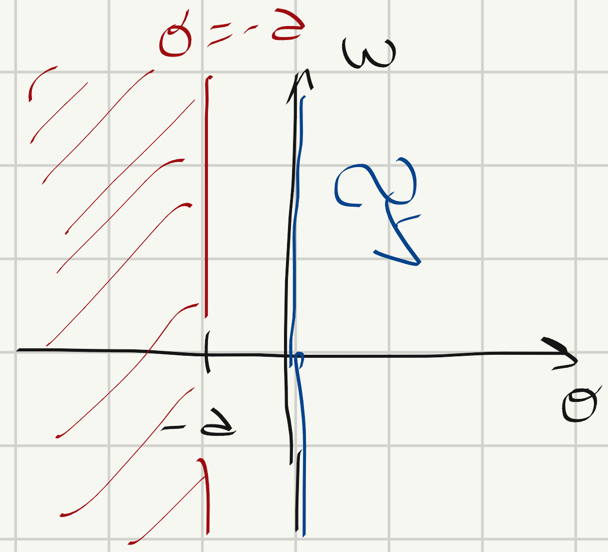
misma expresión algebraica con diferente dominio

Región de Convergencia (ROC)

$a>0$



Ej. 9.1



Ej. 9.2

obs:

* Expresión algebraica + ROC
* si $\sigma=0$, si el eje $\sigma=0$ (eje ω) está incluido en la ROC $\Rightarrow \exists\, \mathcal{F}$
* Dado $a \neq 0$, $\nexists\, \mathcal{F}$ en ambas a la vez.

Def: "ROC"

* $R = \{ s \in \mathbb{C} \,/\, \exists\, X(s) \}$
* Lugar geométrico

Ejemplo 8.3 $\quad x(t) = 3\,u(t)\,e^{-2t} - 2\,u(t)\,e^{-t}$

$$X(s) = \int_{-\infty}^{\infty} \Big(\underbrace{3\,u(t)\,e^{-2t} - 2\,u(t)\,e^{-t}}_{x(t)}\Big)\, e^{-st}\, dt$$

$$X(s) = 3 \underbrace{\int_{-\infty}^{\infty} u(t)\, e^{-2t-st}\, dt}_{\mathcal{L}\{u(t)\,e^{-2t}\}} - 2 \int_{-\infty}^{\infty} \underbrace{u(t)\, e^{-t}}_{\mathcal{L}\{u(t)\,e^{-t}\}}\, e^{-st}\, dt$$

linealidad del integral

$$X(s) = 3\underbrace{\mathcal{L}\{u(t)\,e^{-2t}\}}_{A} - 2\underbrace{\mathcal{L}\{u(t)\,e^{-t}\}}_{B}$$

$$\boxed{X(s) = 3\,\frac{1}{s+2}\ \text{(A)} - 2\,\frac{1}{s+1}\ \text{(B)}}$$

obs: * aparecen funciones racionales

$ROC_A: \ Re\{s\} > -2$

$ROC_B: \ Re\{s\} > -1$

$\Rightarrow ROC = ROC_A \cap ROC_B \Rightarrow \boxed{ROC: Re\{s\} > -1}$

obs: * se combinan las ROCs
* La expresión final es una función racional de segundo grado (abajo)

$$X(s) = \frac{3s+3-2s-4}{(s+2)(s+1)} = \boxed{\frac{s-1}{(s+2)(s+1)}} \rightarrow \text{func. racional}$$

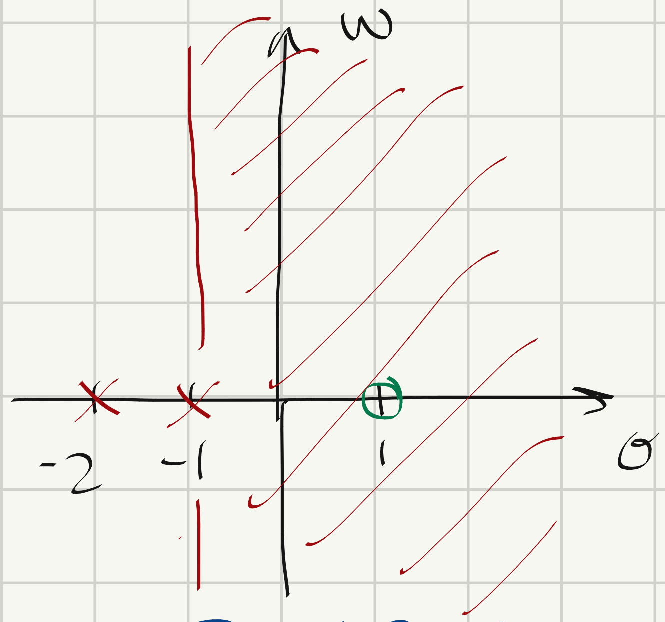
Diagrama de polos y ceros (DPZ)

func. racional: $X(s) = \dfrac{N(s)}{D(s)}$ → los polinomios quedan determinados por sus raíces.

Def: es una rep. gráfica de las raíces de la fun. racional

Notación

* P: Polos ×
* Z: Ceros o
* Singularidades múltiples
  → polos: ⋇
  → ceros: ◎



DPZ + ROC

obs: * los polos quedaron fuera de la ROC

* polos y ceros en infinito: $\lim_{s \to \begin{smallmatrix}0\\ \infty\end{smallmatrix}} X(s) = \begin{cases} 0 & \text{cero} \\ \infty & \text{polo} \end{cases}$

* ejemplos: → $\lim_{s\to\infty} X(s) = \infty \Rightarrow$ polo en infinito

  → $\lim_{s\to\infty} X(s) = 0 \Rightarrow$ cero en infinito

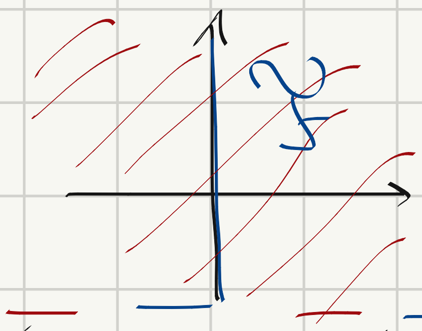
* si $gr\{N(s)\} > gr\{D(s)\} \Rightarrow$ polo en ∞ | cero en 0

* si $gr\{N(s)\} < gr\{D(s)\} \Rightarrow$ cero en ∞ | polo en 0

Ejemplo 9.5 $x(t) = \underbrace{\delta(t)}_{A} - \frac{4}{3}\underbrace{u(t)\,e^{-t}}_{B \quad 9.1} + \frac{1}{3}\underbrace{u(t)\,e^{2t}}_{C \quad 9.1}$

los tengo

A $\mathcal{L}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)\, e^{-st}\, dt = e^{-st}\Big|_{t=0} = \boxed{1}$ $\boxed{ROC: \mathbb{C}}$



$$X(s) = 1 - \frac{4}{3}\cdot\frac{1}{s+1} + \frac{1}{3}\cdot\frac{1}{s-2}$$

$$X(s) = \frac{3(s+1)(s-2) - 4(s-2) + (s+1)}{3(s+1)(s-2)} = \frac{3s^2 - 3s - 6 - 4s + 8 + s + 1}{3(s+1)(s+2)}$$

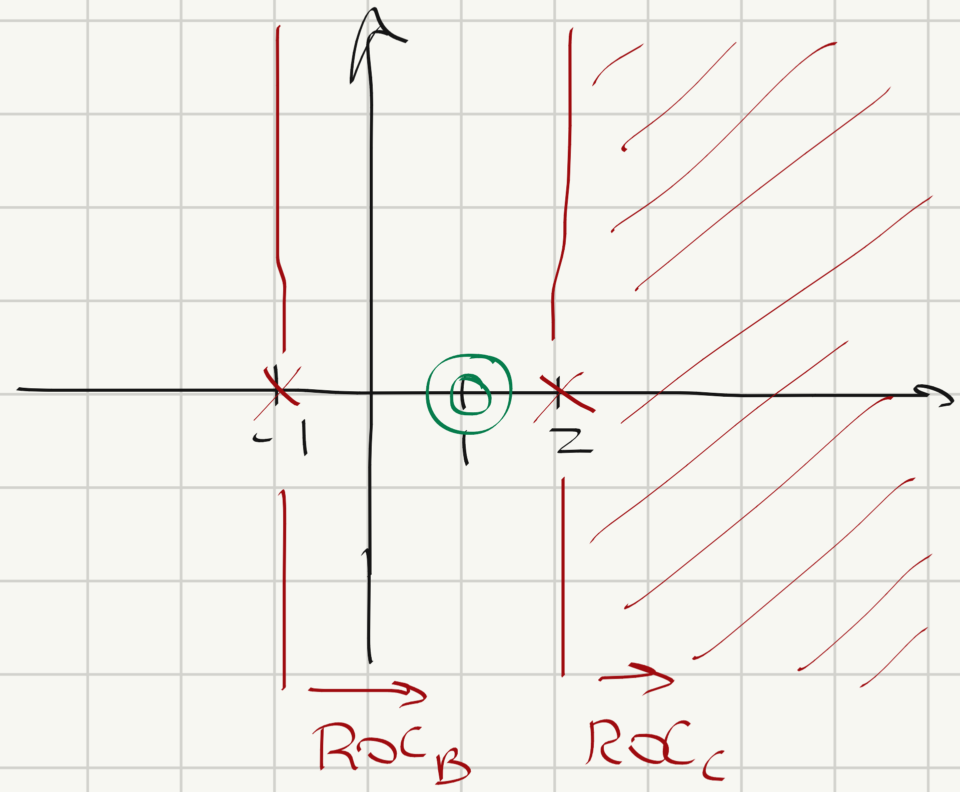
$$X(s) = \frac{\cancel{3}s^2 - \cancel{6}^{\,2}s + \cancel{3}}{\cancel{3}(s+1)(s-2)} = \frac{s^2 - 2s + 1}{(s+1)(s-2)} = \boxed{\frac{(s-1)^2}{(s+1)(s-2)}}$$

ROC_A : $\mathbb{C}$

ROC_B : $Re\{s\} > -1$

ROC_C : $Re\{s\} > 2$

ROC : $\boxed{Re\{s\} > 2}$



Clase 29

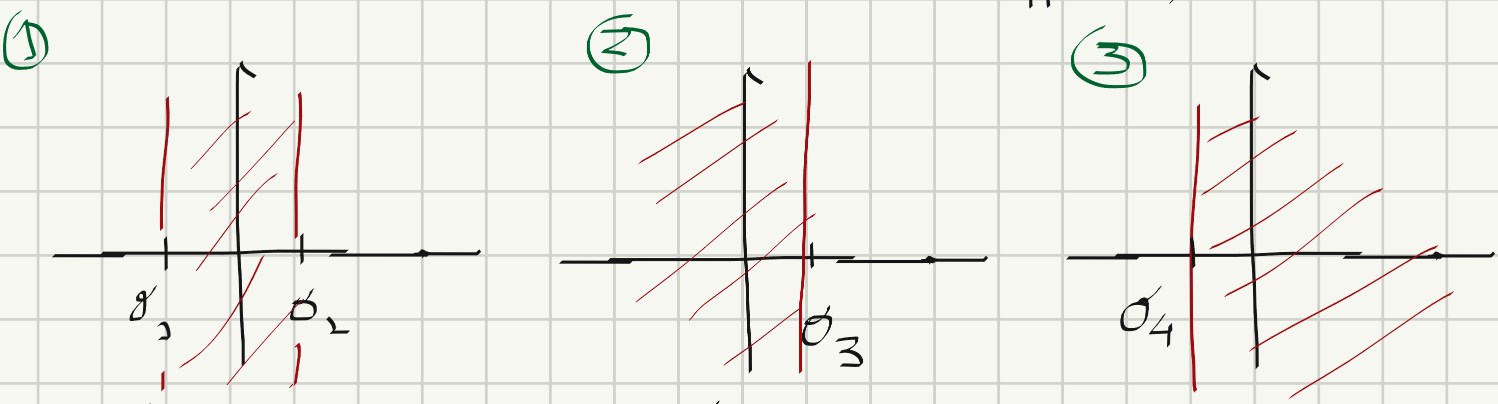
Transformada de Laplace

Región de convergencia

REGIÓN DE CONVERGENCIA

Propiedad 1: La ROC consiste en bandas paralelas al eje $\underline{\underline{w}}$

$$\mathcal{L}\{x(t)\} = \mathcal{F}\{\underbrace{x(t)\,e^{-\sigma t}}_{\text{tamaño}}\} \Rightarrow \text{convergencia depende de } \| x(t)\,e^{-\sigma t} \| \text{ solo } \sigma$$



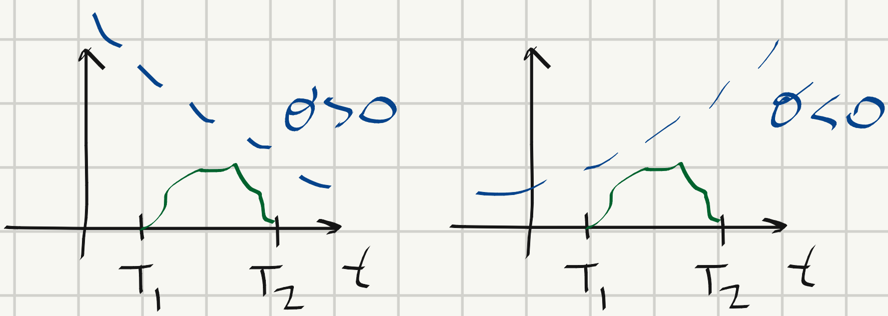
$\sigma_1 < \sigma < \sigma_2$ $\qquad$ $\sigma < \sigma_3$ $\qquad$ $\sigma > \sigma_4$

Propiedad 2 Para $X(s)$ racional, no contiene polos

a polo $X(s) \Rightarrow \boxed{\lim_{s \to a} X(s) = \infty}$

Propiedad 3 $x(t)$ soporte finito y absolutamente integrable

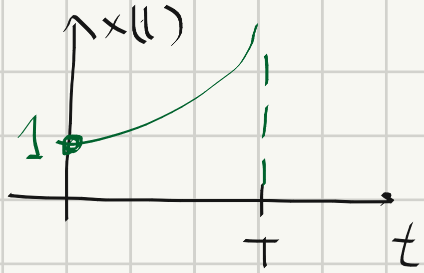
$\Rightarrow \mathrm{ROC} = \mathbb{C}$



$\| x(t)\,e^{-\sigma t} \|$

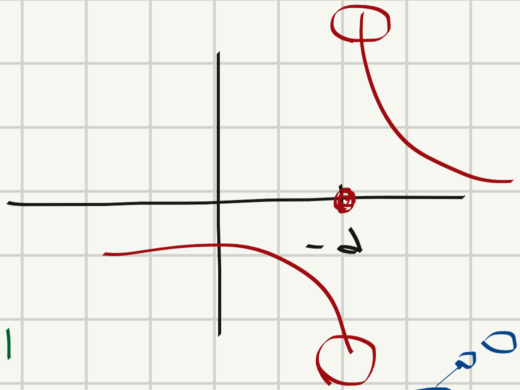
Ejemplo 9.6

$$x(t) = \begin{cases} e^{-at} & 0<t<T \\ 0 & \text{e.o.c.} \end{cases}$$



$$X(s) = \int_0^T e^{-at}\, e^{-st}\, dt = \frac{e^{-(a+s)T} - 1}{-(a+s)} = \boxed{\frac{1 - e^{-(a+s)T}}{a+s}}$$

$\forall s \neq -a$



$s = -a$

$$X(-a) = \int_0^T \underbrace{e^{-at}\, e^{at}}_{1}\, dt = \boxed{T}$$

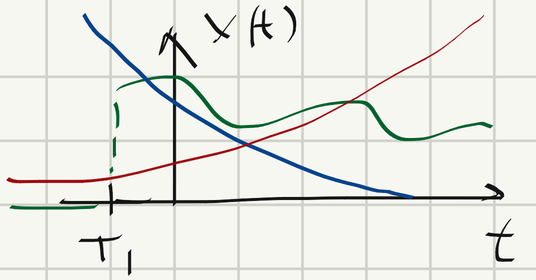
$$\lim_{s\to -a} X(s) = \lim_{s\to -a} \frac{1 - e^{-(a+s)T}}{a+s} = \lim_{s\to -a} \frac{T\, e^{-(a+s)T}}{1} = \boxed{T}$$

indeterminación $\frac{0}{0}$

L'Hopital

$$\boxed{\text{ROC: } \mathbb{C}}$$

Propiedad 4:



x(t) derecha y converge para un cierto σ_0, entonces converge $\forall\, \sigma > \sigma_0$

$$|X(s)| \le \int_{-\infty}^{\infty} |x(t)|\, e^{-\sigma t}\, dt = \int_{T_1}^{\infty} |x(t)|\, e^{-\sigma_0 t}\, e^{-\overbrace{(\sigma-\sigma_0)}^{>0} t}\, dt \le$$

$\sigma > \sigma_0$

$$\int_{T_1}^{\infty} |x(t)|\, e^{-\sigma_0 t}\, e^{-(\sigma-\sigma_0)T_1}\, dt = \underbrace{e^{-(\sigma-\sigma_0)T_1}}_{\text{finito}} \underbrace{\int_{T_1}^{\infty} |x(t)|\, e^{-\sigma_0 t}\, dt}_{<\infty}$$

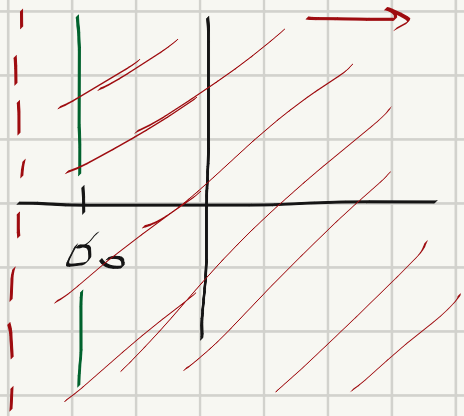
$$\Rightarrow \boxed{|X(s)| < \infty}$$

obs: * no puede crecer infinitamente en la dirección negativa

↳ que es donde puede divergir

* ROC $\supset \{\mathrm{Re}\{s\} > \sigma_0\}$

* ROC es un Π^+

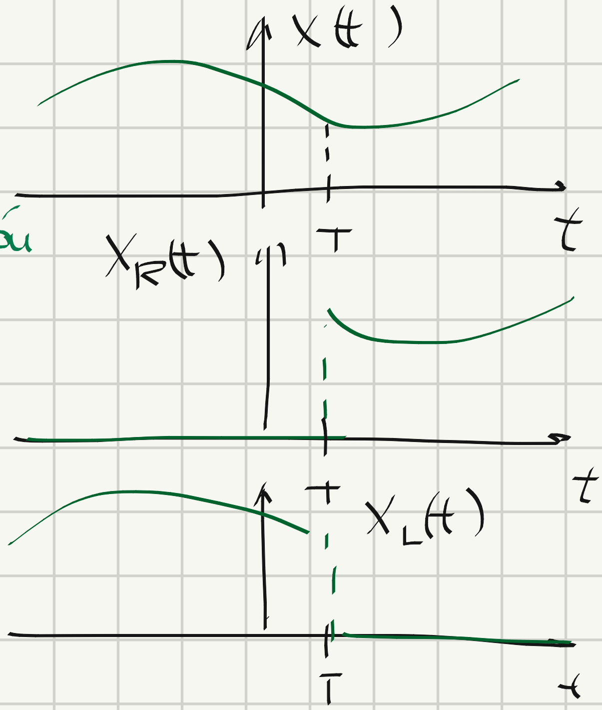


Propiedad 5:

$$\left.\begin{array}{l} x(t) \text{ izquierda} \\ \text{ROC} \supset \{\sigma = \sigma_0\} \end{array}\right\} \Rightarrow \text{ROC} \supset \{\mathrm{Re}\{s\} < \sigma_0\}$$

(ROC es un Π^-)

Propiedad 6 $x(t)$ bilateral



$\exists\ x_R(t)$ y $x_L(t)$ / $x(t) = x_R(t) + x_L(t)$

↑ descomposición

(H) $\sigma = \sigma_0$

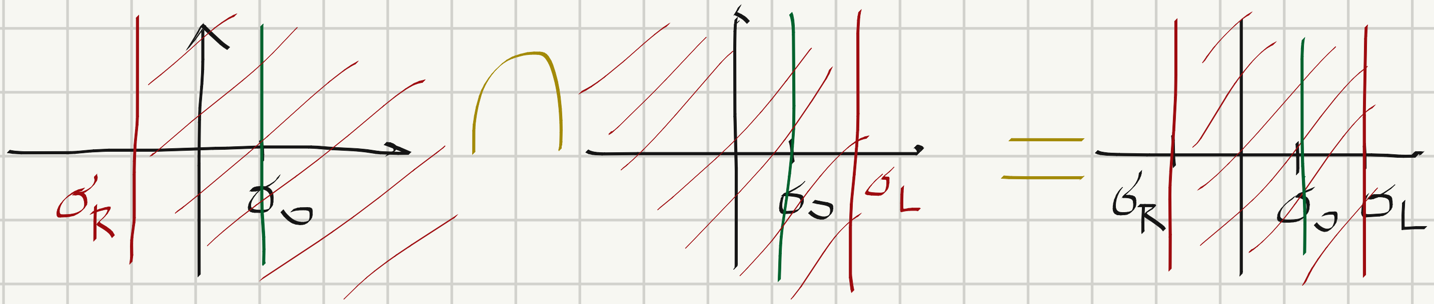
$$L = \{ \mathrm{Re}\{s\} = \sigma_0 \} \subset ROC$$

prop. 4

* $ROC_R \supset \Pi^{+}_{\sigma_0} = \{ \mathrm{Re}\{s\} > \sigma_0 \}$
* $ROC_L \supset \Pi^{-}_{\sigma_0} = \{ \mathrm{Re}\{s\} < \sigma_0 \}$

(T) $ROC = ROC_R \cap ROC_L$

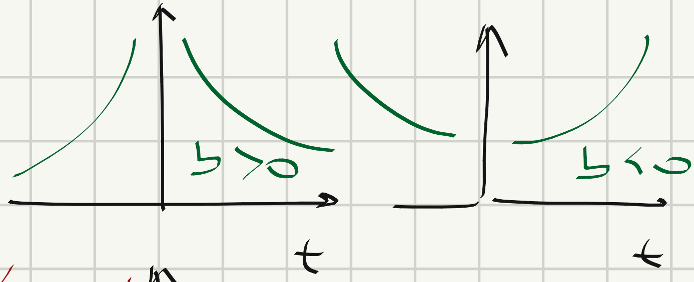
$$ROC = \{ \sigma_R < \mathrm{Re}\{s\} < \sigma_L \}$$



ROC_R ROC_L ROC

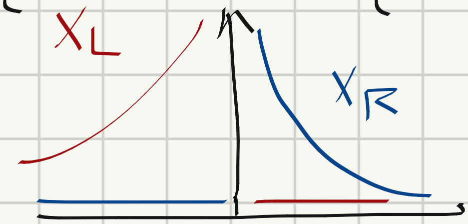
Ejemplo 9.7

$$x(t) = e^{-b|t|}$$



descomposición:

$$x(t) = \underbrace{u(t)\, e^{-bt}}_{x_R(t)} + \underbrace{u(-t)\, e^{+bt}}_{x_L(t)}$$



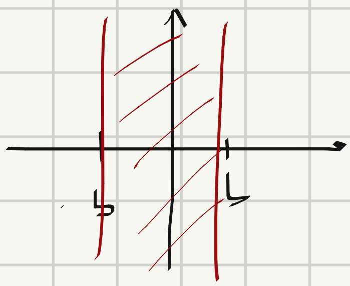
$$\boxed{X_R(s) = \frac{1}{s+b} \quad 9.1 \qquad ROC_R: \ Re\{s\} > -b}$$

$$x_L(t) = \underbrace{-}\Big(\underbrace{-u(-t)\, e^{-(-b)t}}\Big) \rightarrow 9.2$$

$$\boxed{X_L(s) = -\frac{1}{s-b} \qquad \alpha = -b \qquad ROC: \{Re\{s\} < b\}}$$

$$X(s) = \frac{1}{s+b} - \frac{1}{s-b} = \boxed{-\frac{2b}{(s+b)(s-b)}}$$

$$\boxed{ROC: \{-b < Re\{s\} < b\}}$$



obs: 5 casos de ROC

* semiplano derecho
* semiplano izquierdo
* banda
* todo el plano
* conjunto vacío

Propiedad 7:

si $X(s)$ racional, su ROC:
- → se va infinito σ
- → limitada por los polos.

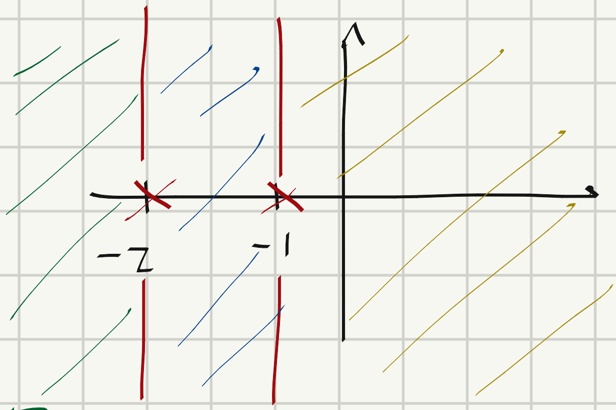
Propiedad 8:

① $X(s)$ racional, $x(t)$ derecha → ROC es el semiplano derecho a la derecha de polo más a la derecha

② $X(s)$ racional, $x(t)$ izquierda → ROC es el semiplano izquierdo a la izquierda del polo de más a la izquierda.

Ejemplo 9.8

$$X(s) = \frac{1}{(s+1)(s+2)}$$



	ROC_1	ROC_2	ROC_3
	TI^-	banda	TI^+
$x(t)$ →	izq.	bilat.	der.

TRANSFORMADA INVERSA DE LAPLACE

$$X(s) = X(\sigma + j\omega) = \mathcal{F}\{x(t)\, e^{-\sigma t}\} \Rightarrow$$

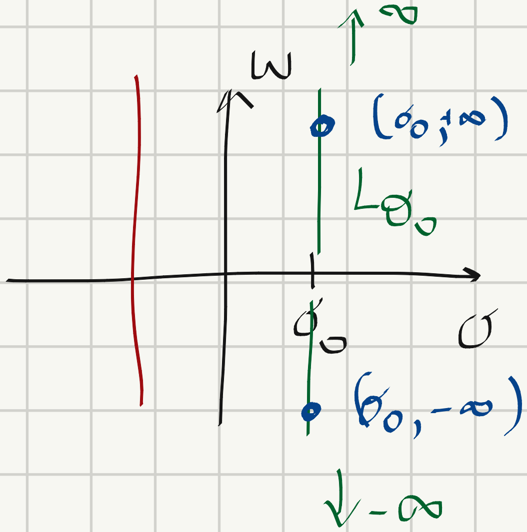
$$\mathcal{F}^{-1}\{X(s)\} = x(t)\, e^{-\sigma t} \Rightarrow \boxed{x(t) = e^{\sigma t}\, \mathcal{F}^{-1}\{X(s)\}}$$

$$\boxed{x(t) = e^{\sigma t} \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)\, e^{j\omega t}\, d\omega}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\underbrace{\sigma + j\omega}_{s})\, e^{\overbrace{(\sigma + j\omega)}^{s} t}\, d\omega$$

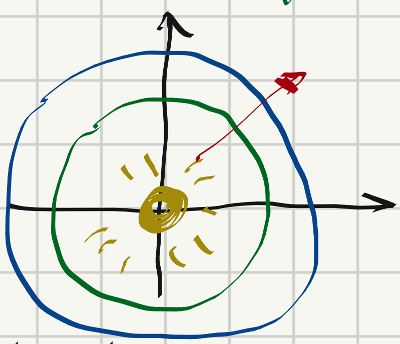
C.V.: $s = \sigma + j\omega$

$$x(t) = \frac{1}{2\pi}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)\, e^{st}\, ds$$



$$\boxed{x(t) = \frac{1}{2\pi}\oint_{L_{\sigma_0}} X(s)\, e^{st}\, ds}$$

Integral de línea



Obs:

* las integrales de línea son difíciles de resolver
* se puede calcular para cualquier σ
* para $X(s)$ racional se pueden resolver por fracciones simples

Ejemplo genérico

* no hay polos múltiples
* $gr(N) < gr(D)$ → sino, hay truco!!

puedo usar fracciones simples

$$X(s) = \sum_{i=1}^{N} \frac{A_i}{s+a_i}$$

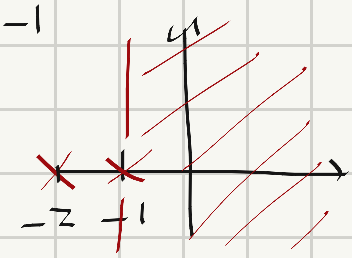
→ N polos a_i, donde cada uno se antitransforma como en 9.1

$$\frac{1}{s+a_i} \overset{\mathcal{L}^{-1}}{\longleftrightarrow} x_i(t) = \begin{cases} u(t)\cdot e^{-a_i t} & \text{si ROC es } \Pi^+ \\ -u(-t)\cdot e^{a_i t} & \text{si ROC es } \Pi^- \end{cases}$$

$$x(t) = \sum_{i=1}^{N} x_i(t) = \boxed{\sum_{i=1}^{N} A_i \cdot u(t) \cdot e^{-a_i t}}$$

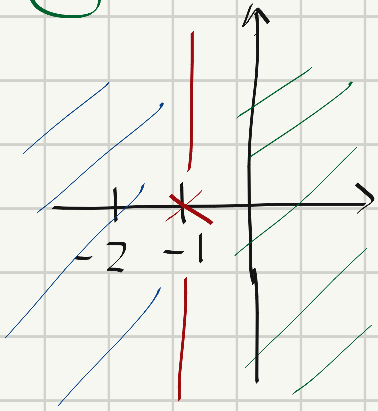
Asumí x_i derecha $\forall i$

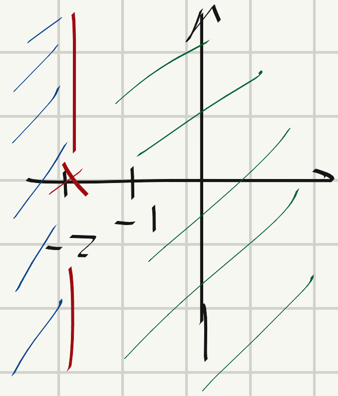
Ejemplo 9.9 $\quad X(s) = \dfrac{1}{(s+1)(s+2)}, \quad Re(s) > -1$



$$X(s) = \frac{A}{s+1} + \frac{B}{s+2}$$

$$X(s) = \underbrace{\frac{1}{s+1}}_{A} - \underbrace{\frac{1}{s+2}}_{B}$$





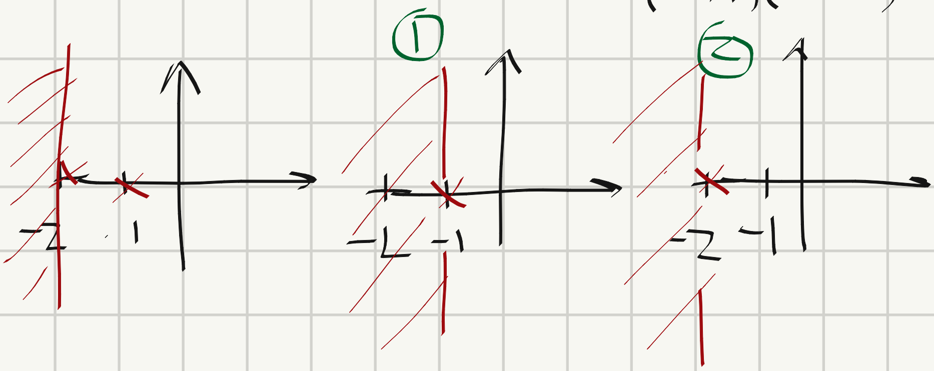
1) $\dfrac{1}{s+1}, \quad Re(s) > -1 \quad \overset{\mathcal{L}^{-1}}{\longleftrightarrow} \quad u(t)\, e^{-t}$

2) $\dfrac{1}{s+2}, \quad Re(s) > -2 \quad \overset{\mathcal{L}^{-1}}{\longleftrightarrow} \quad u(t)\, e^{-2t}$

$$\boxed{x(t) = u(t) \cdot \left[e^{-t} - e^{-2t} \right]}$$

Ejemplo 9.10

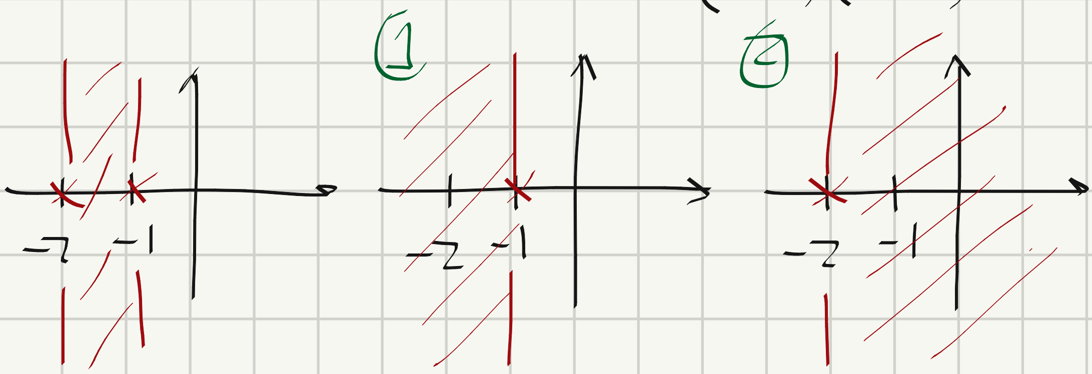
$$X(s) = \frac{1}{(s+1)(s+2)}, \quad Re(s) < -2$$



$$\boxed{x(t) = -u(-t)\left[e^{-t} - e^{-2t}\right]}$$

Ejemplo 9.11

$$X(s) = \frac{1}{(s+1)(s+2)} \qquad -2 < Re(s) < -1$$



$$\boxed{x(t) = -u(-t)\, e^{-t} - u(t)\, e^{-2t}}$$

Clase 30

Análisis de sistemas

En tiempo y frecuencia

ANALISIS DE SISTEMAS EN T y F

Representación magnitud-fase

Análisis: * módulo: distribución de energía (forma)
* fase: comportamiento en el tiempo (ret)

Ejemplos: * barco: interferencia constructiva.

* audio: → fase lineal (retardo constante)
→ reverberación (espacialización)
→ flanging
→ phase shifting
* imágenes: → fase: geometría y bordes
→ módulo: direcciones y colores

fase lineal: $\boxed{\phi(\omega) = \tau_0\, \omega}$

retardo de grupo: $\boxed{\tau(\omega) = -\dfrac{d\phi}{d\omega}}$

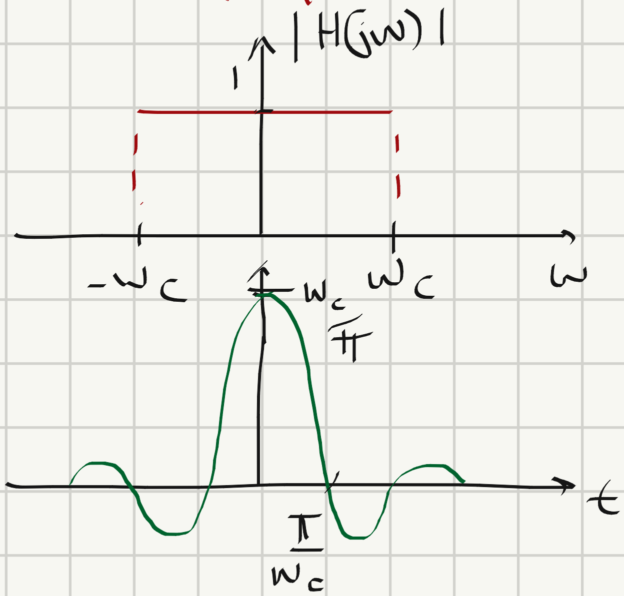
Bode: → escala logarítmica y asintótica
→ producto en logaritmo es aditivo.

Propiedades de filtros en tiempo y frecuencia

Filtro ideal: (pasabajos)

$$H(j\omega)=\begin{cases}1 & |\omega|<\omega_c\\ 0 & |\omega|\geq\omega_c\end{cases}$$

$$h(t)=\text{senc}(\omega_c t)$$

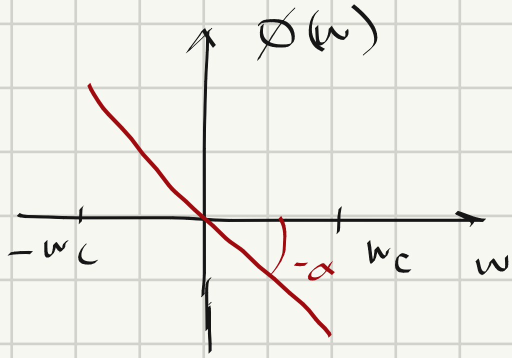


Fase lineal:

$$\phi(\omega)=-\alpha\omega$$

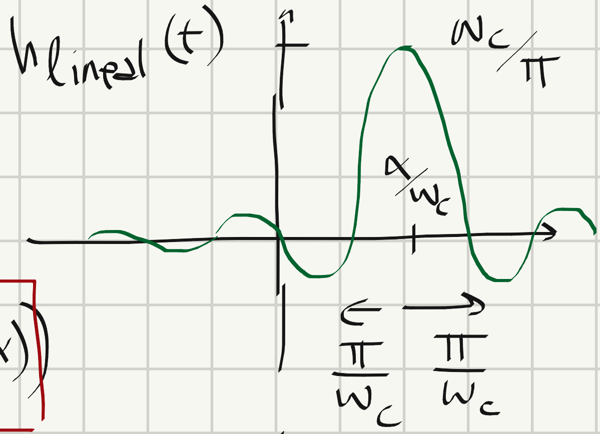
$$H_2(j\omega)=e^{-j\alpha\omega},\quad h_2(t)=\delta(t-\alpha)$$

$$H_{lineal}(j\omega)=H(j\omega)\,H_2(j\omega)$$



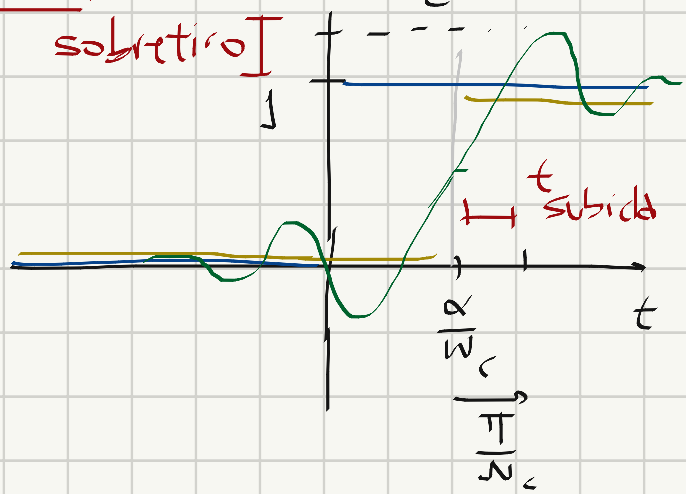
$$H_{lineal}=\begin{cases}e^{-j\alpha\omega} & |\omega|<\omega_c\\ 0 & |\omega|\geq\omega_c\end{cases}$$

$$h_{lineal}(t)=h(t)*h_2(t)=\boxed{\text{sen}(\omega_c(t-\alpha))}$$



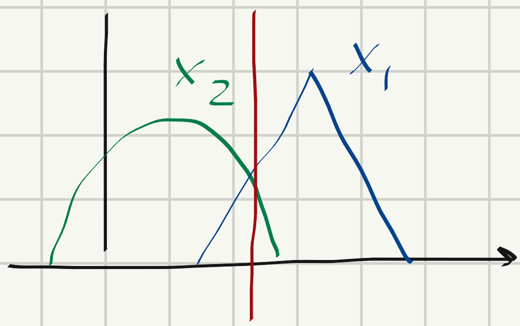
respuesta al escalón:

$$s(t)=\int_{-\infty}^{t}h(\tau)\,d\tau$$



Propiedades en tiempo y frecuencia de filtros reales

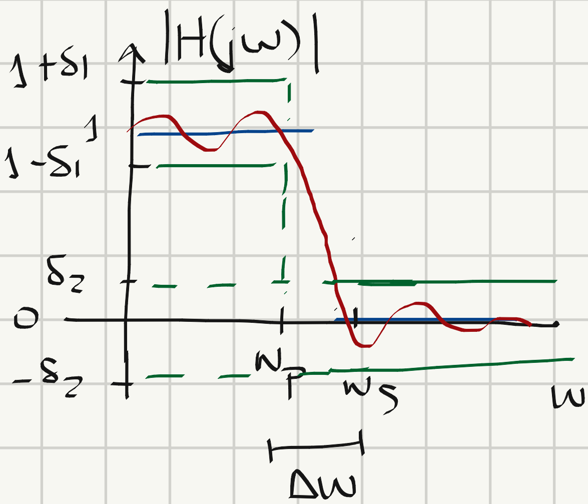
obs: propiedades no deseadas de filtros reales
* cuando hay solapamiento: mejor transiciones suaves
* oscilaciones: discontinuidad
* facilidad de construcción
* causalidad



obs: * necesitamos flexibilizar los requerimientos
* no puedo fijar todos parámetros a la vez

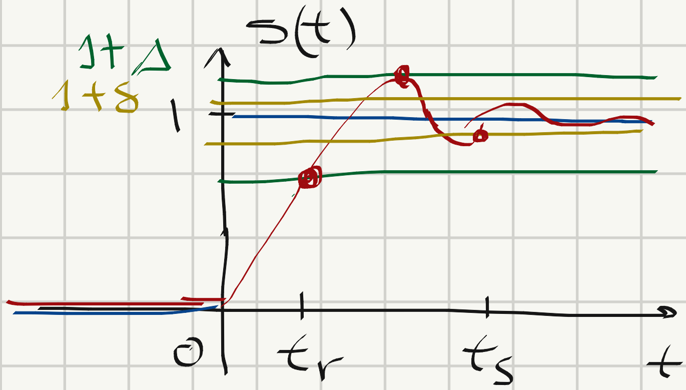
Frecuencia

* δ_1: oscilación permitida BP
* δ_2: oscilación permitida BS
* W_p: ancho de la BP
* W_s: ancho de la BS
* W_c: frecuencia de corte
* Δw: tolerancia en la fc



Tiempo

* Δ: sobretiro
* δ: precisión
* t_r: t. de subida
* t_s: t. de asentamiento



Ejemplos de filtros

Ejemplo 6.3

$f_c = 500\,Hz$, tolerancia 0,05
$N=5$ (orden)
Comparación Butterworth vs elíptico

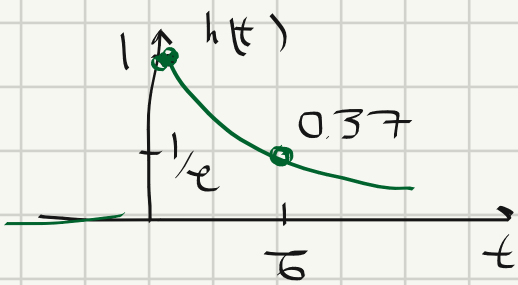
Obs: * respuesta de 5to orden [slides 2 y 3]

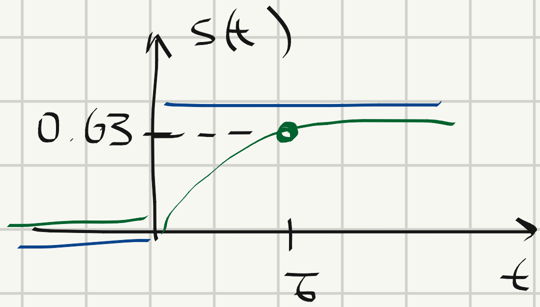
ANALISIS DE SISTEMAS DE 1er y 2do ORDEN en T y F

* muchos sistemas reales de interés que se pueden modelar así
* en general son fáciles de construir
* sistemas de orden mayor se pueden descomponer en combinaciones de 1er y 2do orden.

Primer orden

* modelo: $\tau \cdot y'(t) + y(t) = x(t)$
* tiempo: $\boxed{h(t) = \frac{1}{\tau} \cdot e^{-\frac{t}{\tau}}}$ $\boxed{s(t) = \left(1 - e^{-\frac{t}{\tau}}\right) u(t)}$





* frecuencia: $\boxed{H(jw) = \frac{1}{1 + jw\tau}}$

$$\boxed{|H(jw)|^2 = \frac{1}{w^2\tau^2 + 1}} \qquad \boxed{\theta(w) = -\operatorname{atan} w\tau}$$

Bode:

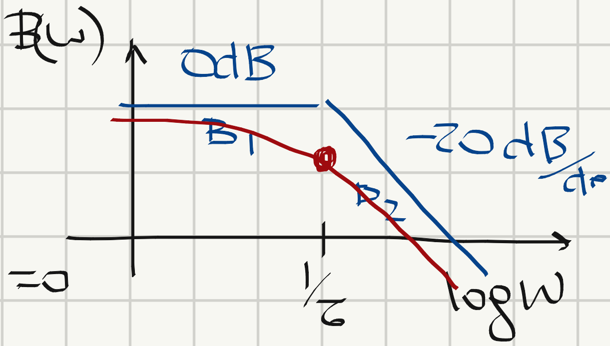
$$B(w) = 20 \cdot \log_{10} |H(jw)| = -10 \log\left((w\tau)^2 + 1\right)$$

a) si $w\tau \ll 1$ $\left(w \ll \frac{1}{\tau}\right) \Rightarrow \boxed{B(w) = 0 \text{ dB}}$

b) si $w\tau \gg 1$ $\left(w \gg \frac{1}{\tau}\right) \Rightarrow$

$$\boxed{B(w) = \underbrace{-20}_{a} \underbrace{\log_{10} w}_{x} \underbrace{-20 \log_{10} \tau}_{b}}$$

* verif. recta:



$$B_1\left(\frac{1}{\tau}\right) = 0$$

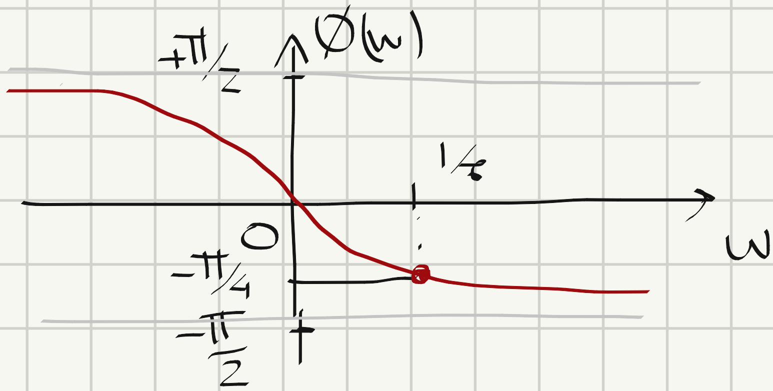
$$B_2\left(\frac{1}{\tau}\right) = -20 \cdot \cancel{\log\left(\frac{1}{\tau}\right)} \cancel{-20 \log \tau} = 0$$

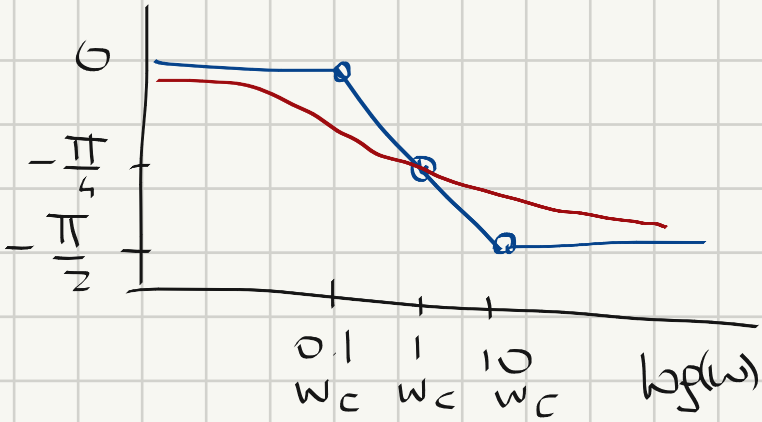
$$B\left(\frac{1}{\tau}\right) = -10 \cdot \log\left(\left(\frac{1}{\tau}\tau\right)^2 + 1\right) = -10 \cdot \log_{10}(2) \neq \boxed{-3.02 \text{ dB}}$$

fase:

$$\phi(\omega) = -\operatorname{atan}(\omega\tau)$$

$$\Theta(\omega) \simeq \begin{cases} 0 & \omega \ll \omega_c \qquad \omega < 0.1\frac{1}{\tau} \\ -\frac{\pi}{4}\left[\log(\omega) + \log(\tau) + 1\right] & 0.1\frac{1}{\tau} < \omega < 10\frac{1}{\tau} \\ -\frac{\pi}{2} & \omega \gg \omega_c \qquad \omega > 10\frac{1}{\tau} \end{cases}$$





Clase 31

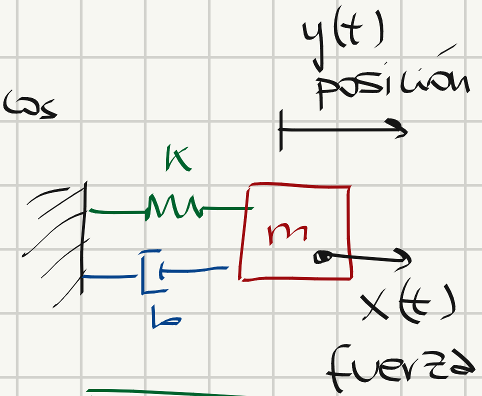
Análisis de sistemas

En tiempo y frecuencia

SISTEMAS DE SEGUNDO ORDEN

modelo gral: $y''(t) + 2\xi\omega_n y'(t) + \omega_n^2 y(t) = \omega_n^2 x(t)$

Obs: * sistemas amortiguados o viscosos
* ejemplos: → circuitos RLC
→ sistemas mecánicos



modelado (ejemplo):

$$m\,y''(t) = x(t) - k\,y(t) - b\,y'(t)$$

$$\boxed{y''(t) + \frac{b}{m}\,y'(t) + \frac{k}{m}\,y(t) = \frac{1}{m}\,x(t)}$$

$$\boxed{\omega_n = \sqrt{\frac{k}{m}}}$$

$$\boxed{\xi = \frac{b}{\sqrt{2km}}}$$

$$\boxed{y''(t) + 2\xi\omega_n y'(t) + \omega_n^2 y(t) = \frac{\omega^2}{k}\,x(t)}$$

k → sólo ganancia

Análisis en frecuencia

$$x(t) = e^{j\omega t} \longrightarrow y(t) = H(j\omega)\,e^{j\omega t}$$

$$H(j\omega)\,(j\omega)^2\,\cancel{e^{j\omega t}} + 2\xi\omega_n\,j\omega\,\cancel{e^{j\omega t}}\,H(j\omega) + \omega_n^2\,H(j\omega)\,\cancel{e^{j\omega t}} = \omega_n^2\,\cancel{e^{j\omega t}}$$

$$H(j\omega)\left[(j\omega)^2 + 2\xi\omega_n\,j\omega + \omega_n^2\right] = \omega_n^2$$

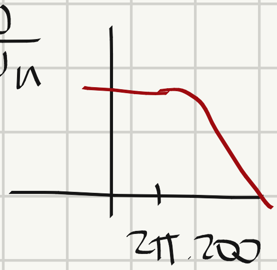
$$\boxed{H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\xi\omega_n j\omega + \omega_n^2}}$$

$$\boxed{H(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n\,s + \omega_n^2}}$$

$s = j\omega$

$$\boxed{H(jw)=\frac{1}{\left(j\frac{w}{w_n}\right)^2+2\varepsilon j\frac{w}{w_n}+1}}$$

obs:
- * w_n: frecuencia natural del sistema (escala la frecuencia)
- * ε: razón de amortiguamiento
- * $H(jw)$: depende sólo de la razón $\frac{w}{w_n}$

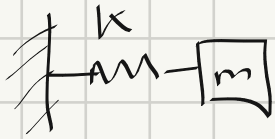


→ $w_n = 2\pi \cdot 100 \qquad w = 2\pi \cdot 200$

→ $w_n = 2\pi \cdot 1 \qquad w = 2\pi \cdot 2$

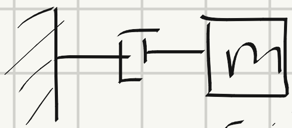
→ $\tilde{w} = \frac{w}{w_n} = 2$

obs:
- * $\varepsilon = 0$, $\varepsilon = \sqrt{\frac{b}{2km}} \Rightarrow \boxed{b=0} \Rightarrow$



no frena ⇒ oscila

- * $w_n = 0$, $w_n = \sqrt{\frac{K}{m}} \Rightarrow \boxed{K=0}$



no oscila, sólo frena

Análisis en el tiempo

$$H(s) = \frac{w_n^2}{(s-c_1)\cdot(s-c_2)} \qquad c_i = -\xi w_n \pm \sqrt{\xi^2-1}$$

obs: * dos casos: → $\xi = 1$, raíz doble
→ $\xi \neq 1$, raíces distintas.

$\xi \neq 1$ $\quad c_1 \neq c_2$ (reales o complejos)

$$H(s) = \frac{M}{\underbrace{s}_{\text{coefs iguales}}-c_1} - \frac{M}{s-c_2} \qquad \boxed{M = \frac{w_n}{2\sqrt{\xi^2-1}}}$$

$$\boxed{h(t) = M \cdot u(t) \cdot \left(e^{c_1 t} - e^{c_2 t}\right)}$$

$\xi = 1$ $\quad \boxed{c_1 = c_2 = -w_n}$ $\qquad H(s) = \dfrac{w_n^2}{(s+w_n)^2}$

Cuadrado de $\frac{1}{s}$ es su derivada

$$\boxed{h(t) = w_n^2 \cdot t \cdot e^{-w_n t} \cdot u(t)}$$

tabla 4.2

caso $\varepsilon \neq 1$:

a) $0 < \varepsilon < 1 \rightarrow$ subamortiguado

$\varepsilon \in \mathbb{C}$

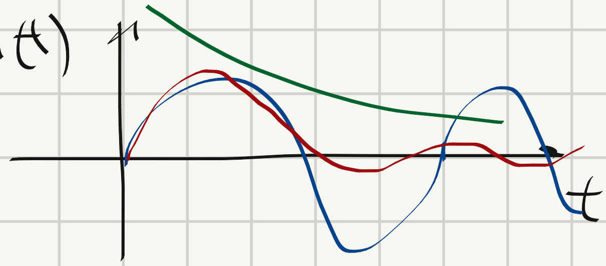
$$h(t) = \frac{w_n}{2\sqrt{\varepsilon^2-1}} \cdot e^{-\varepsilon w_n t} \cdot \left(e^{\sqrt{\varepsilon^2-1}\, w_n t} - e^{-\sqrt{\varepsilon^2-1}\, w_n t} \right)$$

<0

$$h(t) = \frac{w_n}{2j\sqrt{1-\varepsilon^2}} \cdot e^{-\varepsilon w_n t} \cdot \left(e^{j\sqrt{1-\varepsilon^2}\, w_n t} - e^{-j\sqrt{1-\varepsilon^2}\, w_n t} \right)$$

seno

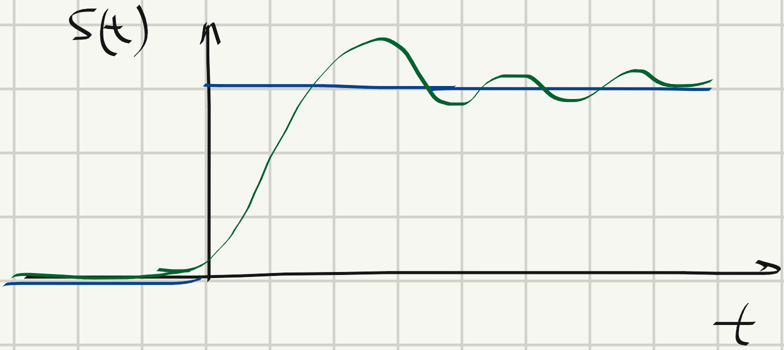
$$\boxed{h(t) = \frac{w_n}{\sqrt{1-\varepsilon^2}} \cdot \underbrace{e^{-\varepsilon w_n t}}_{\text{decaimiento}} \cdot \underbrace{\operatorname{sen}\left(\sqrt{1-\varepsilon^2}\, w_n t\right)}_{\text{oscilatorio}}}$$



obs: * raíces complejas conjugadas
* comportamiento oscilatorio
* "no tiene lo suficiente"

$$\boxed{s(t) = 1 - \frac{e^{-\varepsilon w_n t}}{\sqrt{1-\varepsilon^2}} \cdot \operatorname{sen}\left(\sqrt{1-\varepsilon^2}\, w_n t + a\cos(\varepsilon)\right)}$$

$$\boxed{w_{eff} = \sqrt{1-\varepsilon^2} \cdot w_n}$$



Caso $\xi \neq 1$:

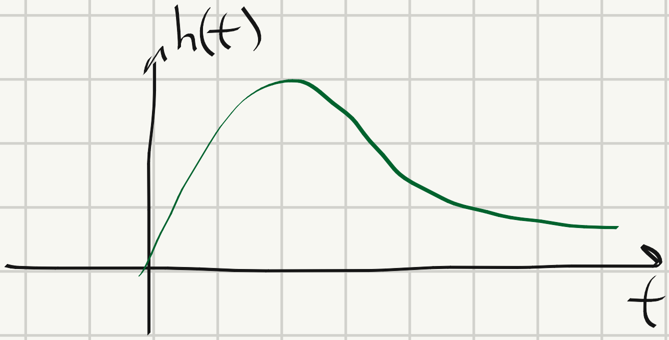
b) $\xi > 1 \rightarrow$ sobre amortiguado

$$\boxed{h(t) = \frac{\omega_n}{2\sqrt{\xi^2-1}} \cdot \underbrace{e^{-\xi\omega_n t}} \left(\underbrace{e^{\overbrace{\sqrt{\xi^2-1}}^{\in \mathbb{R}}\,\omega_n t}}_{} - e^{-\sqrt{\xi^2-1}\,\omega_n t} \right)}$$

este término siempre gana

$\rightarrow \xi > \sqrt{\xi^2-1}$ ✓

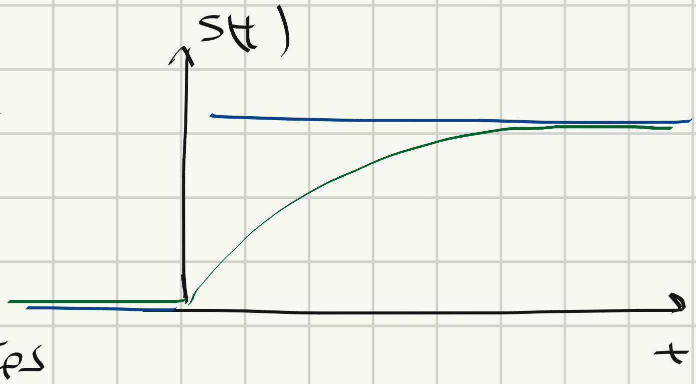
$\xi^2 > \xi^2 - 1$



$$\boxed{s(t) = u(t) \left[1 + \frac{\omega_n}{2\sqrt{\xi^2-1}} \left(\frac{e^{\sqrt{\xi^2-1}\,\omega_n t}}{c_1} - \frac{e^{-\sqrt{\xi^2-1}\,\omega_n t}}{c_2} \right) \right]}$$

$$c_1 = -\xi\omega_n + \sqrt{\xi^2-1}\,\omega_n$$

$$c_2 = -\xi\omega_n - \sqrt{\xi^2-1}\,\omega_n$$

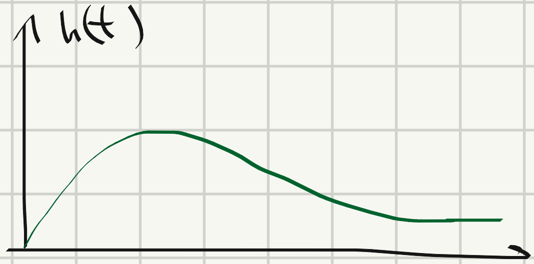


obs:
* raíces reales
* no hay oscilaciones
* exponenciales decrecientes
* "lo frené demasiado"

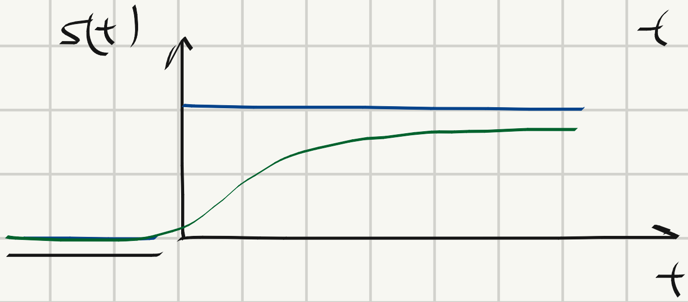
Caso $\xi=1$ amortiguamiento crítico

$$\boxed{h(t) = w_n^2 \, t \, e^{-w_n t} \, u(t)}$$

$$\boxed{s(t) = 1 - e^{-w_n t} - w_n t \, e^{-w_n t}}$$



obs:
- * raíz doble
- * no hay oscilaciones
- * más rápido que el sobreamortiguado



Análisis en frecuencia #2

Bode:

$$H(j\omega) = \frac{1}{\left(j\frac{\omega}{\omega_n}\right)^2 + 2\varepsilon j\frac{\omega}{\omega_n} + 1}$$

1) Módulo

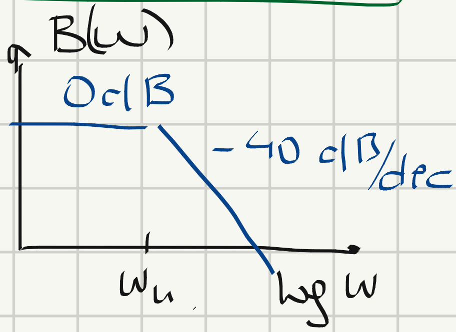
$$|H(j\omega)|^2 = \frac{1}{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + 4\varepsilon^2\left(\frac{\omega}{\omega_n}\right)^2}$$

$$B(\omega) = -10 \log_{10}\left[\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + 4\varepsilon^2\left(\frac{\omega}{\omega_n}\right)^2\right]$$

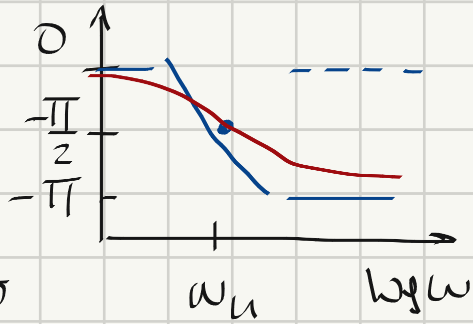
a) $\omega \ll \omega_n$, $B_1(\omega) = -10 \log_{10} 1 = \boxed{0\,dB}$

b) $\omega \gg \omega_n$, $B_2(\omega) = -10 \log_{10}\left[\left(\frac{\omega}{\omega_n}\right)^4 + \cancel{4\varepsilon^2\left(\frac{\omega}{\omega_n}\right)^2}\right]$

$$B_2(\omega) = -40 \log_{10}\left(\frac{\omega}{\omega_n}\right) \Rightarrow \boxed{B_2(\omega) = \underbrace{-40 \log \omega}_{A} + \underbrace{40 \log \omega_n}_{B}}$$



2) fase: $\phi(\omega) = -\operatorname{atan}\left(\dfrac{2\xi\omega/\omega_n}{1-\dfrac{\omega^2}{\omega_n^2}}\right)$



$$\phi(\omega) = \begin{cases} a)\ 0 & \omega < 0.1\,\omega_n \\ b)\ -\dfrac{\pi}{2}\left[\log\left(\dfrac{\omega}{\omega_c}\right)+1\right] & 0.1 < \dfrac{\omega}{\omega_n} < 10 \\ c)\ -\pi & \omega > 10\,\omega_n \end{cases}$$

b) $\phi(\omega) = A\log_{10}\left(\dfrac{\omega}{\omega_n}\right) + B = A\,\underbrace{\cancel{\log_{10}(1)}}_{0} + B$

$\phi(\omega_n) = \boxed{B = -\dfrac{\pi}{2}}$

$\phi(10\,\omega_n) = A\,\underbrace{\cancel{\log_{10}(10)}}_{1} + B = -\pi \Rightarrow \boxed{A = -\dfrac{\pi}{2}}$

$$\boxed{\phi(\omega) = -\dfrac{\pi}{2}\left[\log_{10}\left(\dfrac{\omega}{\omega_n}\right)+1\right]}$$

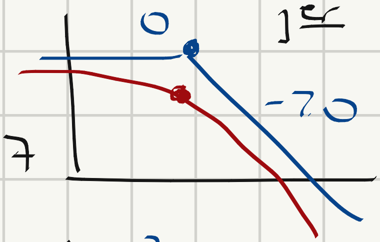
c) $\phi(\omega) \simeq -\operatorname{atan}\left(\dfrac{2\xi\omega/\omega_n}{\dfrac{\omega^2}{\omega_n^2}}\right) = -\operatorname{atan}\left(2\xi\underbrace{\dfrac{\omega_n}{\omega}}_{0^+}\right)$ $\boxed{\neq -\pi}$

Clase 32

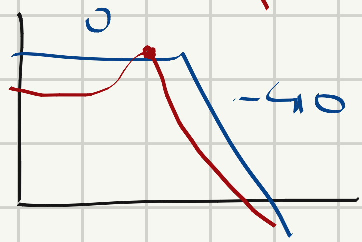
Análisis de sistemas

En tiempo y frecuencia

Respuesta en frecuencia real



si derivo $|H(j\omega)|^2$, para $\varepsilon < \frac{\sqrt{2}}{2} \simeq 0.7$



tiene máximo!!

$$\boxed{\omega_{max} = \overbrace{\sqrt{\underbrace{1-2\varepsilon^2}_{>0}}}^{<1}\,\omega_n}$$

$$|H(j\omega_{max})| = \frac{1}{2\varepsilon\sqrt{1-\varepsilon^2}}$$

si $\varepsilon = 0.5 \rightarrow \omega_{máx} = 0.7\,\omega_n \Rightarrow |H(j\omega_{max})|^2 = 2.3$

obs: * cambio cualitativo

* pico: → amplifique una frecuencia o banda
  → calidad: $\boxed{Q = \frac{1}{2\varepsilon}}$

obs: * $\varepsilon < 0$: sist. inestable

Bode para respuestas racionales

obs: * sabemos analizar polos de 1er y 2do orden

* los ceros se deducen fácilmente.

1er y 2do orden

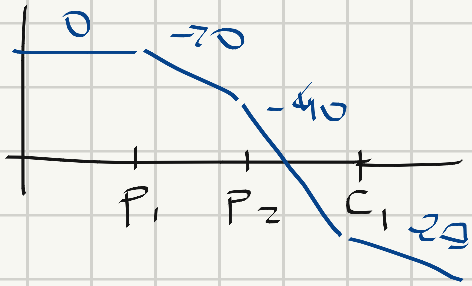
$$\left.\begin{aligned} H_1^c(j\omega) &= 1 + j\omega\tau \\ H_2^c(j\omega) &= 1 + 2\varepsilon j\frac{\omega}{\omega_n} + \left(\frac{\omega}{\omega_n}\right)^2 \end{aligned}\right\} \Rightarrow$$

son los inversos de $H_1(j\omega)$ y $H_2(j\omega)$

* módulo:

$$20\log_{10}|H_1^c(j\omega)| = 20\log_{10}\frac{1}{|H_1(j\omega)|} = -20\log_{10}|H_1(j\omega)|$$

* fase: $\angle H_1^c(j\omega) = -\angle H_1(j\omega)$



caso constante

$$H(j\omega) = K \Rightarrow 20\log_{10}|K| = cte$$

0 dB/dec

$$\angle H(j\omega) = n\pi$$

polo orden alto

* $\dfrac{1}{(s+a)^2} = -\dfrac{d}{ds}\left(\dfrac{1}{s+a}\right)$ → derivadas en frecuencia

↓

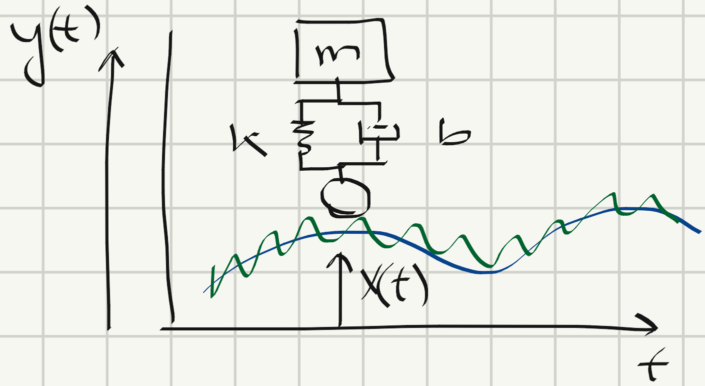
factor t^n en el tiempo.

* $\dfrac{1}{(s+a)^n} = -\dfrac{d^{(n)}}{ds}\left(\dfrac{1}{s+a}\right)$ →

* caída frecuencia $-20n$ dB/dec

Ejemplo de análisis en el dominio del tiempo

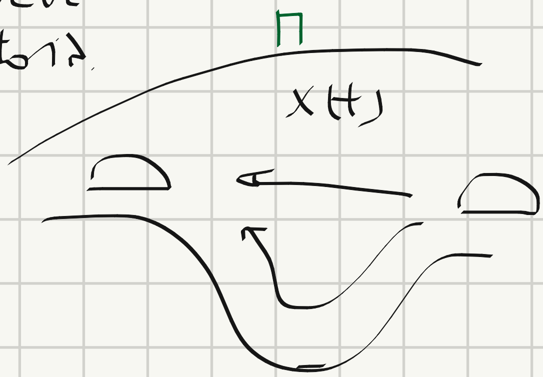
suspensión auto



* X(t) posición del piso
  - topografía
  - rugosidad
* y(t) posición del auto

requerimientos

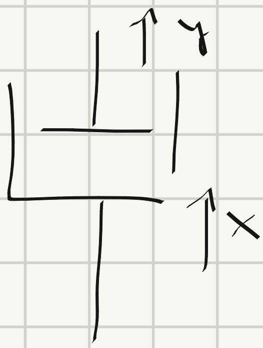
* objetivo: filtrar la rugosidad del suelo
  - transición suave
* no queremos respuesta oscilatoria



* transición suave entre las frecuencias (sin corte abrupto)
* sistema usado en la práctica: bajo costo
  - prueba: apretar y soltar

Modelado

$$m y''(t) = -k\,\overbrace{(y(t)-x(t))}^{\text{importa la diferencia}} - b\,\overbrace{(y'(t)-x'(t))}^{\text{velocidad relativa}}$$



$$\underbrace{m y''(t) + b y'(t) + k y(t)}_{\text{CI salidas}} = \underbrace{b\,x'(t) + k\,x(t)}_{\text{entradas}}$$

$$H(jw) = \frac{bjw + k}{m(jw)^2 + bjw + k}$$

cero

$$\boxed{w_n = \sqrt{\frac{k}{m}}}$$

$$\boxed{H(jw) = \frac{1 + 2\varepsilon j \frac{w}{w_1}}{\left(j\frac{w}{w_n}\right)^2 + 2\varepsilon j \frac{w}{w_n} + 1}}$$

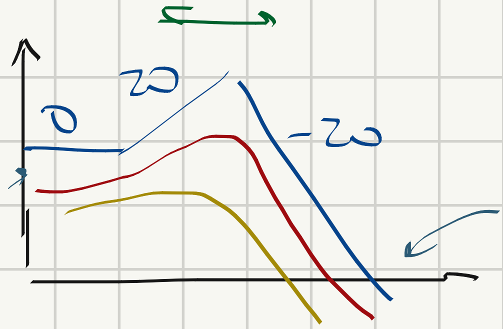
* ceros: $2\varepsilon\, s + w_n = 0 \Rightarrow$ $\boxed{c = -\frac{w_n}{2\varepsilon}}$

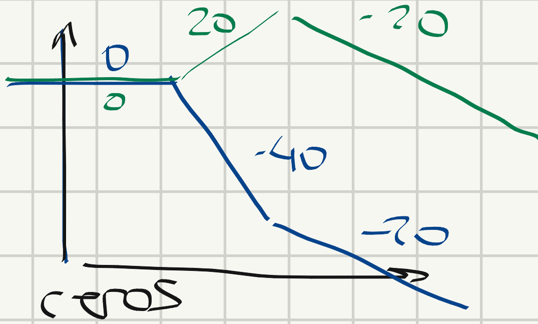
* polos: $\rightarrow p_1 = -\varepsilon w_n + \sqrt{\varepsilon^2 - 1}\; w_n$

  $\rightarrow p_2 = -\varepsilon w_n - \sqrt{\varepsilon^2 - 1}\; w_n$

$\varepsilon = 0.5 \Rightarrow c = -w_n$

$p_1 = -0.5 w_n \pm j\sqrt{0.75}\; w_n$





obs:

* asintótico no funciona (están cerca)
* depende del orden de p'-. y ceros
* asintótico funciona a los ex. os

RESUMEN

* Análisis en frecuencia: - módulo
  - fase: - retardo grupo
    - lineal
* Análisis temporal: - transitorios
  - tiempos
* Especifica en tiempo y frecuencia: compromiso
* probamos las herramientas en sistemas de segundo orden.
* Factores: → restricciones T y F
  → causalidad
  → estabilidad
  → construcción (real)

Clase 33

Transformada de Laplace

Propiedades

PROPIEDADES (tabla 9.1)

Linealidad

$$a\,x(t)+b\,y(t) \xleftrightarrow{\mathcal{L}} a\,X(s)+b\,Y(s), \quad ROC = R_1 \cap R_2$$

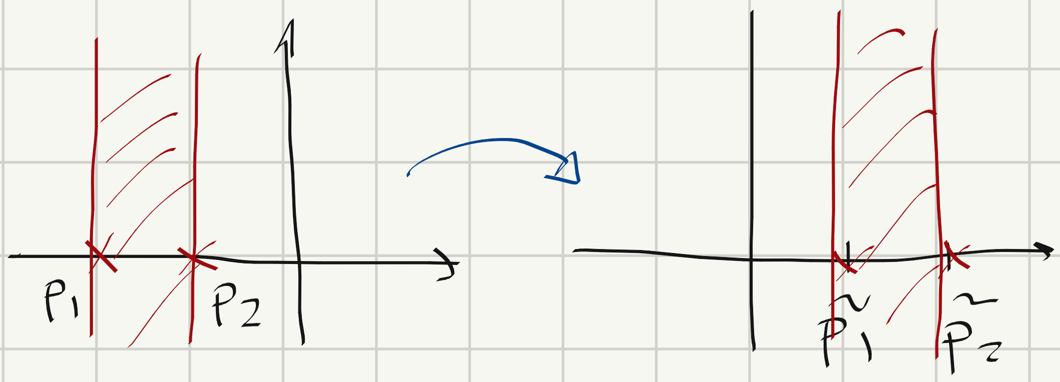
R_1 R_2

Desplazamiento en s

$$X(s) = \frac{N(s)}{D(s)} \quad D(s)=0$$

$$e^{s_0 t}\,x(t) \xleftrightarrow{\mathcal{L}} X(s-s_0), \quad ROC = \{ s + Re\{s_0\} \,/\, s \in R \}$$

$$= \text{"} R + Re\{s_0\} \text{"}$$



ej: $e^{-\sigma_0 t}\, e^{-at}\, e^{-\sigma t}$

$e^{-t}\, e^{-\sigma t}$ $\quad \sigma > -1$

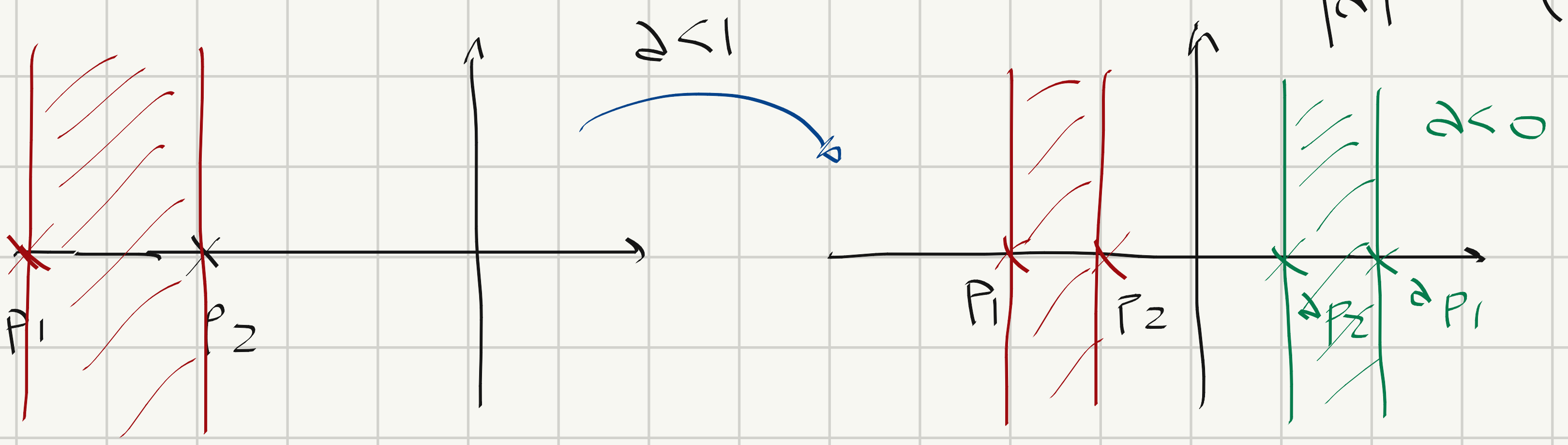
$e^{-3t}\, e^{-t}\, e^{-\sigma t}$ $\quad \boxed{\sigma > -4}$

$D(p_1) = 0 \quad \tilde{D}(p_1 + s_0) = 0$

$\tilde{p}_1 = p_1 + Re\{s_0\}$

Escalamiento temporal

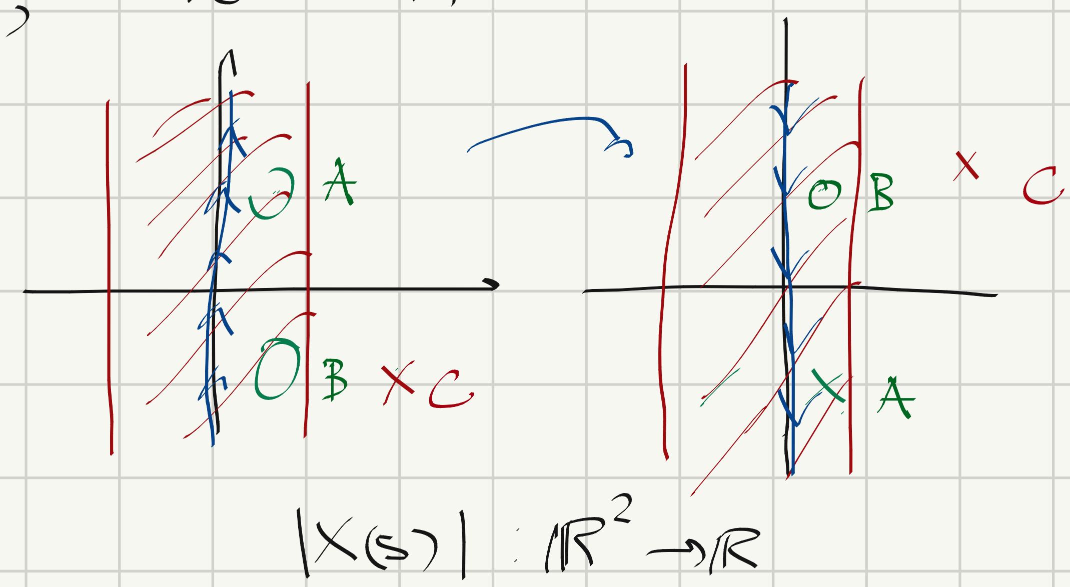
$$x(at) \xleftrightarrow{\mathcal{L}} \frac{1}{|a|}\, X\left(\frac{s}{a}\right)$$



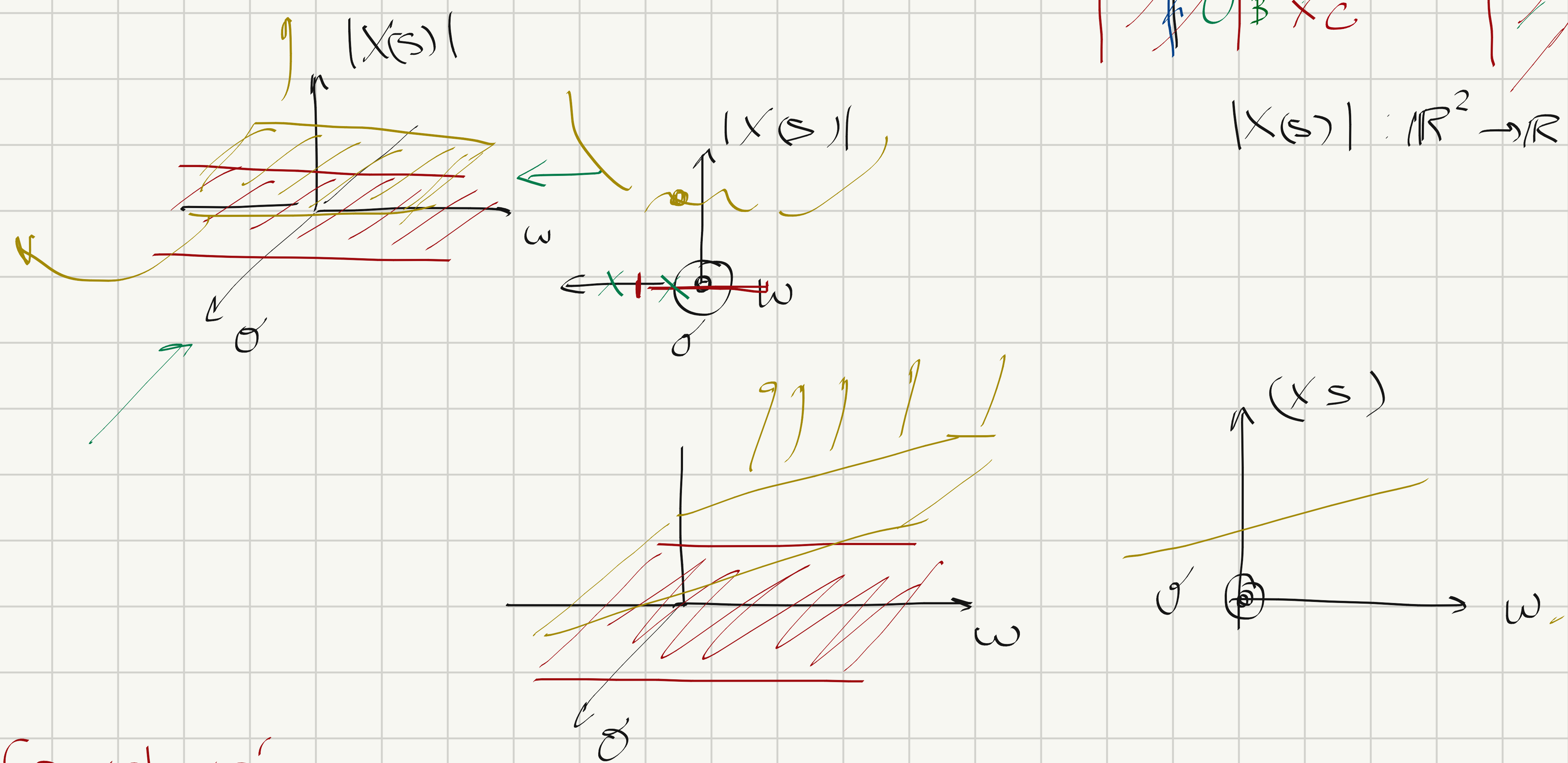
ej: $e^{-t}\, e^{-\sigma t}$, $\exists$ si $\sigma > -1$

$e^{-2t}\, e^{-\sigma t}$, $\exists$ si $\boxed{\sigma > -2}$

Conjugación $x^*(t) \overset{\mathcal{L}}{\longleftrightarrow} X^*(s^*)$, ROC = R

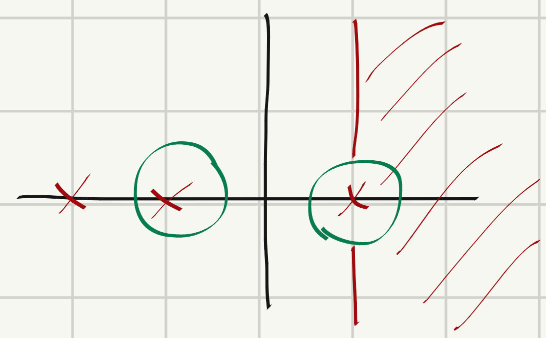


* $x(t) \in \mathbb{R} \Rightarrow X^*(s^*) = X(s)$



Convolución

$$x_1(t) * x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s) \cdot X_2(s)$$



$\boxed{\text{ROC} \supset (R_1 \cap R_2)}$

Obs: * no se agregan polos
* pueden eliminarse polos
* ROC es igual o mayor.

Diferenciación en t

$$x'(t) \overset{\mathcal{L}}{\longleftrightarrow} s \cdot X(s) \quad , \text{ROC} \supset R$$

* puedo eliminar un polo en s=0

Diferenciación en s

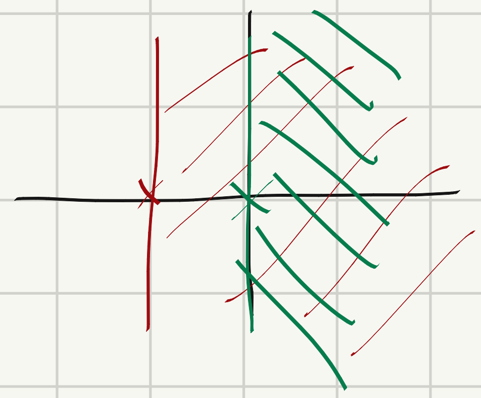
$$-t \cdot x(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{d\,X(s)}{ds} \quad , \text{ROC} = R \qquad \frac{1}{(s+a)} \rightarrow \frac{1}{(s+a)^2}$$

* a lo sumo aumenta la multiplicidad

Integral en el tiempo

$$\int_{-\infty}^{t} x(\tau)\cdot d\tau \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s}\cdot X(s) \,, \quad ROC \supset \{ R \cap \Pi_0^+ \}$$

polo en cero



$$\Pi_\lambda^+ = \{ s \in \mathbb{C} \,/\, Re(s) > \lambda \}$$

Teorema del valor inicial y final

(H) $x(t) = 0 \,, \; \forall t < 0$ y $x(t)$ no tiene singularidades en cero.

$x(t)$ es derecha

(T) * $x(0^+) = \lim_{s \to \infty} s \cdot X(s)$

* $\lim_{t \to \infty} x(t) = \lim_{s \to 0} s \cdot X(s)$

Ejemplo 9.16 (9.4) $\quad x(t) = u(t)\cdot e^{-2t} + u(t)\cdot e^{-t}\cdot \cos(3t)$

$$X(s) = \frac{2s^2 + 5s + 12}{(s^2 + 2s + 10)(s+2)}$$

* evalúo en t: $\boxed{x(0^+) = 2}$

* $\lim_{s \to \infty} s\cdot X(s) = \lim_{s \to \infty} \frac{s\,(2s^2 + 5s + 12)}{(s^2 + 2s + 10)(s+2)} = \frac{2\cancel{s^3}}{\cancel{s^3}} = \boxed{2}$

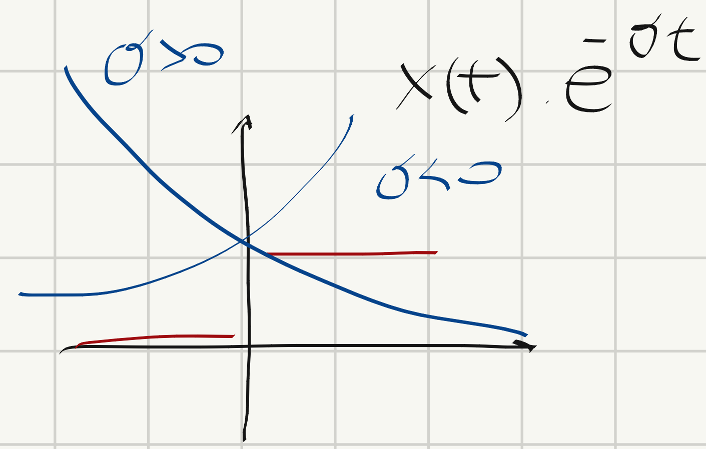
PROPIEDADES (APLICADAS A SISTEMAS)

$$Y(s) = H(s) \cdot X(s)$$

función de tranferencia

$$s = j\omega \rightarrow Y(j\omega) = H(j\omega) \cdot X(j\omega)$$

resp. en frecuencia



Prop. 4.

Causalidad: $h(t) = 0, \ \forall t < 0$ (derecha)

$$H(s) = \int_{-\infty}^{\infty} h(t) \cdot e^{-st} \cdot dt = \int_{0}^{\infty} h(t) \cdot e^{-st} \cdot dt$$

1) H sistema causal T ROC es un Π^+

obs: al revés no pasa.

2) H ROC es un Π^+ T sistema causal
X(s) racional

Ejemplo 9.17

$h(t) = u(t)\, e^{-t}$ → derecha → causal ✓

$H(s) = \dfrac{1}{s+1}$, $\text{Re}\{s\} > -1$ → causal ✓

Ejemplo 9.18

$u(t) = e^{-|t|}$, bilateral, $\underbrace{h(t) \neq 0, \forall t < 0}$ → no causal ✗

$$H(s) = -\frac{2}{(s+1)(s-1)} = \frac{1}{s+1} - \frac{1}{s-1}, \quad \underbrace{-1 < \text{Re}(s) < 1}$$

→ no causal ✗

Estabilidad

$$\mathcal{F}\{x(t)\, e^{-\sigma t}\} = \mathcal{L}\{x(t)\}$$

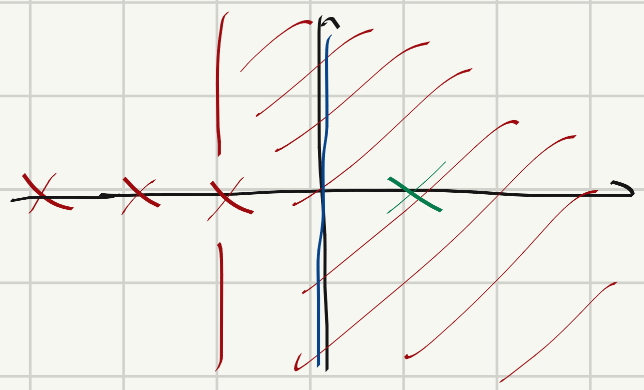
$\mathcal{S}$ es estable $\iff$ $H(s)$ tiene una ROC $\supset L_0$

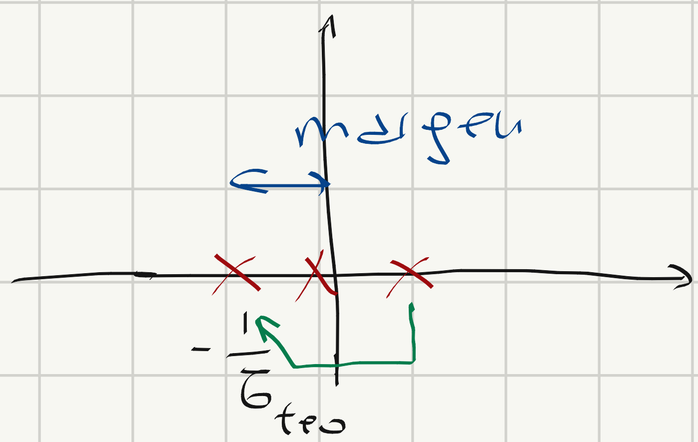
$$L_0 = \{ s \in \mathbb{C} \,/\, \text{Re}(s) = 0 \}$$

Lema:

Un sistema causal es estable $\iff$ sus polos $p_i \in \Pi_0^-$

$\tau = R.C$



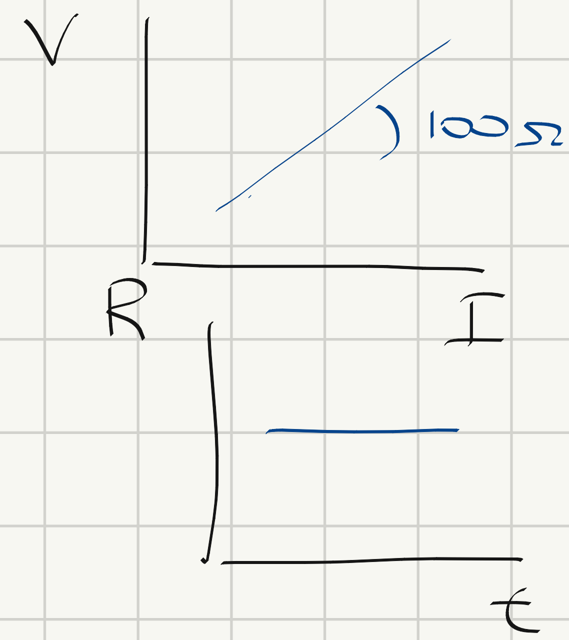


obs:

* manera concreta de estudiar la estabilidad

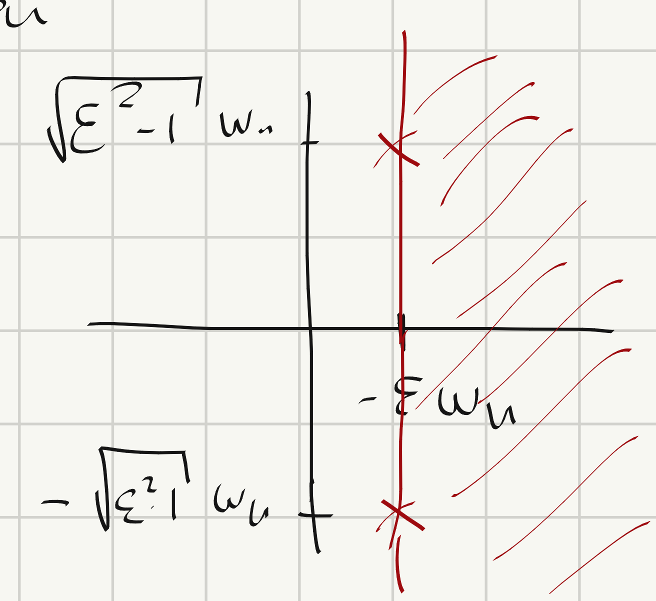
* que tan estable es?
  ↳ margen de estabilidad

* Diseño de sistemas
  ↳ filtros: PDS
  ↳ modificación de sistemas: control (MyS)



Ejemplo 9.22: "Sist. de segundo orden"

$$p_i = -\varepsilon w_n \pm \sqrt{\varepsilon^2 - 1}\; w_n$$



si $\varepsilon < 0 \Rightarrow Re\{p_i\} > 0$

$\Rightarrow$ sist. inestable

Obs: $\varepsilon = \sqrt{\dfrac{b}{2km}}$

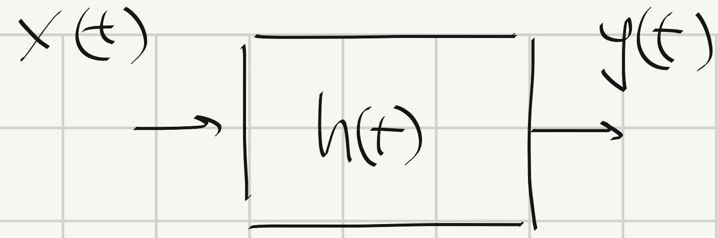
Sistemas basados en ecuaciones diferenciales

$$\underbrace{\sum_{n=0}^{N_p} a_n \cdot y^{(n)}(t)}_{\text{salida}} = \underbrace{\sum_{k=0}^{N_z} b_k \cdot x^{(k)}(t)}_{\text{entradas}} \Rightarrow H(s) = \frac{\overbrace{\sum_{k=0}^{N_z} b_k \cdot s^k}^{\text{ceros}}}{\underbrace{\sum_{n=0}^{N_p} a_n \cdot s^n}_{\text{polos}}}$$

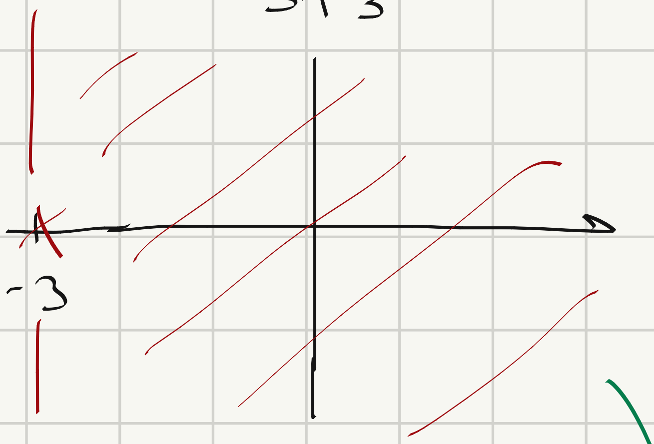
función racional

Ejemplo 9.25

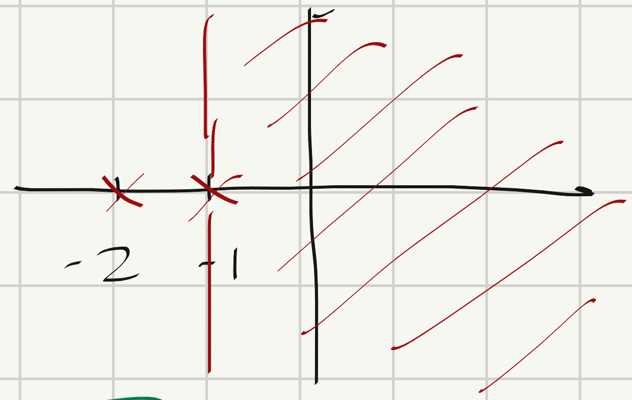
$$\begin{cases} x(t) = u(t)\, e^{-3t} \\ y(t) = u(t)\left(e^{-t} - e^{-2t}\right) \end{cases}$$

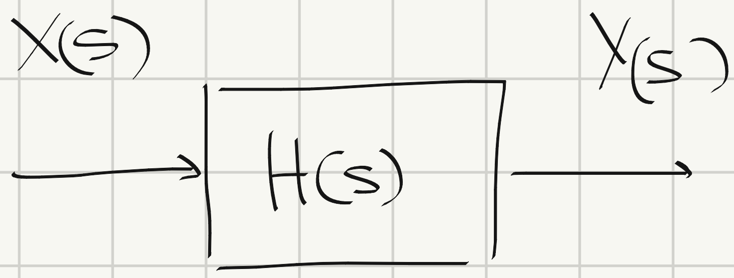


$$X(s) = \frac{1}{s+3}, \quad Re(s) > -3$$

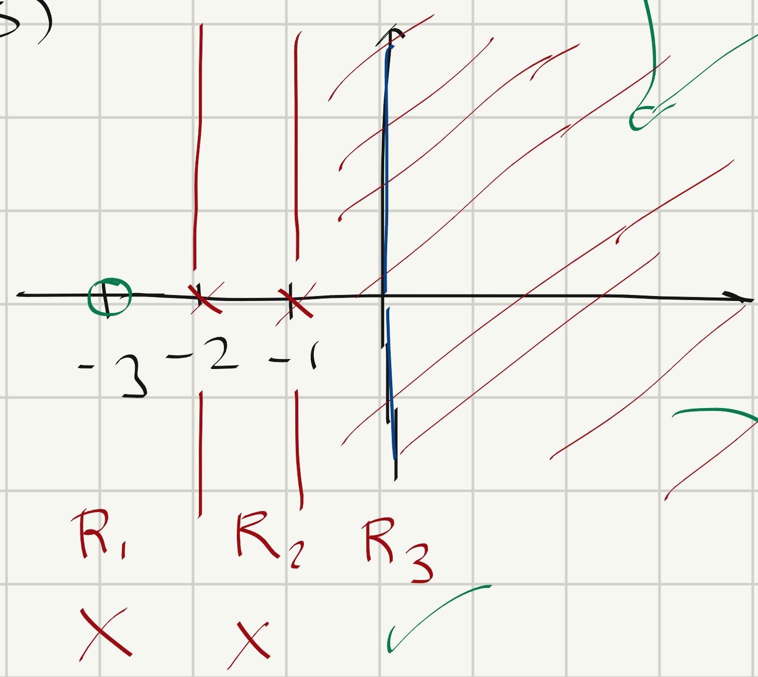


$$Y(s) = \frac{1}{s+1} - \frac{1}{s+2} = \frac{1}{(s+1)(s+2)}, \quad Re(s) > -1$$





H(s)



prop. convolución

$Y(s) = H(s)\, X(s)$

$\text{ROC} \supset \{R_1 \cap R_2\}$

$$\boxed{H(s) = \frac{Y(s)}{X(s)} = \frac{s+3}{(s+1)(s+2)}}$$

$$\boxed{Re(s) > -1}$$

obs: * identificación de sistemas

* $\text{ROC} \supset \Pi^+ \Rightarrow$ causal

* $Re\{p_i\} < 0, \forall i \Rightarrow$ estable

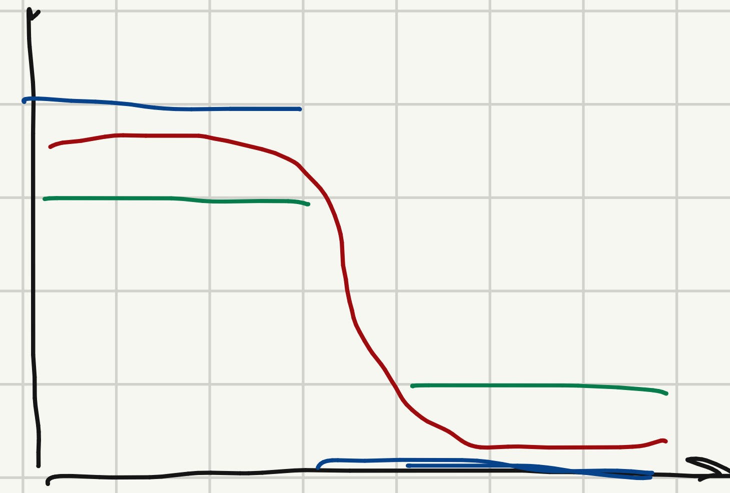
Clase 34

Transformada de Laplace

Aplicación: diseño de filtros

DISEÑO DE SISTEMAS (FILTROS)

Filtro Butterworth (pasabajos)
(p 703)



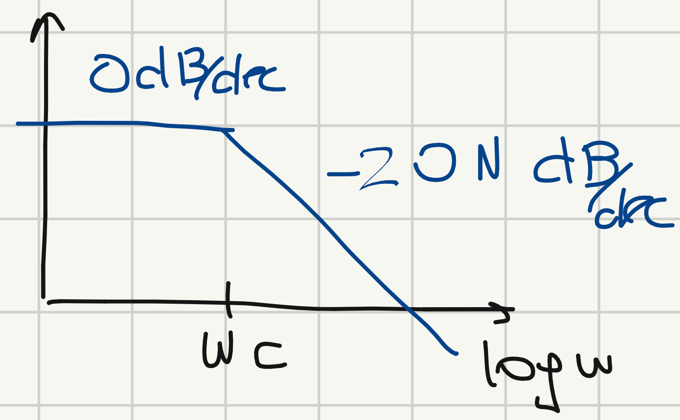
$$|B(j\omega)|^2 = \frac{1}{1+\left(\frac{j\omega}{j\omega_c}\right)^{2N}}$$

Bode

$$10 \log |B(j\omega)|^2 = -20 \log \left[1+\left(\frac{j\omega}{j\omega_c}\right)^{2N}\right]$$

a) $\omega \gg \omega_c \Rightarrow -20 \log \left(\frac{\omega}{\omega_c}\right)^{2N} = \boxed{-20N \log\left(\frac{\omega}{\omega_c}\right)}$

b) $\omega \ll \omega_c \Rightarrow -10 \log(1) = \boxed{0}$



Transferencia

* Hay infinitas $B(j\omega)$ que dan ese módulo

→ $|B(j\omega)|^2 = B(j\omega)\, B^*(j\omega)$

→ Asumo que $b(t) \in \mathbb{R} \Rightarrow B^*(j\omega) = B(-j\omega)$

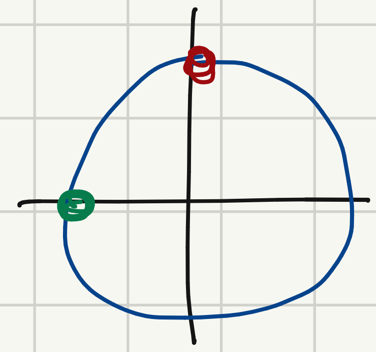
$$B(j\omega)\ B(-j\omega) = \frac{1}{1+\left(\frac{j\omega}{j\omega_c}\right)^{2N}}$$ y sabemos que $s = j\omega$

$$B(s)\ B(-s) = \frac{1}{1+\left(\frac{s}{j\omega_c}\right)^{2N}} = P(s)$$ → función racional con 2N polos

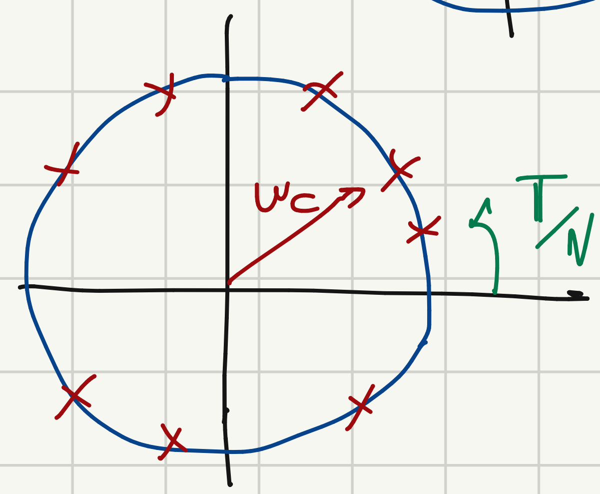
Polos P(s)

$$1+\left(\frac{s}{jw_c}\right)^{2N}=0 \Rightarrow \frac{s}{jw_c}=\sqrt[2N]{-1} \Rightarrow s_p=\sqrt[2N]{-1}\; jw_c \qquad , \; jw_c = 1\, e^{j\frac{\pi}{2}}$$

$$s_p=\sqrt[2N]{1\, e^{j(\pi+2k\pi)}}\; e^{j\frac{\pi}{2}}\, w_c$$



$$\boxed{s_p = w_c\, e^{j\left(\frac{\pi}{2}+\frac{\pi}{2N}+k\frac{\pi}{N}\right)}}$$

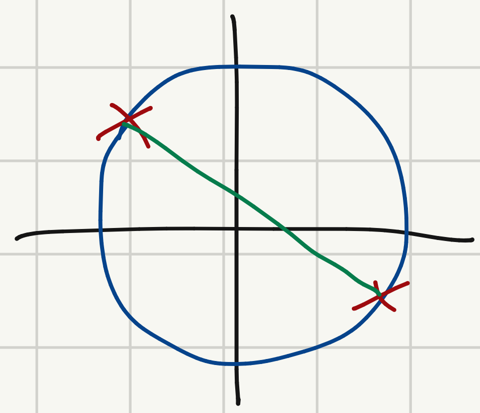


obs

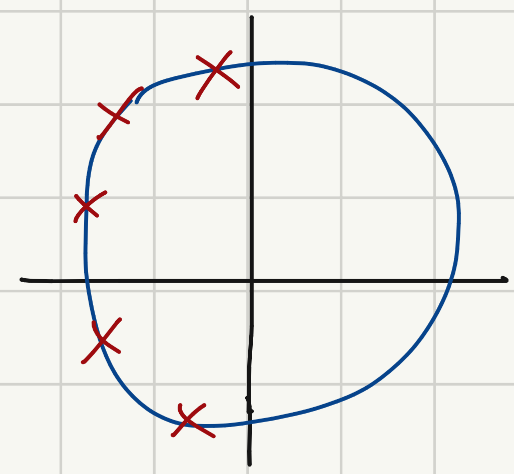
* hay 2N polos equiespaciados
* módulo w_c y ángulo $\frac{\pi}{N}$
* nunca cae en $j\omega$ y en σ para N impar

Polos de B(s)

* Si s_p es polo de B(s) $\Rightarrow$ $-s_p$ es polo de B(−s)
* tengo que elegir N polos
* si quiero que sea causal y estable
  ↳ elijo $s_p \,/\, Re(s_p)<0$



* con los polos puedo armar el B(s)



Transferencia (exp. algebraica)

$N=4$

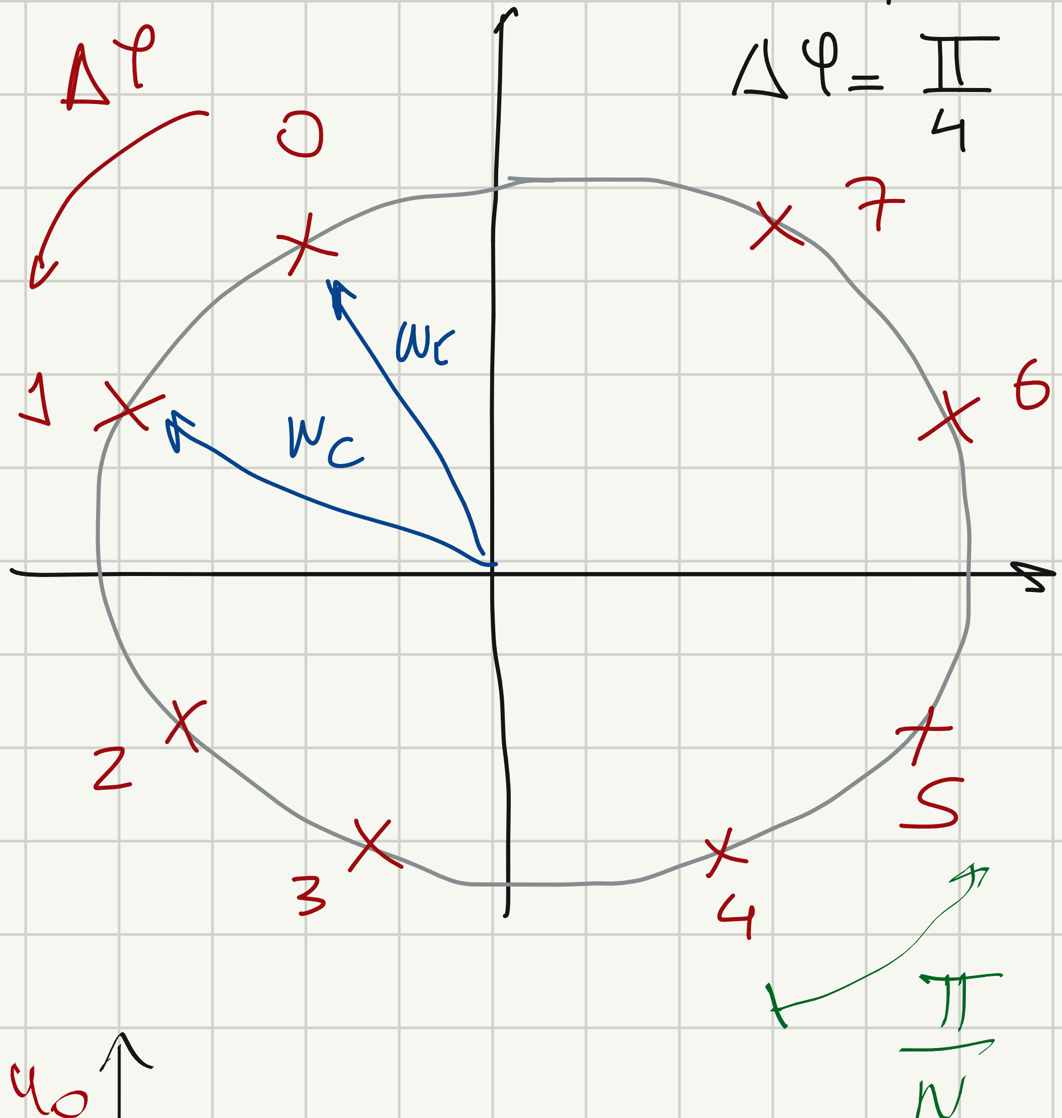
$\Delta\varphi = \frac{\pi}{4}$

* polos $\rightarrow \varphi_0 = \frac{\pi}{2} + \frac{\pi}{2N}$, $\Delta\varphi = \frac{\pi}{N}$

$\rightarrow \varphi_k = \varphi_0 + k\,\Delta\varphi$

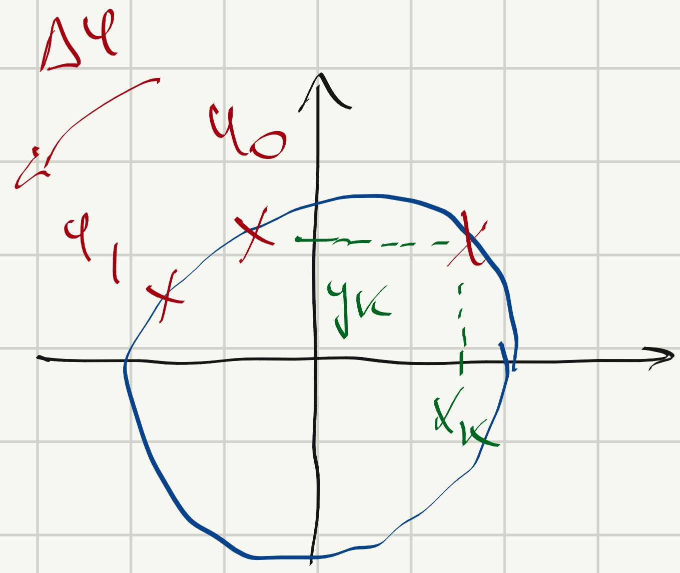
$\rightarrow \boxed{s_p = \Omega_c\, e^{j\varphi_k}}$

$\boxed{k = 0 \quad 3}$



* $s_p^k = x_k + j\, y_k$

$$\boxed{\begin{aligned} x_k &= \Omega_c \cos\varphi_k \\ y_k &= \Omega_c \operatorname{sen}\varphi_k \end{aligned}}$$



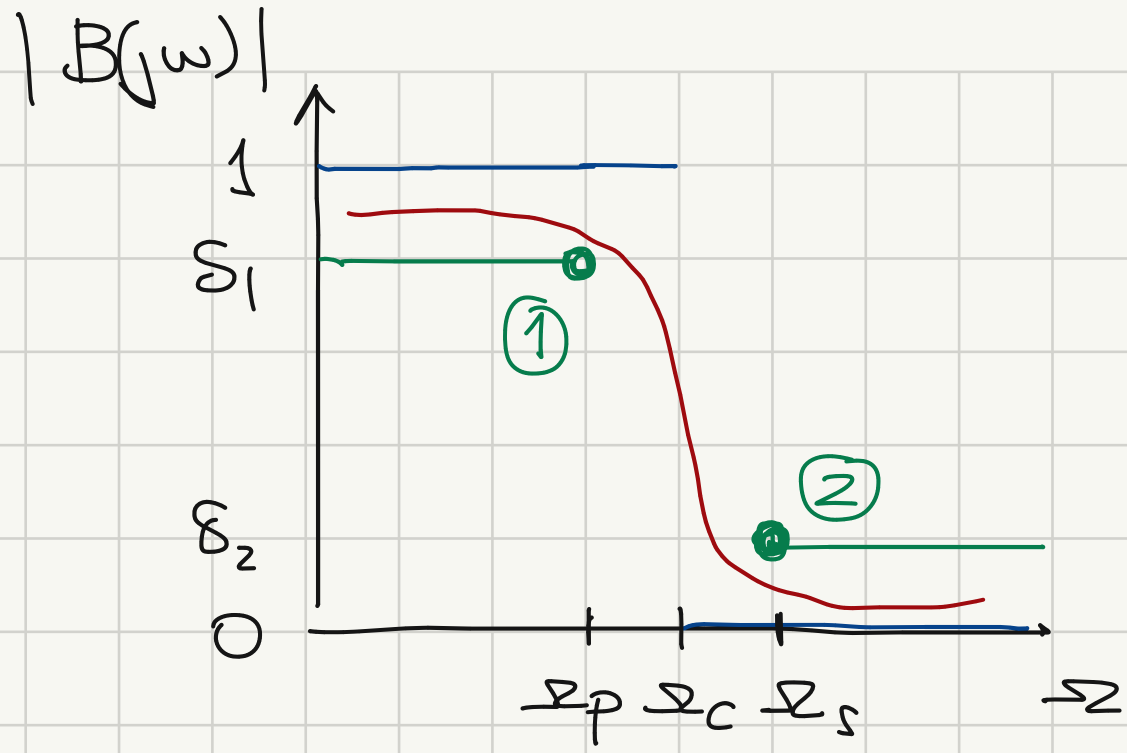
*

$$\boxed{B(s) = \frac{1}{\prod_{k=0}^{N-1} (s - s_p^k)}}$$

Ejemplo

$\delta_1 = 0.8$, $\delta_2 = 0.15$

$f_p = 10\,Hz$, $f_s = 30\,Hz$

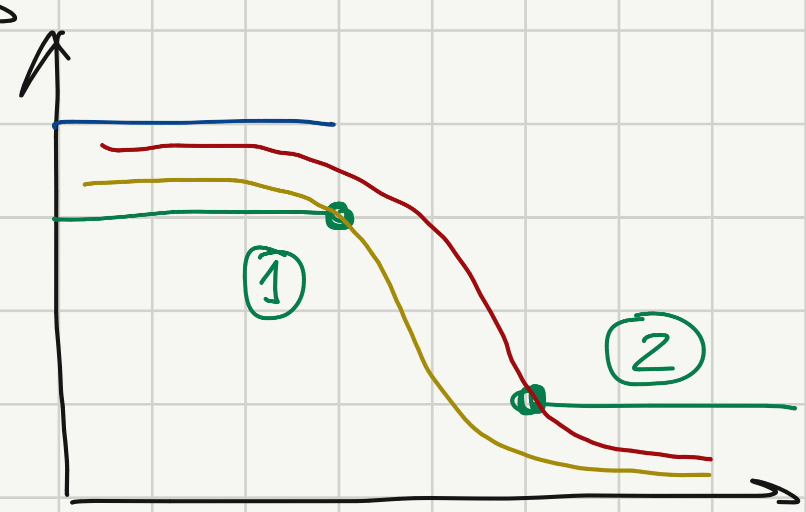


$\Omega_p = 2\pi\, 10$ rad/s , $\Omega_s = 2\pi\, 30$ rad/s

(1) $\quad |B(j\Omega_p)|^2 \geq \delta_1^2$

(2) $\quad |B(j\Omega_s)|^2 \leq \delta_2^2$

obs
* incógnitas N, Ω_c
* sistema de 2x2 inecuaciones



pasos
* cálculo N
* cálculo Ω_c

Diseño

* cálculo de N

1 impongo la igualdad $|B(j\Omega p)|^2 = \dfrac{1}{1+\left(\frac{\Omega p}{\Omega c}\right)^{2N}} = \delta_1^2$

$$1+\left(\frac{\Omega p}{\Omega s}\right)^{2N} = \delta_1^{-2} \Rightarrow 2N \log\left(\frac{\Omega p}{\Omega s}\right) = \underbrace{\log\left(\delta_1^{-2}-1\right)}_{cte}$$

cte ⟶ defino $d_1 = \log\left(\delta_1^{-2}-1\right)$

$$2N \log\left(\frac{\Omega p}{\Omega c}\right) = d_1 \quad A$$

2 $$2N \log\left(\frac{\Omega s}{\Omega c}\right) = d_2 \quad B$$

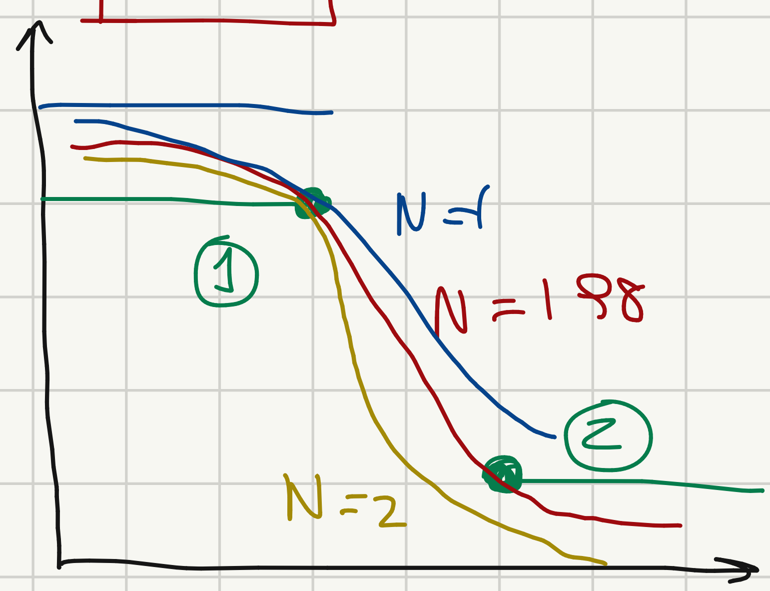
$$A - B = 2N\left(\log \Omega p - \cancel{\log \Omega c} - \log \Omega s + \cancel{\log \Omega c}\right) = d_1 - d_2$$

$$2N \log\left(\frac{\Omega p}{\Omega s}\right) = d_1 - d_2$$

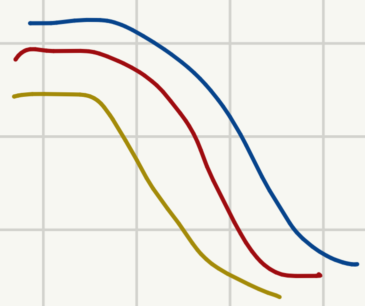
$$\boxed{N = \frac{d_1 - d_2}{2\log\left(\frac{\Omega p}{\Omega s}\right)}} \Rightarrow \boxed{N = 1.98}$$

N = 1

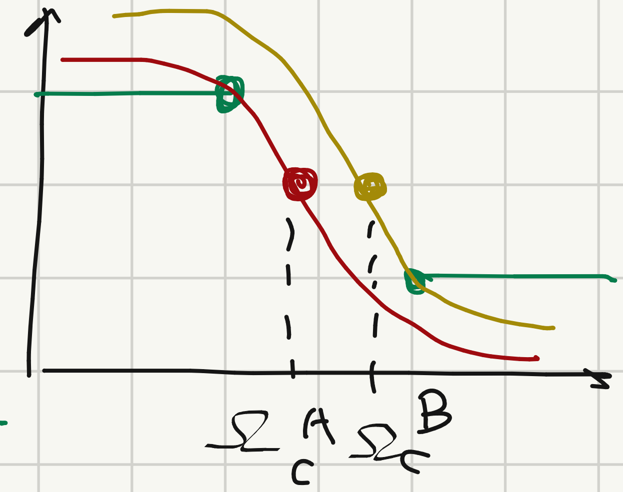
$\boxed{N = 2}$ ✓



Obs $|B(j\omega)|$ decrece con N



* Cálculo de Ω_c (lo hago en Ⓐ)



$$2N \log\left(\frac{\Omega_p}{\Omega_c}\right) = d_1 \Rightarrow \log\left(\frac{\Omega_p}{\Omega_c}\right) = \frac{d_1}{2N}$$

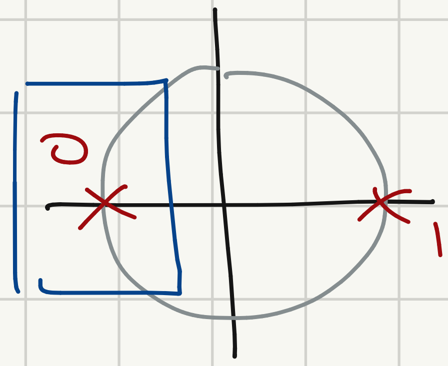
$$\log\left(\frac{\Omega_c}{\Omega_p}\right) = -\frac{d_1}{2N} \Rightarrow \boxed{\Omega_c = \Omega_p \, e^{-\frac{d_1}{2N}}}$$

$$\boxed{\Omega_c = 7255 \text{ rad/s}} \longrightarrow \boxed{f_c = 1155 \text{ Hz}}$$

obs si resuelvo en Ⓑ $\boxed{f_c = 1169}$

Transferencia

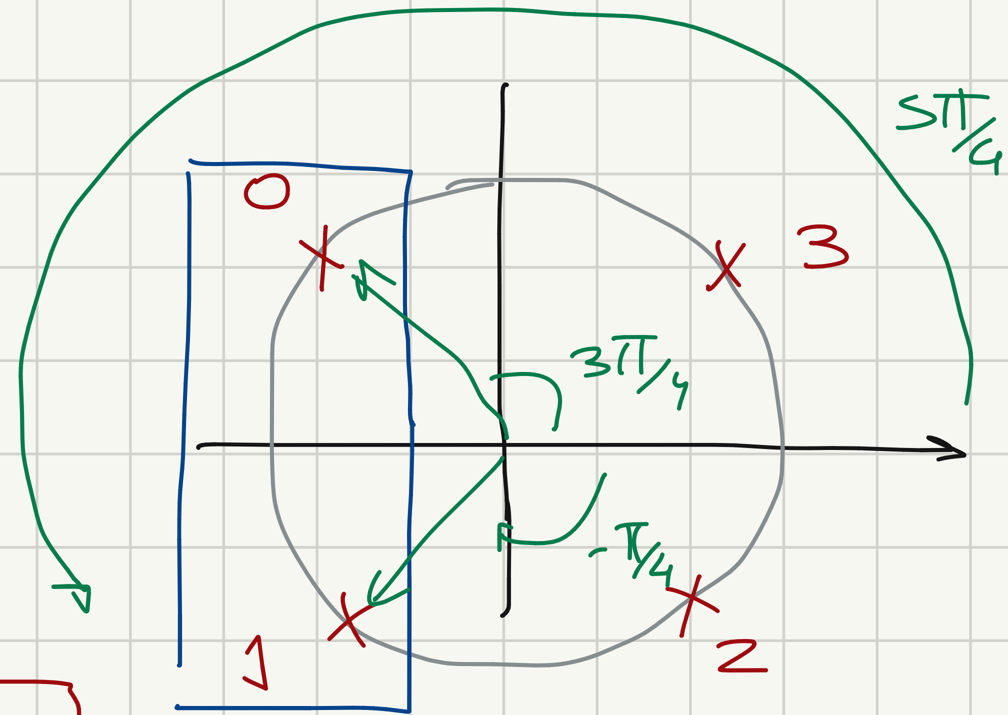
1) $N=1$, $s_p^1 = \Omega_c e^{j\pi} = -\Omega_c$



$s_p^2 = \Omega_c e^{j\pi + \pi}$

$$\boxed{B(S) = \frac{1}{1+\frac{s}{\Omega_c}}} \qquad \frac{1}{1+\tau s} \Rightarrow \boxed{\tau = \frac{1}{\Omega_c}}$$

2) Sistema de segundo orden $N=2$



$N=2$, $\Delta\varphi = \frac{\pi}{2}$, $\varphi_0 = \frac{\pi}{2}+\frac{\pi}{4} = \frac{3}{4}\pi$

$s_p^0 = \Omega_c e^{j\varphi_0} = \Omega_c e^{j\frac{3\pi}{4}}$

$s_p^1 = \Omega_c e^{j(\varphi_0+\frac{\pi}{2})} = \Omega_c e^{j\frac{5\pi}{4}}$

$$\boxed{B(S) = \frac{1}{\left(S+\Omega_c e^{j\frac{3\pi}{4}}\right)\left(S+\Omega_c e^{j\frac{5\pi}{4}}\right)}}$$

$e^{j\frac{3\pi}{4}} + e^{j\frac{5\pi}{4}}$

$e^{j\frac{3\pi}{4}} + e^{-j\frac{3\pi}{4}}$

$$B(S) = \frac{1}{s^2 + 2\Omega_c \underbrace{\cos\left(\frac{3\pi}{4}\right)}_{\varepsilon} s + \Omega_c^2}$$

$\varepsilon = \frac{\sqrt{2}}{2} \boxed{\simeq 0.7} \rightarrow$ subamortiguado

Implementación

Ejemplo

* desarrollo $\overset{-1}{\downarrow}\;\overset{-2}{\downarrow}$ $(s+1)(s+2) = s^2+3s+2$

$$[1\ \ 1]\,[1\ \ 2] = [1\ \ 3\ \ 2]$$

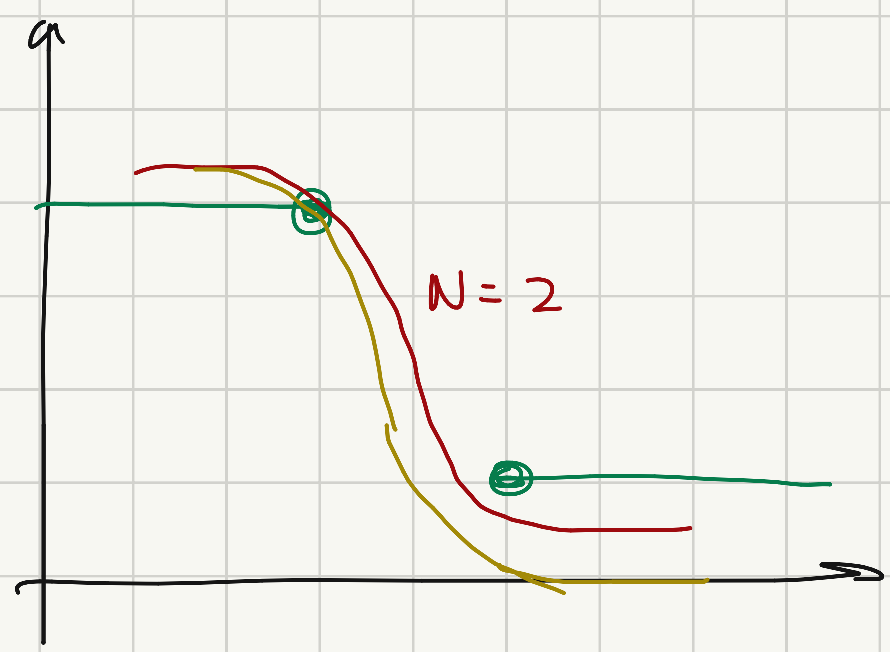
$$[1\ \ 1]\,[1\ \ 2] \xrightarrow{\text{convolve}} [1\ \ 3\ \ 2]$$

* raíces $[1\ 3\ 2] \rightarrow [-1\ ,-2]$

$$[1\ 3\ 2] \xrightarrow{\text{roots}} [-1\ ,-2]$$

Ejemplo

$$\boxed{B(s) = \frac{1}{s^2 + 1026\,s + 52639}}$$



Implementación futura:

* butter ( )

RESUMEN

Laplace

* generalización de $\mathcal{F}$ → sistemas más grales
* completa el análisis → estabilidad y causalidad
  → diseño
* DPC + ROC estudio completo del sistema

RESUMEN

Curso

"Aprender a describir señales y sistemas"

* Definición de señales y sistemas
* Estudio y caracterización de sistemas
* Análisis de SLITs en el tiempo
* Análisis espectral de señales periódicas
  → tiempo continuo y discreto
* Análisis espectral de señales no-periódicas
  → tiempo continuo y discreto
* Muestreo relación entre TC y TD
* Análisis general (Laplace)
  → tiempo continuo
  → sistemas inestables
  → respuestas transitorias
* Análisis combinado tiempo y frecuencia
  → requerimientos de filtros reales
  → diseño de sistemas

RESUMEN

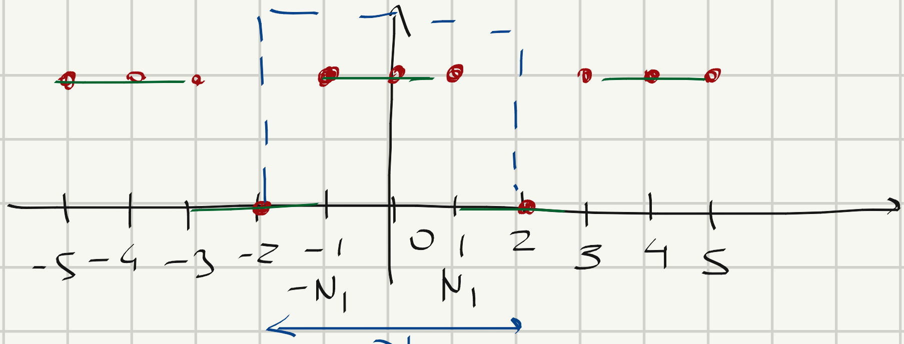
Futuro

* señales no estacionarias — transformadas Tiempo Frecuencia — PDS
* señales no determinísticas — procesos estocásticos — PDS
* sistemas no lineales — aproximación — CONTROL
* Completar el análisis para tiempo discreto — Trans Z — PDS
* Implementación Digital — PC, sistema embebido, E Digital — PDS
* Procesamiento de señales — Diseño de filtros — PDS
* Control → Diseño de sistemas
  → Diseño óptimo — CONTROL
* Modelos y simulación → Modelado
  → Identificación — MSC
  → Simulación

Fin

Ejemplo 3.12

$N=4\ ,\ N_1 = 1$



$$a_k = \frac{1}{N} \sum_{n=-\frac{N}{2}}^{N/2} x[n]\, e^{-jk\omega_0 n}$$

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} e^{-jk\omega_0 n} = \frac{1}{N} \sum_{m=0}^{2N_1} e^{-jk\omega_0 (m-N_1)} = \frac{1}{N} \sum_{n=0}^{2N_1} e^{-jk\omega_0 m}\, e^{+jk\omega_0 N_1}$$

$m = n + N_1$

$$a_k = \frac{1}{N}\, e^{jk\omega_0 N_1} \sum_{n=0}^{2N_1} \Big(\underbrace{e^{-jk\omega_0}}_{r}\Big)^m = \frac{1}{N}\, e^{j\omega_0 k N_1}\, \frac{1 - r^{2N_1+1}}{1-r}$$

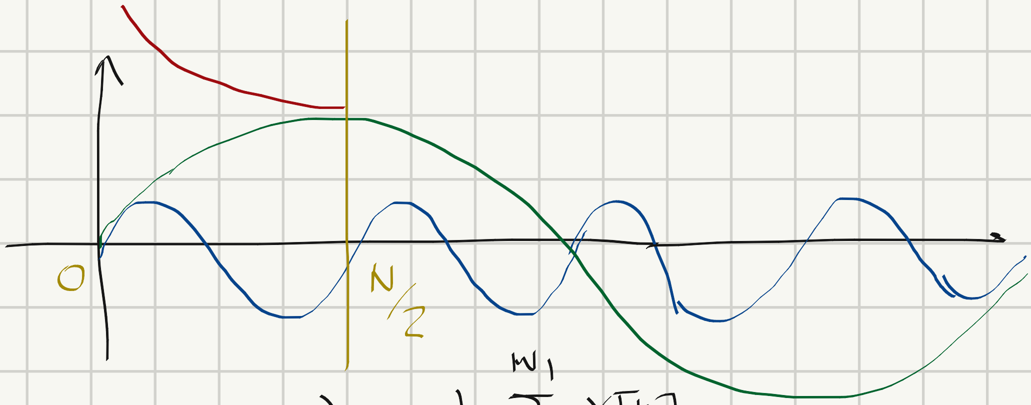
$$a_k = \frac{1}{N}\, e^{jk\omega_0 N_1}\, \frac{1 - e^{-jk\omega_0(2N_1+1)}}{1 - e^{-jk\omega_0}} = \frac{1}{N}\, \frac{e^{jk\omega_0 N_1} - e^{-jk\omega_0 (N_1+1)}}{\left(1 - e^{-jk\omega_0}\right) e^{jk\frac{\omega_0}{2}}}\, e^{jk\frac{\omega_0}{2}}$$

$$a_k = \frac{1}{N}\, \frac{e^{jk\omega_0 (N_1+\frac{1}{2})} - e^{-jk\omega_0 (N_1+\frac{1}{2})}}{e^{jk\frac{\omega_0}{2}} - e^{-jk\frac{\omega_0}{2}}} \cdot \frac{2j}{2j}$$

$$\boxed{a_k = \frac{1}{N}\, \frac{\operatorname{sen}\left[\left(N_1+\frac{1}{2}\right)\omega_0\, k\right]}{\operatorname{sen}\left[\frac{\omega_0}{2}\, k\right]}}$$

$$a_k = \frac{1}{4} \cdot \frac{\operatorname{sen}\left[\frac{3}{2}\cdot\frac{2\pi}{4}k\right]}{\operatorname{sen}\left[\frac{2\pi}{4.2}k\right]} = \boxed{\frac{1}{4}\cdot\frac{\operatorname{sen}\left[\frac{3\pi}{4}k\right]}{\operatorname{sen}\left[\frac{\pi}{4}k\right]}} \quad \begin{matrix} * \\ * \end{matrix}$$

k	*	*	a_k
0	—	—	$\frac{3}{4}$
1	$\sqrt{2}/2$	$\sqrt{2}/2$	$1/4$
2	1	-1	$-\frac{1}{4}$
3	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{4}$
4			
5			
6			

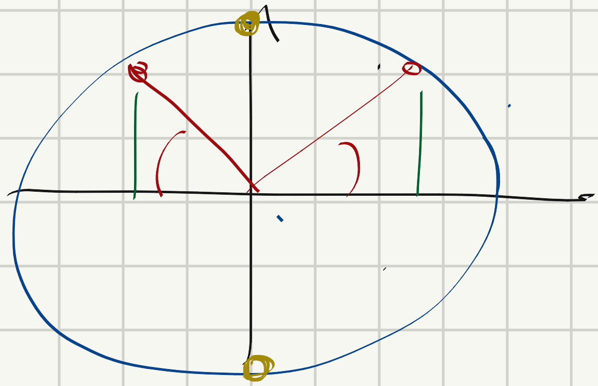


$$a_0 = \frac{1}{N}\cdot\sum_{n=-N_1}^{N_1} x[n]$$

$$a_0 = \frac{1}{N}\cdot\sum_{-N_1}^{N_1} 1 = \frac{2N_1+1}{N} = \boxed{\frac{3}{4}}$$

$$\frac{9\pi}{4} \qquad \frac{8\pi}{4} = 2\pi$$

$$\frac{1}{4}\pi$$



$$\operatorname{sen}\left[\frac{w_0}{2}\cdot k\right] = \operatorname{sen}\left[\frac{2\pi}{N.2}\cdot k\right]$$

$$\operatorname{sen}\left[\frac{\pi}{N}\cdot k\right] \qquad \frac{\pi}{2} = \frac{\pi}{N}\cdot k$$

$$\| \quad \frac{\pi}{2} \qquad \boxed{k = \frac{N}{2}}$$

Anexo