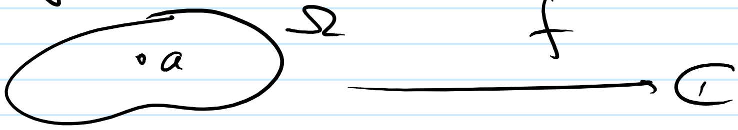
Clase 20

Series de potencias y funciones analíticas

Singularidades evitables



holomorfa en $f:\Omega-\{a\}\longrightarrow \mathbb{C}$

se dice que a es una singularidad de f

Def: Si $\lim_{z\to a}(z-a)f(z)=0$ se dice que la singularidad es evitable

Teorema $f:\Omega-\{a\}\longrightarrow \mathbb{C}$ una función analítica en Ω con una singularidad evitable $\exists\, F:\Omega\longrightarrow\mathbb{C}$ analítica: $F|_{\Omega-\{a\}}=f$

Dem

Ejemplo $\frac{\sin x}{x}=f(x)$ $\lim_{x\to 0}\frac{\sin x}{x}=1$

$$\lim_{x\to 0} x f(x) = 0$$

$$\begin{cases} \frac{\sin x}{x} & x\neq 0 \\ 1 & x=0 \end{cases}$$

$\lim_{z\to a}(z-a)f(z)=0$

D círculo alrededor de a $\quad D \subset \Omega \quad \gamma = \partial D$

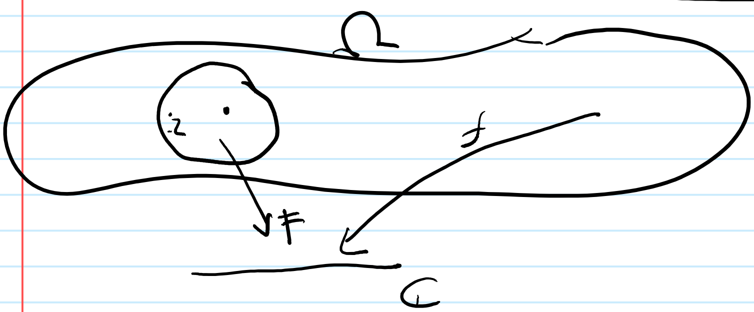
$z \subset D \quad \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta$ está definida en todo D

z → analítica

$z \neq a$ $\qquad z = a$

$z \neq a \quad f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta$ (F. int. de Cauchy)

$$F(z) := \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta \quad \forall z \in D \quad \begin{cases} z \neq a & f(z) \\ z = a & \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - a} d\zeta \end{cases}$$



$F|_{D - \{a\}} = f \qquad F \quad D - \{a\}$

$\hat{F} = \{ f$ afuera

Desarrollo de Taylor

$f : \Omega \longrightarrow \mathbb{C} \quad a \in \Omega$ analítica en Ω $\quad F(z) = \frac{f(z) - f(a)}{z - a}$

$F : \Omega - \{0\} \longrightarrow \mathbb{C}$

la singularidad en a es evitable

$$(z-a) F(z) = f(z) - f(a) \xrightarrow[z \to a]{} 0$$

Existe una función $f_1 : \Omega \longrightarrow \mathbb{C}$ $\therefore$ $f_1 \Big|_{\Omega - (a)} = +$

Obs $f_1(a) = f'(a)$

$$\boxed{f_1(z) = \left(f(z) - f(a)\right)/(z-a) \quad \forall z \neq a \quad z \in \Omega} \quad f_1$$

$$\Downarrow$$

$$\boxed{f(z) = f(a) + (z-a) f_1(z)} \quad \text{es analítica } f_1$$

$z \neq a$

$\boxed{z = a \quad \text{también.}}$

$f(z) = f(a) + (z-a) f_1(z)$ $\quad$ f_1 es analítica en lo mismo región que f

$\Omega \to \mathbb{C}$

$f(z) ; f(a) + (z-a) f(z)$

$\cancel{f_1(z) = \frac{f(z) - f(a)}{z-a}}$ $\qquad f_1(z) = \frac{f(z) - f(a)}{z-a}$

$f'(a)$

$f'(z) = f_1(z) + (z-a) f_1'(z)$

$$\boxed{f'(a) = f_1(a)}$$

$f(z) = f(a) + (z-a) f_1(z)$

$f_1(z) = f'(a) + (z-a) f_2(z) = f_1(a) + (z-a) f_2(z)$

$f_2(z) = f_2(a) + (z-a) f_3(z)$

$f_{n-1}(z)$

$\cancel{f_{n-1}(z) = f_{n-1}(a) + (z-a) f}$

$f_{n-1}(z) = f_{n-1}(a) + (z-a) f_n(z)$

$$f_{n-1}(z) = f_{n-1}(a) + (z-a)\, f_n(z)$$

$$\boxed{f(z) = f(a) + (z-a) f_1(a) + (z-a)^2 f_2(a) + \dots + (z-a)^{n-1} f_{n-1}(a) + (z-a)^n f_n(z)}$$

$$e^z = 1 + z + z^2/2 + z^3/3! + \dots + \frac{z^{n-1}}{(n-1)!} + \frac{z^n}{n!} + \frac{z^{n+1}}{(n+1)!} + \dots$$

$a = 0$

$$e^z = 1 + z + z^2/2 + \dots + \frac{z^{n-1}}{(n-1)!} + z^n \underbrace{\left(\frac{1}{n!} + \frac{z}{(n+1)!} + \frac{z^2}{(n+2)!} + \dots \right)}_{f_n(z)}$$

$$f(z) = f(a) + f_1(a)(z-a) + f_2(a)(z-a)^2 + \dots + (z-a)^n f_n(z)$$

$$f'(z) = f_1(a) + f_2(a)\, 2(z-a) + \dots \qquad f'(a) = f_1(a)$$

$$f''(z) = 2 f_2(a) + 3(z-a) f_3(a) + \dots \qquad f''(a) = 2 f_2(a)$$

$$\vdots$$

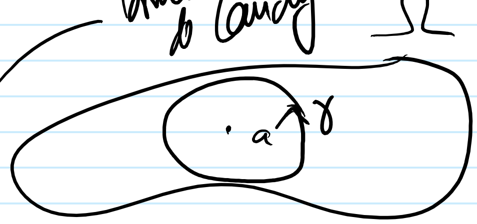
$$f^{(n)}(a) = n!\, f_n(a)$$

$$\boxed{f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2} f''(a) + \dots + \frac{(z-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \underbrace{(z-a)^n f_n(z)}_{R_n(f)(z)}}$$

Queremos controlar $f_n(z)$

$$f_n(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)\, d\zeta}{(\zeta-a)^n (\zeta - z)}$$

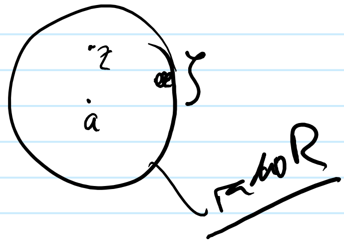
Fórmula integral de Cauchy



$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)} d\zeta \qquad n = 0$$

$$f_n(z) = \frac{1}{2\pi i} \int \frac{f(\zeta)}{(\zeta-a)^n (\zeta - z)} d\zeta$$

$$f_n(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-a)^n(\zeta-z)}\,d\zeta$$

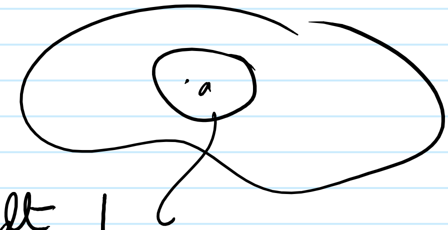


$$|f_n(z)| = \frac{1}{2\pi}\left|\int_\gamma \frac{f(\zeta)}{(\zeta-a)^n(\zeta-z)}\,d\zeta\right|$$

$$\gamma(t) = a + Re^{it} \qquad d\zeta = \gamma'(t)\,dt$$

$$\Updownarrow$$

$$|\gamma(t)-a| = R$$



$$|f_n(z)| = \frac{1}{2\pi}\left|\int_0^{2\pi}\frac{f(a+Re^{it})Rie^{it}\,dt}{R^n e^{int}\,\underset{(\gamma(t)-z)}{\cancel{(\zeta-z)}}}\right|$$

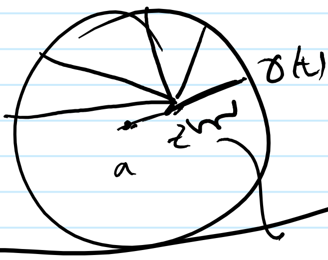
$$|f_n(z)| = \frac{1}{2\pi}\left|\int_0^{2\pi}\frac{f(a+Re^{it})}{(\gamma(t)-z)}\,\frac{Rie^{it}}{R^n e^{int}}\,dt\right|$$

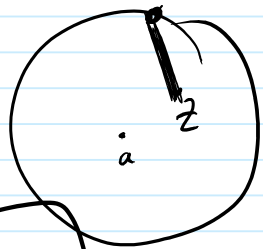
$$M = \max_{\partial D}|f|, \qquad \partial D = \text{traza}(\gamma)$$

$$\le \frac{1}{2\pi}\frac{M}{R^{n-1}}\int_0^{2\pi}\frac{dt}{|\gamma(t)-z|}$$

$$\boxed{|f_n(z)| \le \frac{M}{2\pi R^{n-1}}\int_0^{2\pi}\frac{dt}{|\gamma(t)-z|}}$$

$$|f_n(z)| \le \frac{M}{2\pi R^{n-1}}\int_0^{2\pi}\frac{dt}{|\gamma(t)-z|} \le \frac{M}{2\pi R^{n-1}}\,\frac{2\pi}{R-|z-a|}$$





$$\cancel{f_n(z) \le \frac{M}{2\pi R^{n-1}}}$$

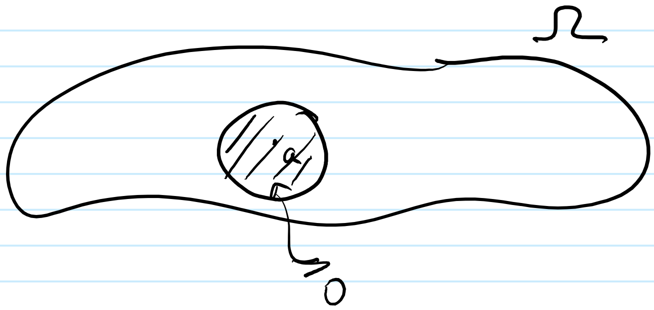
$$\boxed{R - |z-a| = \min|\gamma(t)-z|}$$

$$|f_n(z)| \le \frac{M}{R^{n-1}\,(R-|z-a|)}$$

Teorema Si f es analítica en Ω y $\exists\, a \in \Omega$:

$f^{(k)}(a)=0 \ \forall k \Rightarrow f=0$ en un entorno de a.

$$\boxed{\begin{array}{l} e^{-1/x^2} = f(x) \\ f'(0)=f''(0)= \\ \dots = 0 \end{array}}$$



Dem Tomamos un disco de radio R centrado a contenido en Ω

$$\cancel{f_k(z)} \quad f_n(z) \le \frac{M}{R^{n-1}} \frac{1}{R-|z-a|}$$

en el disco

Como $\forall k$ $f^{(k)}(a)=0 \Rightarrow$

$$f(z) = (z-a)^n f_n(z)$$

$$|f(z)| \le \frac{|z-a|^n}{R^{n-1}} \frac{1}{(R-|z-a|)} = \left(\frac{|z-a|}{R}\right)^n \frac{R}{(R-|z-a|)}$$

$\forall n \quad n \to \infty$ $\quad \downarrow 0$

$\frac{|z-a|}{R} < 1$

$$\boxed{\begin{array}{c} f(z)=0 \\ \forall z \\ \text{en el disco} \end{array}}$$