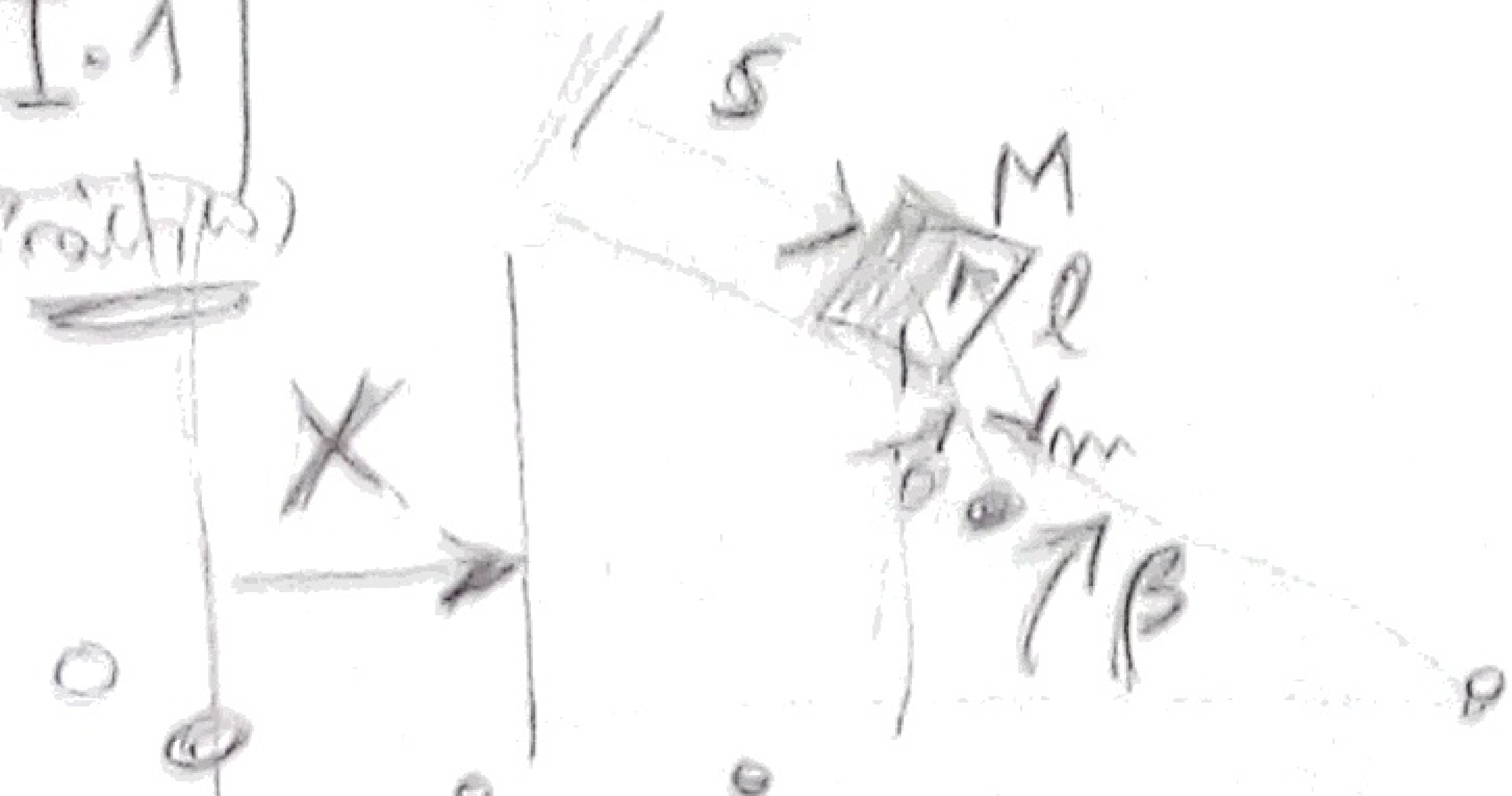
I.1




EXAMEN DINÁMICA CLÁSICA CURSO 2017 AGOSTO

1

$x_M = X + s\cos\beta$

$y_M = -s\,\mathrm{sen}\,\beta$

$x_m = x_M + l\,\mathrm{sen}\,\theta$

$y_m = y_M - l\cos\theta$

$\dot{x}_M = \dot{X} + \dot{s}\cos\beta$

$\dot{y}_M = -\dot{s}\,\mathrm{sen}\,\beta$

$\dot{x}_m = \dot{X} + \dot{s}\cos\beta + l\dot{\theta}\cos\theta$

$\dot{y}_m = -\dot{s}\,\mathrm{sen}\,\beta + l\dot{\theta}\,\mathrm{sen}\,\theta$

$v_M^2 = \dot{s}^2 + \dot{X}^2 + 2\dot{s}\dot{X}\cos\beta$

$v_m^2 = v_M^2 + l^2\dot{\theta}^2 + 2l\dot{\theta}\dot{s}\cos(\theta+\beta)$
$+ 2l\dot{\theta}\dot{X}\cos\theta$

$\dot{x}_m^2 = \dot{x}_M^2 + l^2\dot{\theta}^2\cos^2\theta + 2l\dot{\theta}\dot{x}_M\cos\theta$

$\dot{y}_m^2 = \dot{y}_M^2 + l^2\dot{\theta}^2\,\mathrm{sen}^2\theta + 2\dot{y}_M l\dot{\theta}\,\mathrm{sen}\,\theta$

$v_m^2 = v_M^2 + l^2\dot{\theta}^2 + 2l\dot{\theta}\left[\dot{x}_M\cos\theta + \dot{y}_M\,\mathrm{sen}\,\theta\right]$

$\dot{x}_M\cos\theta + \dot{y}_M\,\mathrm{sen}\,\theta = \left[\dot{X} + \dot{s}\cos\beta\right]\cos\theta - \dot{s}\,\mathrm{sen}\,\beta\,\mathrm{sen}\,\theta =$

$= \dot{X}\cos\theta + \dot{s}\left[\cos\beta\cos\theta - \mathrm{sen}\,\beta\,\mathrm{sen}\,\theta\right] =$

$= \dot{X}\cos\theta + \dot{s}\cos[\theta+\beta]$

IRRELEVANTE

$= \frac{M v_M^2}{2} + \frac{m v_m^2}{2} = \frac{(M+m)}{2}\left[\dot{s}^2 + 2\dot{s}\dot{X}\cos\beta + \dot{X}^2\right]$

$\frac{m l^2\dot{\theta}^2}{2} + m l\dot{\theta}\dot{s}\cos(\theta+\beta) + m l\dot{\theta}\dot{X}\cos\theta$

$V = +M g y_M + m g y_m = -(M+m) s g\,\mathrm{sen}\,\beta - m g l\cos\theta$

$= \frac{(M+m)}{2}\left[\dot{s}^2 + 2\dot{s}\dot{X}\cos\beta + \dot{X}^2\right] + \frac{m l^2\dot{\theta}^2}{2} + m l\dot{\theta}\dot{s}\cos(\theta+\beta)$

$m l\dot{\theta}\dot{X}\cos\theta + (M+m) g s\,\mathrm{sen}\,\beta + m g l\cos\theta$

$$L = \frac{(M+m)}{2}\left[\dot{s}^2 + 2\dot{s}\dot{X}\cos\beta + \dot{X}^2\right] + \frac{ml^2}{2}\dot{\theta}^2 + ml\dot{\theta}\dot{s}\cos(\theta+\beta)$$
$$+ ml\dot{\theta}\dot{X}\cos\theta + (M+m)gs\,\mathrm{sen}\beta + mgl\cos\theta$$

$$\frac{\partial L}{\partial \dot{s}} = (M+m)\dot{s} + (M+m)\dot{X}\cos\beta + ml\dot{\theta}\cos(\theta+\beta)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) = (M+m)\ddot{s} + (M+m)\ddot{X}\cos\beta + ml\ddot{\theta}\cos(\theta+\beta)$$
$$- ml\dot{\theta}^2\,\mathrm{sen}(\theta+\beta)$$

$$\frac{\partial L}{\partial s} = (M+m)g\,\mathrm{sen}\beta$$

**s:** $$\boxed{(M+m)\ddot{s} + (M+m)\ddot{X}\cos\beta + ml\ddot{\theta}\cos(\theta+\beta) - ml\dot{\theta}^2\,\mathrm{sen}(\theta+\beta) - (M+m)g\,\mathrm{sen}\beta = 0}$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} + ml\dot{s}\cos(\theta+\beta) + ml\dot{X}\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta} + ml\ddot{s}\cos(\theta+\beta) - ml\dot{s}\dot{\theta}\,\mathrm{sen}(\theta+\beta) + ml\ddot{X}\cos\theta - ml\dot{X}\dot{\theta}\,\mathrm{sen}\theta$$

$$\frac{\partial L}{\partial \theta} = -ml\dot{\theta}\dot{s}\,\mathrm{sen}(\theta+\beta) - ml\dot{\theta}\dot{X}\,\mathrm{sen}\theta - mgl\,\mathrm{sen}\theta$$

**θ:** $$\boxed{ml^2\ddot{\theta} + ml\ddot{s}\cos(\theta+\beta) + ml\ddot{X}\cos\theta + mgl\,\mathrm{sen}\theta = 0}$$

$\boxed{a.}$ $\boxed{X = 0 \,\|\, \dot{X} = 0 \,\|\, \ddot{X} = 0}$ Si $\boxed{\theta = \theta_0 = cte} \rightarrow \boxed{\begin{matrix}\dot{\theta} = 0 \\ \ddot{\theta} = 0\end{matrix}}$

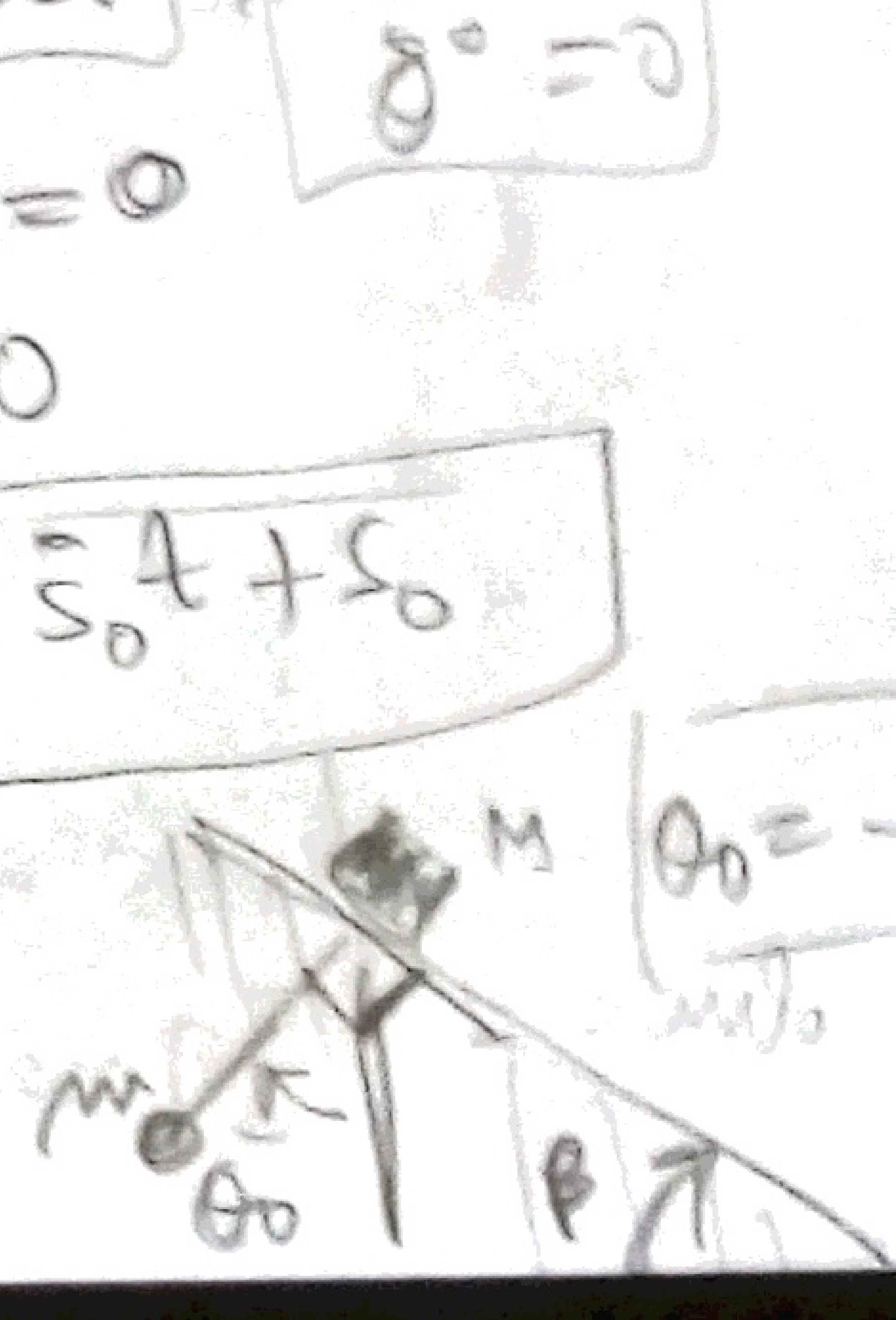
En ese caso $\boxed{s:}$ $\cancel{(M+m)}\ddot{s} - \cancel{(M+m)}g\,\mathrm{sen}\beta = 0$

$\boxed{\theta:}$ $\cancel{ml}\ddot{s}\cos(\theta_0+\beta) + \cancel{m}g\cancel{l}\,\mathrm{sen}\,\theta_0 = 0$

$\rightarrow \boxed{\ddot{s} = g\,\mathrm{sen}\beta} \rightarrow \boxed{s(t) = g\frac{\mathrm{sen}\beta}{2}t^2 + \dot{s}_0 t + s_0}$

$\cancel{g}\,\mathrm{sen}\beta\cos(\theta_0+\beta) + \cancel{g}\,\mathrm{sen}\,\theta_0 = 0$

$\hookrightarrow \boxed{\theta_0 = -\beta}$

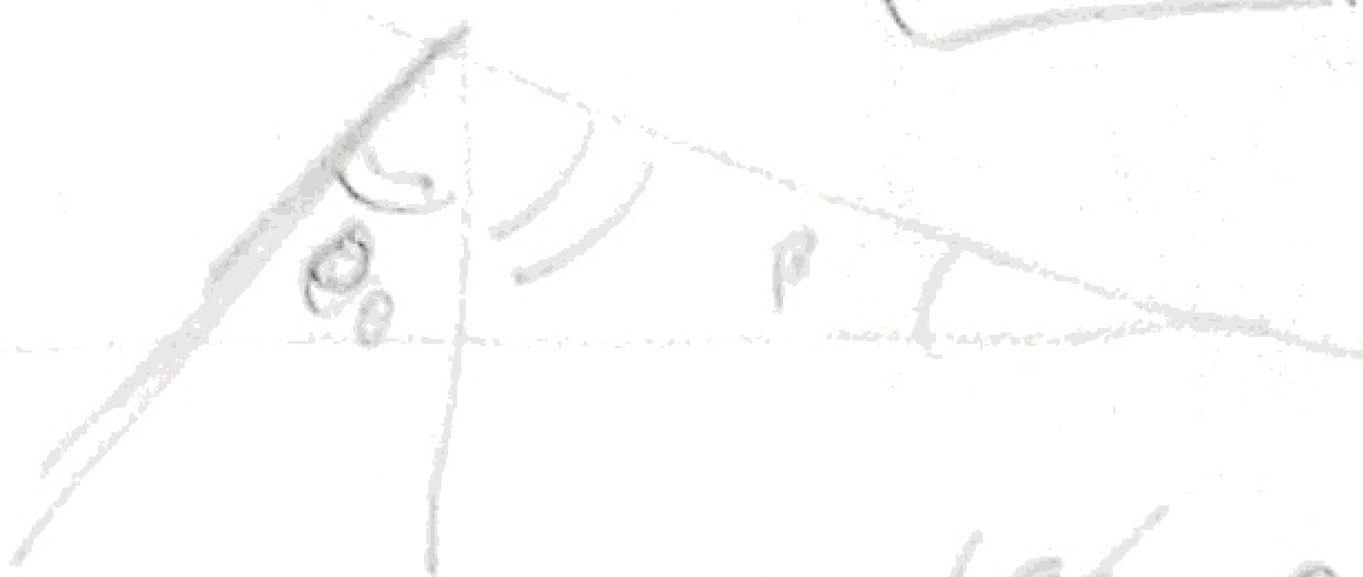



$\theta_0 = -$

$\operatorname{sen}\beta \cos(\theta_0+\beta) + \operatorname{sen}\theta_0 = 0$

$\boxed{\theta_0 = -\beta}$

2




$\theta_0 = 90 - (\cancel{90} - \beta)$

$\boxed{\theta_0 = -\beta}$

$\operatorname{sen}\beta\,[\cos\theta_0 \cos\beta - \operatorname{sen}\theta_0 \operatorname{sen}\beta] + \operatorname{sen}\theta_0 = 0$

$[\cos\theta_0 \operatorname{sen}\beta\cos\beta - \cancel{\operatorname{sen}\theta_0}(1-\cos^2\beta)] + \cancel{\operatorname{sen}\theta_0} = 0$

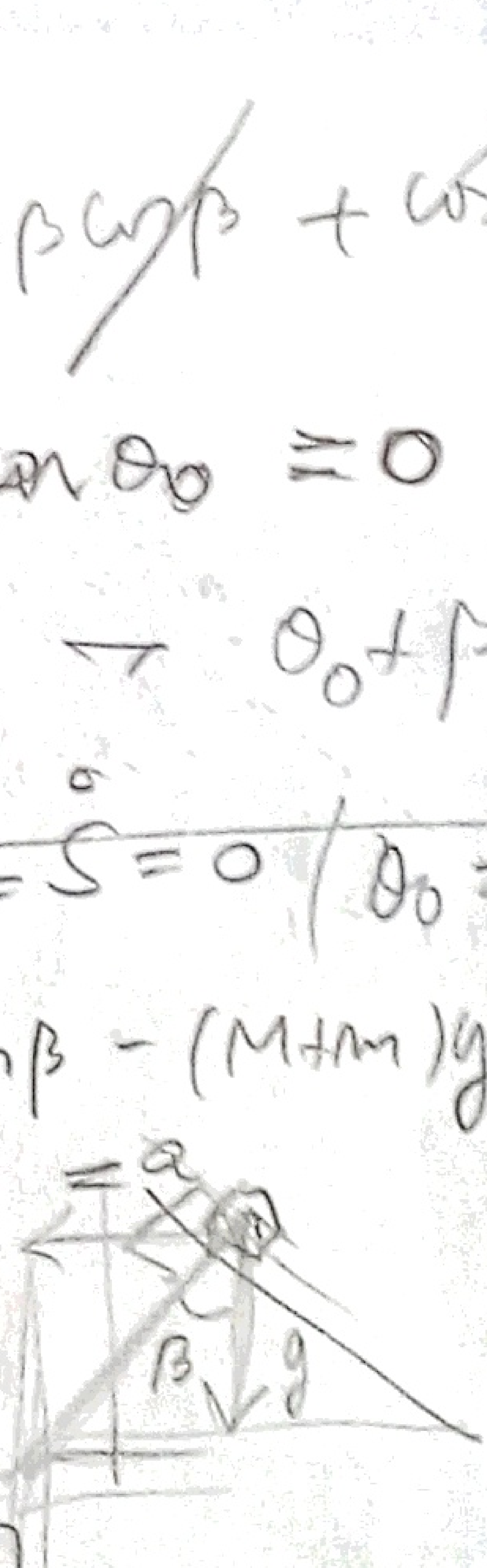
$\cos\theta_0 [$ $\quad \cos\theta_0 \operatorname{sen}\beta\cancel{\cos\beta} + \cos^{\cancel{2}}\beta \operatorname{sen}\theta_0 = 0$

$\cos\theta_0 \operatorname{sen}\beta + \cos\beta \operatorname{sen}\theta_0 = 0$

$\operatorname{sen}(\theta_0+\beta) = 0 \rightarrow \theta_0+\beta = \{0, \pi$

C. Si $\ddot{x} = a \;||\; \dot{x} = at \;||\; \ddot{s} = \dot{s} = 0 \,/\, \theta_0 = cte$

$\dot{\theta} = \ddot{\theta} = 0 \;||$ S: $(M+m)\,a\cos\beta - (M+m)\,g\operatorname{sen}\beta = 0$




$\theta$: $\cancel{m l}\, a\cos\theta_0 + \cancel{m}g\cancel{l}\operatorname{sen}\theta_0 = 0$

S: $\frac{a}{g} = tg\,\beta \rightarrow \boxed{a = g\,tg\,\beta}$

$\theta$: $tg\,\beta = -tg\,\theta_0 \longrightarrow \boxed{\theta_0 = -\beta}$

EXAMEN DE DINÁMICA CLÁSICA - CURSO - 2012
AGOSTO

**I-2** (Práctico)

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - V(x,y) \quad ①$$

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - V(r,\theta)$$

a.

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r}$$

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - \frac{\partial V}{\partial r}$$

$r$: $$m[\underbrace{\ddot{r} - r\dot{\theta}^2}_{=a_r}] = \underbrace{-\frac{\partial V}{\partial r}}_{= -(\nabla V)^r}$$

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -\frac{\partial V}{\partial \theta} \rightarrow m[r^2\ddot{\theta} + 2mr\dot{r}\dot{\theta}] = -\frac{\partial V}{\partial \theta}$$

$$\boxed{m[\underbrace{r\ddot{\theta} + 2\dot{r}\dot{\theta}}_{=a_\theta}] = \underbrace{-\frac{1}{r}\frac{\partial V}{\partial \theta}}_{= -(\nabla V)_\theta}}$$

b.

Si $V(r,\theta) = V(r) \longrightarrow \theta$ es ignorable

$$\boxed{p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = \text{cte.}} \quad p_\theta = mr\underbrace{r\dot{\theta}}_{v_\theta} = L_z$$

I.2 c.

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + V(r,\theta)$$ POLARES

$p_r = m\dot{r}$

$p_\theta = mr^2\dot{\theta} \rightarrow \dot{\theta} = \frac{p_\theta}{mr^2}$

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + V(x,y)$$ CARTESIANAS

$p_x = m\dot{x} \parallel p_y = m\dot{y}$

$r = \sqrt{x^2+y^2} \parallel \theta = \operatorname{arctg}\left(\frac{y}{x}\right)$

$$\dot{r} = \frac{x\dot{x}}{\sqrt{x^2+y^2}} + \frac{y\dot{y}}{\sqrt{x^2+y^2}} \rightarrow \boxed{p_r = \frac{xp_x + yp_y}{\sqrt{x^2+y^2}}}$$

$$\frac{d}{d\xi}\operatorname{arctg}(\xi) = \frac{1}{1+\xi^2}$$

$$\rightarrow \dot{\theta} = \frac{\partial\theta}{\partial x}\dot{x} + \frac{\partial\theta}{\partial y}\dot{y} = \frac{\dot{x}}{\left(1+\left(\frac{y}{x}\right)^2\right)}\left(-\frac{y}{x^2}\right) +$$

$$+ \frac{\dot{y}}{\left(1+\left(\frac{y}{x}\right)^2\right)}\frac{1}{x} = \frac{-y\dot{x}}{x^2+y^2} + \frac{x\dot{y}}{x^2+y^2}$$

$$\rightarrow \dot{\theta} = \frac{x\dot{y} - y\dot{x}}{x^2+y^2} \rightarrow \boxed{p_\theta = \frac{xp_y - yp_x}{x^2+y^2}}$$

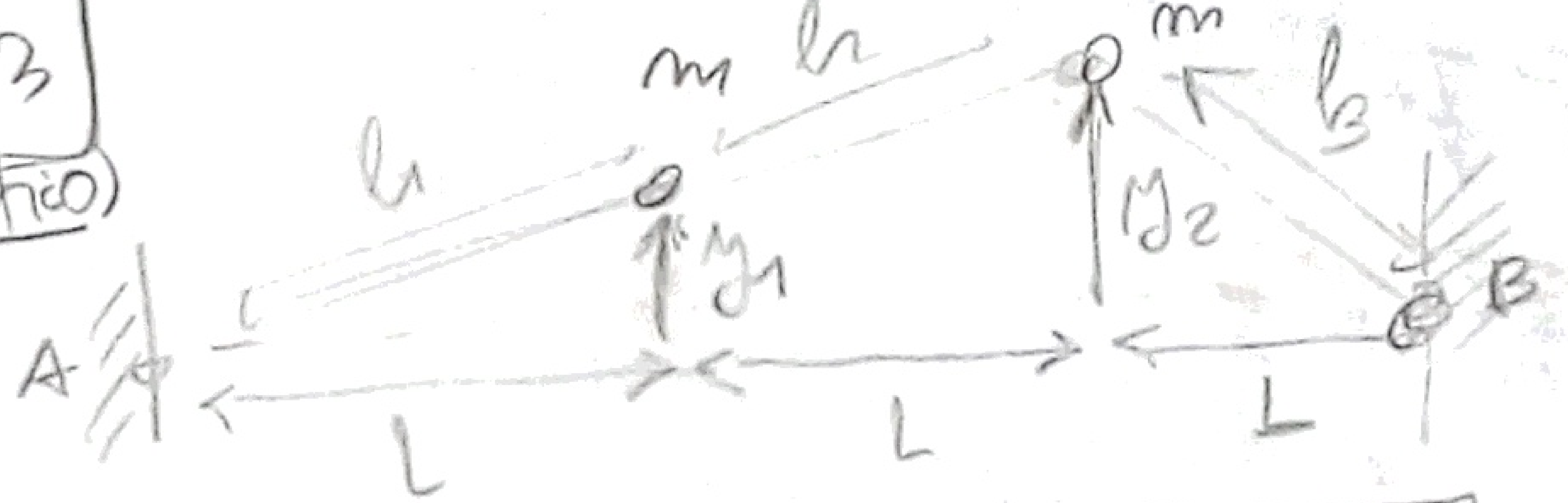
$\{\theta,\theta\} = \{r,r\} = \{p_r,p_r\} = \{p_\theta,p_\theta\} = 0$

$\{r,\theta\} \overset{?}{=} 0 \parallel \{r,p_r\} = 1 \parallel \{r,p_\theta\} = 0 \parallel \{\theta,p_r\} = 0$

$\{\theta,p_\theta\} = 1$

I.3 (r chico)

EXÁMEN DINÁMICA CLÁSICA - CURSO 2017 AGOSTO




$$L = \frac{m}{2}\left[\dot{y}_1^2 + \dot{y}_2^2\right] - T\left[\sqrt{y_1^2+L^2} + \sqrt{y_2^2+L^2} + \sqrt{(y_2-y_1)^2+L^2} - 3L\right]$$

$$L \cong \frac{m}{2}\left[\dot{y}_1^2 + \dot{y}_2^2\right] - T\left[L\left(1+\frac{y_1^2}{2L^2}\right) + L\left(1+\frac{y_2^2}{2L^2}\right) + L\left(1+\frac{(y_2-y_1)^2}{2L^2}\right) - 3L\right]$$

$$L \cong \frac{m}{2}\left[\dot{y}_1^2 + \dot{y}_2^2\right] - \frac{T}{2L}\left[y_1^2 + y_2^2 + (y_2-y_1)^2\right]$$

$$L \cong \frac{m}{2}\left[\dot{y}_1^2 + \dot{y}_2^2\right] - \frac{T}{2L}\left[2y_1^2 + 2y_2^2 - 2y_1y_2\right]$$

$$\mathbb{M} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \quad \Big\| \quad \mathbb{K} = \begin{pmatrix} 2T/L & -T/L \\ -T/L & 2T/L \end{pmatrix}$$

$$\boxed{\mathbb{K} - \omega^2\mathbb{M} = \begin{pmatrix} \frac{2T}{L} - \omega^2 m & -T/L \\ -T/L & \frac{2T}{L} - \omega^2 m \end{pmatrix}}$$

$$\det(\mathbb{K} - \omega^2\mathbb{M}) = \left(\frac{2T}{L} - m\omega^2\right)^2 - \frac{T^2}{L^2} = 0$$

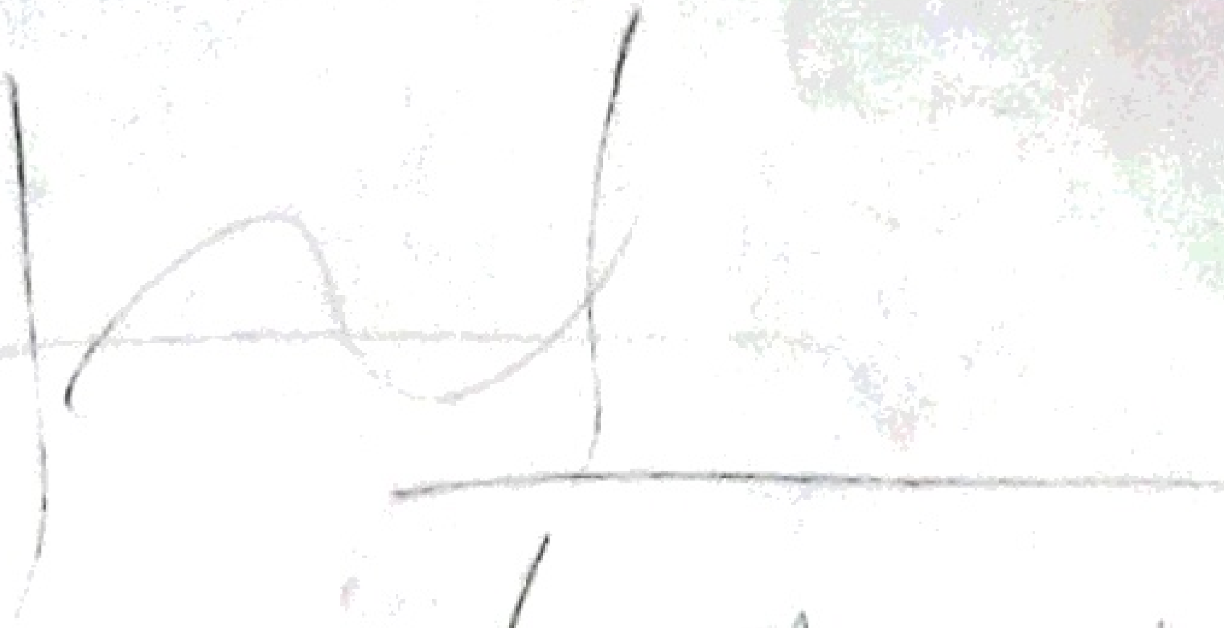
$$m^2\omega^4 - \frac{4T}{L}m\omega^2 + \frac{4T^2}{L^2} - \frac{T^2}{L^2} = 0 \quad \Big\| \quad \boxed{\omega^4 - 4\frac{T}{mL}\omega^2 + 3\frac{T^2}{m^2L^2} = 0}$$

Si $\boxed{\xi = \frac{T}{mL}} \rightarrow \boxed{\omega^4 - 4\xi\omega^2 + 3\xi^2 = 0} \rightarrow$

$$\omega_\pm^2 = \frac{4\xi \pm \sqrt{16\xi^2 - 12\xi^2}}{2} = \left(\frac{4\pm 2}{2}\right)\xi = (2\pm 1)\xi$$

$$\rightarrow \boxed{\omega_\pm^2 = \begin{cases} 3\xi \\ \xi \end{cases}} \longrightarrow \boxed{\omega_\pm^2 = \begin{cases} 3\frac{T}{mL} \\ \frac{T}{mL} \end{cases}}$$

$$\omega_{\pm}^2 = \begin{cases} 3\dfrac{T}{mL} = 3\varrho \\ \dfrac{T}{mL} = \varrho \end{cases}$$



$$\left[\mathbb{K} - \omega^2 \mathbb{M}\right] = \begin{pmatrix} \frac{2T}{L} - \omega^2 m & -\frac{T}{L} \\ -\frac{T}{L} & \frac{2T}{L} - \omega^2 m \end{pmatrix} = m \begin{pmatrix} 2\varrho - \omega^2 & -\varrho \\ -\varrho & 2\varrho - \omega^2 \end{pmatrix}$$

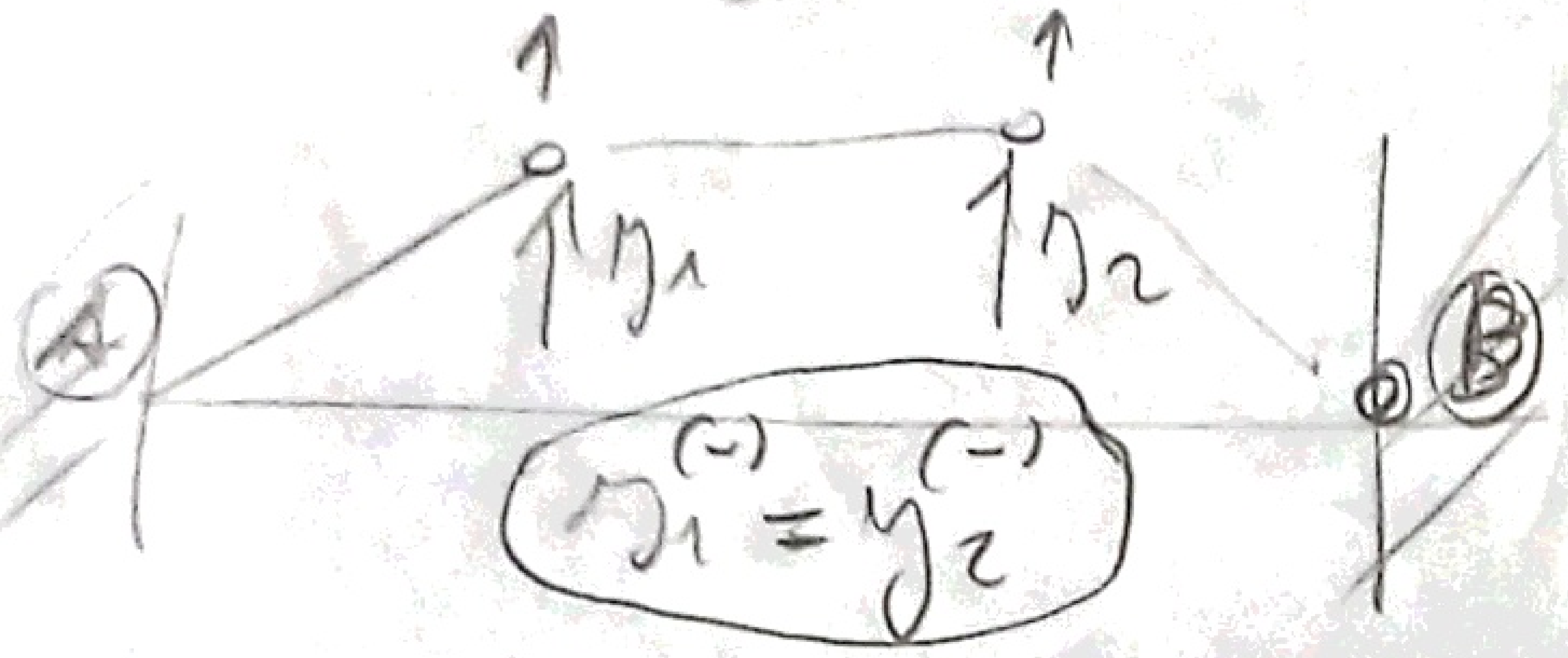
Modo $\omega_-^2$ lento $\Rightarrow$ $\mathbb{K} - \omega_-^2 \mathbb{M} = m \begin{pmatrix} 2\varrho - \varrho & -\varrho \\ -\varrho & 2\varrho - \varrho \end{pmatrix}$

$$\left(\mathbb{K} - \omega_-^2 \mathbb{M}\right) A^{(-)} = m\varrho \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} A_1^{(-)} \\ A_2^{(-)} \end{pmatrix} = 0$$

$\rightarrow$ $\boxed{A_1^{(-)} = A_2^{(-)}}$

$$y_1^{(-)} = A_1^{(-)} \cos\left[\sqrt{\frac{T}{mL}}\, t + \phi_-\right]$$

$$y_2^{(-)} = \underbrace{A_1^{(-)}}_{= A_2^{(-)}} \cos\left[\sqrt{\frac{T}{mL}}\, t + \phi_-\right]$$



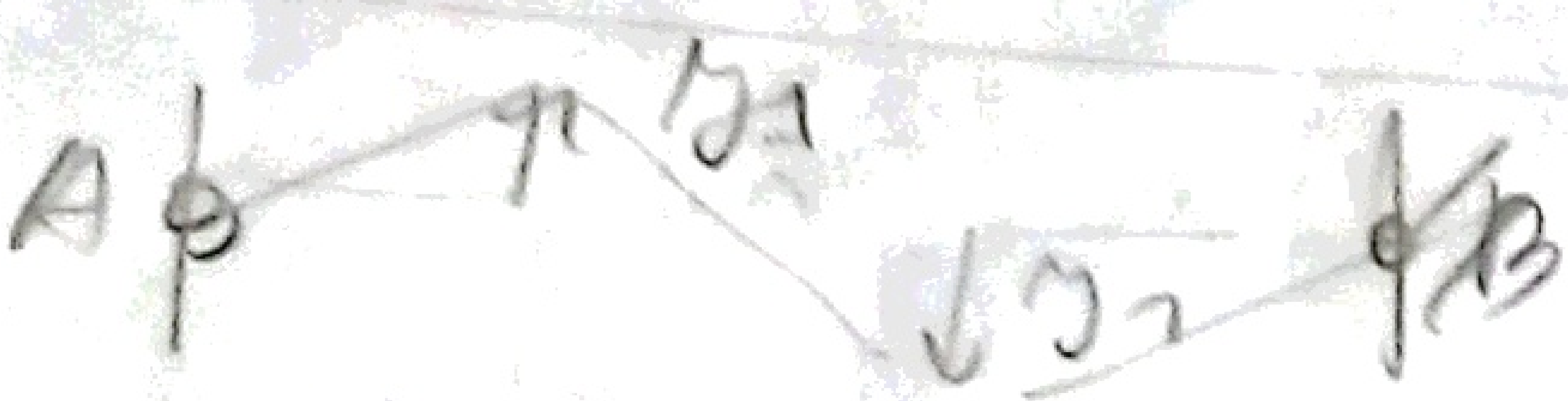

$y_1^{(-)} = y_2^{(-)}$

Modo $\omega_+^2$ rápido

$$\mathbb{K} - \omega_+^2 \mathbb{M} = m \begin{pmatrix} 2\varrho - 3\varrho & -\varrho \\ -\varrho & 2\varrho - 3\varrho \end{pmatrix} = m\varrho \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$$

$$\left[\mathbb{K} - \omega_+^2 \mathbb{M}\right] A^{(+)} = -m\varrho \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} A_1^{(+)} \\ A_2^{(+)} \end{pmatrix} \rightarrow \boxed{A_1^{(+)} = -A_2^{(+)}}$$

$$y_1^{(+)} = A_1^{(+)} \cos\left[\sqrt{\frac{3T}{mL}}\, t + \phi_+\right] \quad \| \quad y_2^{(+)} = -A_1^{(+)} \cos\left[\sqrt{\frac{3T}{mL}}\, t + \phi_+\right]$$




$\boxed{y_2^{(+)} = -y_1^{(+)}}$

Pregunta teórica

(EXAMEN DINÁMICA CLÁSICA - CURSO 2017 AGOSTO)

**II·2** (Teórico)

$$S = \int_{t_0}^{t_1} [p\dot{q} - H]\,dt \qquad \boxed{\delta S = 0} \qquad 2$$

$$\delta q(t_0) = \delta q(t_1) = 0 \quad \Big| \quad \delta S = \int_{t_0}^{t_1} [\delta p\,\dot{q} + \underbrace{p\,\delta(\dot{q})}_{=\frac{d}{dt}(p\delta q) - \dot{p}\delta q} - \delta H]\,dt$$

$$\delta S = \int_{t_0}^{t_1} \left[\dot{q}\,\delta p - \dot{p}\,\delta q - \frac{\partial H}{\partial q}\delta q - \frac{\partial H}{\partial p}\delta p\right] dt + \int_{t_0}^{t_1} \frac{d}{dt}(p\,\delta q)\,dt \rightarrow$$

$$\rightarrow \delta S = \int_{t_0}^{t_1} \left\{\left[\dot{q} - \frac{\partial H}{\partial q}\right]\delta q - \left[\dot{p} + \frac{\partial H}{\partial q}\right]\delta p\right\} dt + \cancel{[p\,\delta q]_{t_0}^{t_1}}^{\,0} = 0 \rightarrow \boxed{\dot{q} = \frac{\partial H}{\partial p} \;\Big\|\; \dot{p} = -\frac{\partial H}{\partial q}}$$

**II·1** (Teórico)

$$\text{a:} \quad \boxed{h = \frac{\partial L}{\partial \dot{q}}\dot{q} - L} \quad \Big| \quad \frac{dh}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\dot{q} + \cancel{\frac{\partial L}{\partial \dot{q}}\ddot{q}} - \frac{\partial L}{\partial q}\dot{q} - \cancel{\frac{\partial L}{\partial \dot{q}}\ddot{q}} - \frac{\partial L}{\partial t}$$

$$\frac{dh}{dt} = \underbrace{\left[\cancel{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q}}^{\,0}\right]}_{=\text{ EdeM}}\dot{q} - \frac{\partial L}{\partial t} \Rightarrow \boxed{\frac{dh}{dt} = -\frac{\partial L}{\partial t}}$$

$$\Big| \text{ si } \frac{\partial L}{\partial t} = 0 \Rightarrow \boxed{\frac{dh}{dt} = 0}$$

II.1 b. (teóricos)

$$L = \frac{mv^2}{2} - e\phi + e\vec{v}\cdot\vec{A}$$

Si $\phi = 0$ || $\vec{A} = \vec{A}(\vec{r})$ ||

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + e\dot{x}A_x(\vec{r}) + e\dot{y}A_y(\vec{r}) + \dot{z}A_z(\vec{r})$$

$$h = \frac{\partial L}{\partial \dot{\vec{r}}}\cdot\dot{\vec{r}} - L = m\dot{\vec{r}}\cdot\dot{\vec{r}} + e\vec{A}\cdot\dot{\vec{r}} - \frac{mv^2}{2} - e\vec{v}\cdot\vec{A} = \frac{mv^2}{2} = T$$

Em este caso $\boxed{h = \frac{mv^2}{2} = T}$

Dado que

$$\boxed{\frac{\partial L}{\partial t} = 0} \rightarrow \boxed{h = T = \text{constante}}$$