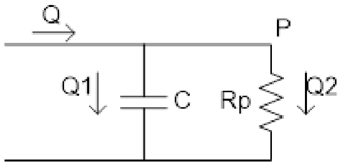
IV. Full Mathematical Development

Two-element Windkessel



By using Kirchhoff's laws and Ohm's law, the circuit can be solved:

$$Q_1(t) = C \cdot \frac{dP(t)}{dt}$$

$$Q_2(t) = \frac{P(t)}{R_p}$$

$$Q(t) = Q_1(t) + Q_2(t)$$

Flow may be expressed as a function of the pressure with the described equation:

$$Q(t) = C \cdot \frac{dP(t)}{dt} + \frac{P(t)}{R_p} \leftrightarrow C \cdot s \cdot P + \frac{P}{R_p} = P\left(s \cdot C + \frac{1}{R_p}\right)$$

The equation may be transformed into Laplace domain, this definition will be used in the next section for performing the impedance analysis. Now, solving for the pressure as a function of previous samples:

$$Q(t) \cong Q(i)$$

$$Q(i) = C \cdot \frac{P(i) - P(i-1)}{\Delta t} + \frac{P(i)}{R_p}$$

$$\frac{R_p \cdot Q(i) \cdot \Delta t}{C} = \frac{\Delta t \cdot P(i)}{C} + P(i) - P(i-1)$$

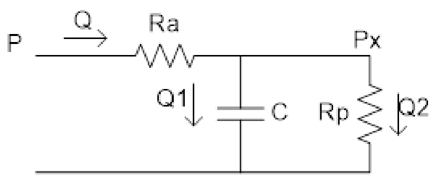
$$\frac{R_p \cdot Q(i) \cdot \Delta t}{C} = P(i) \cdot \left[R_p + \frac{\Delta t}{C}\right] - P(i-1) \cdot R_p$$

The final solution is:

$$P(i) = \frac{1}{R_p + \frac{\Delta t}{C}} \cdot \left[Q(i) \cdot \frac{R_p \cdot \Delta t}{C} + P(i-1) \cdot R_p\right]$$

Three-element Windkessel

The circuit is solved following the same procedure.



$$Q(t) = \frac{P(t) - P_x(t)}{R_a} \rightarrow P_x(t) = P(t) - Q(t) \cdot R_a$$

$$Q_1(t) = C \cdot \frac{dP_x(t)}{dt} = C \cdot \frac{d(P(t) - Q(t) \cdot R_a)}{dt}$$

$$Q_2(t) = \frac{P_x(t)}{R_p} = \frac{P(t) - Q(t) \cdot R_a}{R_p}$$

Assembling these equations:

$$Q(t) = Q_1(t) + Q_2(t)$$

$$Q(t) = C \cdot \frac{d(P(t) - Q(t) \cdot R_a)}{dt} + \frac{P(t) - Q(t) \cdot R_a}{R_p}$$

Rearranging the terms, the relationship between pressure and flow can be found:

$$Q(t) \cdot \left(1 + \frac{R_a}{R_p}\right) + C \cdot R_a \cdot \frac{dQ(t)}{dt} = C \cdot \frac{dP(t)}{dt} + \frac{P(t)}{R_p} \leftrightarrow Q \cdot \left(1 + \frac{R_a}{R_p} + C \cdot R_a \cdot s\right) = P(C \cdot s + \frac{1}{R_p})$$

Again, Laplace definition will be later useful for the impedance analysis.

Transforming every term from a continuous to a discrete function:

$$Q(i) \cdot \left(1 + \frac{R_a}{R_p}\right) + C \cdot R_a \cdot \frac{Q(i) - Q(i-1)}{\Delta t} = C \cdot \frac{P(i) - P(i-1)}{\Delta t} + \frac{P(i)}{R_p}$$

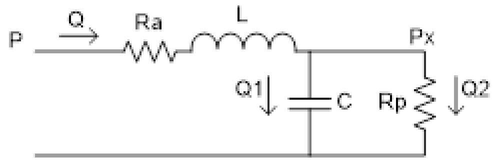
$$R_p \cdot \left\{Q(i) \cdot \left[1 + \frac{R_a}{R_p} + \frac{C \cdot R_a}{\Delta t}\right] - Q(i-1) \cdot \frac{C \cdot R_a}{\Delta t}\right\} = P(i) \cdot \left(\frac{C \cdot R_p}{\Delta t} + 1\right) - P(i-1) \cdot \frac{C \cdot R_p}{\Delta t}$$

Finally:

$$P(i) = \frac{1}{1 + \frac{C \cdot R_p}{\Delta t}} \cdot \left\{P(i-1) \cdot \frac{C \cdot R_p}{\Delta t} + R_p \cdot \left\{Q(i) \cdot \left[1 + \frac{R_a}{R_p} + \frac{C \cdot R_a}{\Delta t}\right] - Q(i-1) \cdot \frac{C \cdot R_a}{\Delta t}\right\}\right\}$$

Four-element Windkessel Series

The circuit is solved following the same procedure.



$$P_L(t) = L \cdot \frac{dQ(t)}{dt}$$

$$Q(t) = \frac{P(t) - P_L(t) - P_x(t)}{R_a} \rightarrow P_x(t) = P(t) - L \cdot \frac{dQ(t)}{dt} - Q(t) \cdot R_a$$

$$Q_1(t) = C \cdot \frac{dP_x(t)}{dt} = C \cdot \frac{d\left(P(t) - L \cdot \frac{dQ(t)}{dt} - Q(t) \cdot R_a\right)}{dt}$$

$$Q_2(t) = \frac{P_x(t)}{R_p} = \frac{P(t) - L \cdot \frac{dQ(t)}{dt} - Q(t) \cdot R_a}{R_p}$$

$$Q(t) = Q_1(t) + Q_2(t)$$

Assembling the previous equations:

$$Q(t) = C \cdot \frac{d\left(P(t) - L \cdot \frac{dQ(t)}{dt} - Q(t) \cdot R_a\right)}{dt} + \frac{P(t) - L \cdot \frac{dQ(t)}{dt} - Q(t) \cdot R_a}{R_p}$$

$$Q(t) = C \cdot \frac{dP(t)}{dt} - C \cdot L \cdot \frac{d^2Q(t)}{dt^2} + C \cdot R_a \cdot \frac{dQ(t)}{dt} + \frac{P(t)}{R_p} - \frac{L}{R_p} \cdot \frac{dQ(t)}{dt} - \frac{R_a}{R_p} \cdot Q(t)$$

Finally arriving to the solution in the temporal domain:

$$Q(t) \cdot \left(1 + \frac{R_a}{R_p}\right) + \frac{dQ(t)}{dt} \cdot \left(\frac{L}{R_p} - C \cdot R_a\right) + C \cdot L \cdot \frac{d^2Q(t)}{dt^2} = P(t) \cdot \left(\frac{1}{R_p}\right) + C \cdot \frac{dP(t)}{dt}$$

And to the solution in the Laplace domain:

$$Q \cdot \left[\left(1 + \frac{R_a}{R_p}\right) + s \cdot \left(\frac{L}{R_p} - C \cdot R_a\right) + C \cdot L \cdot s^2\right] = P \cdot \left(\frac{1}{R_p} + C \cdot s\right)$$

Transforming the solution into a function of the previous samples:

$$Q(i) \cdot \left(1 + \frac{R_a}{R_p}\right) + \frac{Q(i) - Q(i-1)}{\Delta t} \cdot \left(\frac{L}{R_p} - C \cdot R_a\right) + C \cdot L \cdot \frac{Q(i) - 2 \cdot Q(i-1) + Q(i-2)}{\Delta t^2}$$
$$= P(i) \cdot \left(\frac{1}{R_p}\right) + C \cdot \frac{P(i) - P(i-1)}{\Delta t}$$

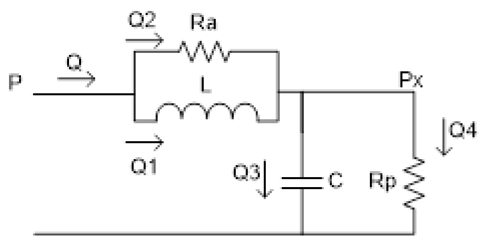
$$Q(i) \cdot \left[\left(1 + \frac{R_a}{R_p}\right) + \frac{\left(\frac{L}{R_p} - C \cdot R_a\right)}{\Delta t} + \frac{C \cdot L}{\Delta t^2}\right] - Q(i-1) \cdot \left[\frac{\left(\frac{L}{R_p} - C \cdot R_a\right)}{\Delta t} + \frac{2 \cdot C \cdot L}{\Delta t^2}\right] + Q(i-2) \cdot \frac{C \cdot L}{\Delta t^2}$$
$$= P(i) \cdot \left(\frac{1}{R_p} + \frac{C}{\Delta t}\right) + P(i-1) \cdot \frac{C}{\Delta t}$$

Finally:

$$P(i) = \frac{1}{\frac{1}{R_p} + \frac{C}{\Delta t}} \cdot \left\{P(i-1) \cdot \frac{C}{\Delta t} + Q(i) \cdot \left[\left(1 + \frac{R_a}{R_p}\right) + \frac{\left(\frac{L}{R_p} - C \cdot R_a\right)}{\Delta t} + \frac{C \cdot L}{\Delta t^2}\right] - Q(i-1) \right.$$
$$\left. \cdot \left[\frac{\left(\frac{L}{R_p} - C \cdot R_a\right)}{\Delta t} + \frac{2 \cdot C \cdot L}{\Delta t^2}\right] + Q(i-2) \cdot \frac{C \cdot L}{\Delta t^2}\right\}$$

Four-element Windkessel Parallel

Due to the increased complexity of this model, it is analyzed directly in the Laplace domain and



later transformed into temporal domain to obtain the final solution.

$$Q(t) = \frac{P(t)}{Z_t}$$

$$Z_t = \frac{s \cdot L \cdot R_a}{s \cdot L + R_a} + \frac{\frac{1}{s \cdot C} \cdot R_p}{\frac{1}{s \cdot C} + R_p} = \frac{s \cdot L}{1 + s \cdot \frac{L}{R_a}} + \frac{\frac{1}{s \cdot C \cdot R_p}}{1 + \frac{1}{s \cdot C \cdot R_p}} =$$

$$= \frac{s \cdot L \cdot \left(1 + \frac{1}{s \cdot C \cdot R_p}\right) + \frac{1}{s \cdot C} \cdot \left(1 + s \cdot \frac{L}{R_a}\right)}{\left(1 + s \cdot \frac{L}{R_a}\right) \cdot \left(1 + \frac{1}{s \cdot C \cdot R_p}\right)} =$$

$$= \frac{s \cdot L \cdot \left(1 + \frac{1}{s \cdot C \cdot R_p}\right) + \frac{1}{s \cdot C} \cdot \left(1 + s \cdot \frac{L}{R_a}\right)}{1 + \frac{1}{s \cdot C \cdot R_p} + s \cdot \frac{L}{R_a} + \frac{L}{C \cdot R_p \cdot R_a}}$$

$$Q(s) \cdot \frac{s \cdot L \cdot \left(1 + \frac{1}{s \cdot C \cdot R_p}\right) + \frac{1}{s \cdot C} \cdot \left(1 + s \cdot \frac{L}{R_a}\right)}{1 + \frac{1}{s \cdot C \cdot R_p} + s \cdot \frac{L}{R_a} + \frac{L}{C \cdot R_p \cdot R_a}} = P(s)$$

$$Q(s) \cdot \left[s \cdot L \cdot \left(1 + \frac{1}{s \cdot C \cdot R_p}\right) + \frac{1}{s \cdot C} \cdot \left(1 + s \cdot \frac{L}{R_a}\right)\right] = \left(1 + \frac{1}{s \cdot C \cdot R_p} + s \cdot \frac{L}{R_a} + \frac{L}{C \cdot R_p \cdot R_a}\right) \cdot P(s)$$

$$Q(s) \cdot \left[s \cdot L \cdot \left(1 + \frac{1}{s \cdot C \cdot R_p}\right) + \frac{1}{s \cdot C} \cdot \left(1 + s \cdot \frac{L}{R_a}\right)\right]$$
$$= \left[\frac{(s \cdot C \cdot R_p + 1) \cdot R_a + L \cdot s \cdot (s \cdot C \cdot R_p + 1)}{s \cdot C \cdot R_p \cdot R_a}\right] \cdot P(s)$$

$$Q(s) \cdot \left[s \cdot L \cdot \left(s + \frac{1}{C \cdot R_p}\right) + \frac{1}{C} \cdot \left(1 + s \cdot \frac{L}{R_a}\right)\right] \cdot C \cdot R_p \cdot R_a$$
$$= \left((s \cdot C \cdot R_p + 1) \cdot R_a + L \cdot s \cdot (s \cdot C \cdot R_p + 1)\right) \cdot P(s)$$

$$Q(s) \cdot \left[s^2 \cdot L + s \cdot \left(\frac{L}{C \cdot R_a} + \frac{L}{C \cdot R_p}\right) + \frac{1}{C}\right] \cdot C \cdot R_p \cdot R_a$$
$$= (s \cdot C \cdot R_p \cdot R_a + R_a + L \cdot s^2 \cdot C \cdot R_p + s \cdot L) \cdot P(s)$$

Applying the transformation: $s^n = \frac{d^n}{dt^n}$

$$\frac{d^2Q(t)}{dt^2} \cdot L \cdot C \cdot R_p \cdot R_a + \frac{dQ(t)}{dt} \cdot \left(L \cdot (R_p + R_a)\right) + Q(t) \cdot R_p \cdot R_a$$
$$= \frac{d^2P(t)}{dt^2} \cdot L \cdot C \cdot R_p + \frac{dP(t)}{dt} \cdot (C \cdot R_p \cdot R_a + L) + P(t) \cdot R_a$$

After transforming every term from a continuous to a discrete function:

$\frac{Q(i)-2\cdot Q(i-1)+Q(i-2)}{\Delta t^2}\cdot L\cdot C\cdot R_p\cdot R_a+\frac{Q(i)-Q(i-1)}{\Delta t}\cdot\left(L\cdot\left(R_p+R_a\right)\right)+Q(i)\cdot R_p\cdot R_a=\frac{P(i)-2\cdot P(i-1)+P(i-2)}{\Delta t^2}\cdot L\cdot C\cdot R_p+\frac{V(i)-V(i-1)}{\Delta t}\cdot\left(C\cdot R_p\cdot R_a+L\right)+P(i)\cdot R_a$

$Q(i)\cdot\left[\frac{L\cdot C\cdot R_p\cdot R_a}{\Delta t^2}+\frac{L\cdot\left(R_p+R_a\right)}{\Delta t}+R_p\cdot R_a\right]-Q(i-1)\cdot\left[\frac{2\cdot L\cdot C\cdot R_p\cdot R_a}{\Delta t^2}+\frac{L\cdot\left(R_p+R_a\right)}{\Delta t}\right]+Q(i-2)\cdot\frac{L\cdot C\cdot R_p\cdot R_a}{\Delta t^2}=P(i)\cdot\left[\frac{L\cdot C\cdot R_p}{\Delta t^2}+\frac{C\cdot R_p\cdot R_a+L}{\Delta t}+R_a\right]-P(i-1)\cdot\left[\frac{2\cdot L\cdot C\cdot R_p}{\Delta t^2}+\frac{C\cdot R_p\cdot R_a+L}{\Delta t}\right]+P(i-2)\cdot\frac{L\cdot C\cdot R_p}{\Delta t^2}$

Finally:

$$P(i)=\frac{1}{\frac{L\cdot C\cdot R_p}{\Delta t^2}+\frac{C\cdot R_p\cdot R_a+L}{\Delta t}+R_a}\cdot\left\{P(i-1)\cdot\left[\frac{2\cdot L\cdot C\cdot R_p}{\Delta t^2}+\frac{C\cdot R_p\cdot R_a+L}{\Delta t}\right]-P(i-2)\cdot\frac{L\cdot C\cdot R_p}{\Delta t^2}+Q(i)\cdot\left[\frac{L\cdot C\cdot R_p\cdot R_a}{\Delta t^2}+\frac{L\cdot\left(R_p+R_a\right)}{\Delta t}+R_p\cdot R_a\right]-Q(i-1)\cdot\left[\frac{2\cdot L\cdot C\cdot R_p\cdot R_a}{\Delta t^2}+\frac{L\cdot\left(R_p+R_a\right)}{\Delta t}\right]+Q(i-2)\cdot\frac{L\cdot C\cdot R_p\cdot R_a}{\Delta t^2}\right\}$$