Clase 16

viernes, 20 de mayo de 2016 19:09

Clase 16

f analítica $f^{(n+1)}(z)=0 \quad \forall z \in U$

$\Longrightarrow f$ es un polinomio de grado $\leq n$

$n=0 \quad f'=0 \quad \forall z \in U \qquad f = cte \qquad | \quad f'=0$

$f'(z) = u_x(x,y) + i v_x(x,y)$

$f'(z) = (u_x(x,y), v_x(x,y))$

$u_x = v_x = 0 \qquad U$

$v_y = 0 \quad u_y = 0$

$u_x = 0 \quad v_y = 0 \Longrightarrow u = cte$

$v_x = 0 \quad v_y = 0 \Longrightarrow v = cte$

$f = u + iv = cte$

$f^{(n+1)}(z) = 0 \Longrightarrow f$ es un pol de grado $\leq n$

$$\begin{cases} n=0 \\ n-1 \quad f^{(n)}=0 \Longrightarrow f \text{ es un pol de grado } \leq n-1 \\ n \end{cases}$$

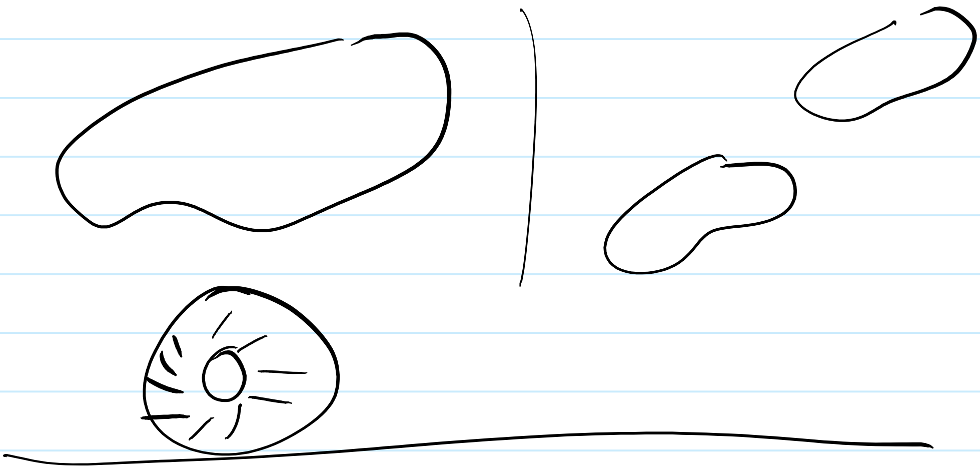
$g = f' \qquad g^{(n)}(z) = 0 \quad \forall z \in U$

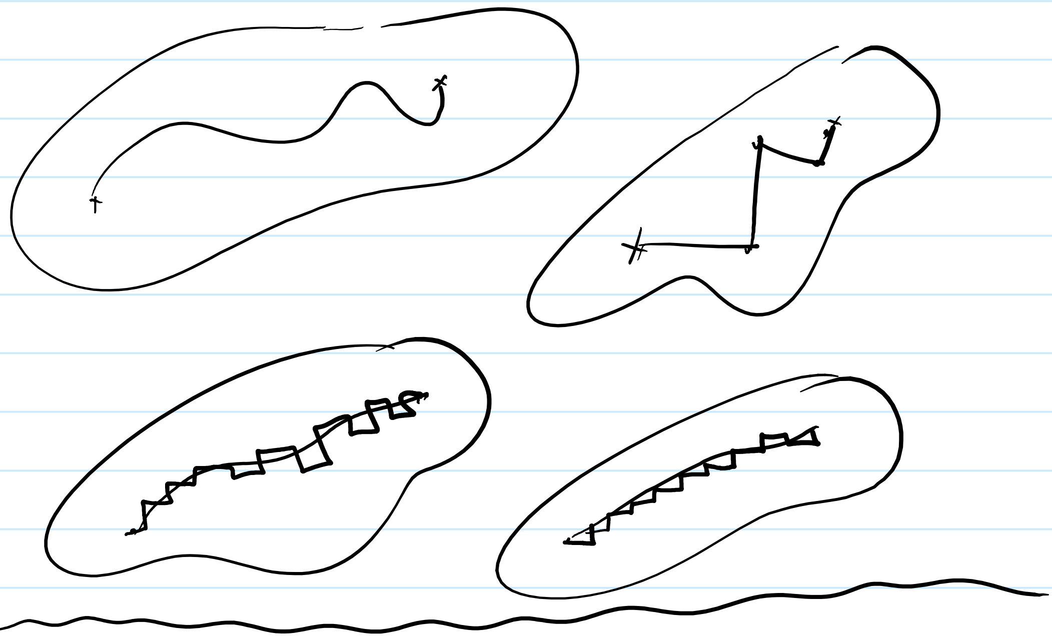
$g = a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \dots + a_1 z + a_0 = f'(z)$

$h = f - \frac{1}{n} a_{n-1} z^n - \frac{1}{n-1} a_{n-2} z^{n-1} - \dots - \frac{a_1}{2} z^2 - a_0 z$

$h' = f' - g = 0 \Longrightarrow h = cte = b_0$

$f = \frac{1}{n} a_{n-1} z^n + \frac{1}{n-1} a_{n-2} z^{n-1} + \dots + \frac{a_1}{2} z^2 + a_0 z + b_0$





$f: \mathbb{C} \to \mathbb{C}$ $\qquad a\,u(x,y)^2 + b\,v(x,y)^2 = \cancel{0}\ \text{cte}$ $\qquad$ 4c

$\Rightarrow f$ es constante

<u>Teorema</u> $f: \mathbb{C} \to \mathbb{C}$ C^0 en una región

$\exists\, g : \mathbb{R}^2 \to \mathbb{R}$. $\qquad g(u(x,y), v(x,y)) = 0$ $\qquad$ <u>General</u>

$\Rightarrow f$ es constante

<u>g para 4c $\qquad ax^2 + by^2 = g(x,y) \qquad g(s,t)$</u>

$$a u^2(x,y) + b\, v^2(x,y) = \text{cte} \longrightarrow a s^2 + b t^2 = \text{cte}$$

$$\left.\begin{array}{l} 2\,a u u_x + 2 b v v_x = 0 \\ 2\,a u u_y + 2 b v v_y = 0 \end{array}\right\} \qquad \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix} \begin{pmatrix} au \\ bv \end{pmatrix} = 0$$

$2auu_x + 2bvv_x = 0$

$$\left.\begin{array}{l} 2auu_x + 2bvv_x = 0 \\ 2auu_y + 2bvv_y = 0 \end{array}\right\} \quad (u_y \;\; v_y)(bv)$$

$$\begin{pmatrix} u_x & -u_y \\ u_y & u_x \end{pmatrix}\begin{pmatrix} au \\ bv \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$u_x^2 + u_y^2 = u_x^2 + v_x^2 = |f_x|^2 = |f'|^2 = 0$$

$f = cte$

$|f'| \neq 0 \quad au = 0 \quad bv = 0$

$\to a = 0 \quad v = 0$

$\to a = 0 \quad b = 0$

$\to u = 0 \quad b = 0$

$u = 0 \quad v = 0 \qquad f = cte$

$\cancel{g}(u, v) = \cancel{cte}\; 0$

$\cancel{g_u u_x + g_v v_x = 0}$

$g_u u_y$,

$f : U \to \mathbb{R}^2 \xrightarrow{g} \mathbb{R}$ $\quad f\,g$ componibles

$U \subset \mathbb{R}^2$, (x, y), (s, t)

$f(x,y) \qquad g(s,t)$

$$g(u(x,y), v(x,y)) = 0 \leftarrow \frac{\partial}{\partial x}, \frac{\partial}{\partial y}$$

$$g_s u_x + g_t v_x = 0$$

$$g_s u_y + g_t v_y = 0$$

$$\begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix}\begin{pmatrix} g_s \\ g_t \end{pmatrix} = \vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad |z^2| = |z|^2$$

$$\begin{pmatrix} u_x & -u_y \\ u_y & u_x \end{pmatrix}\begin{pmatrix} g_s \\ g_t \end{pmatrix} = 0$$

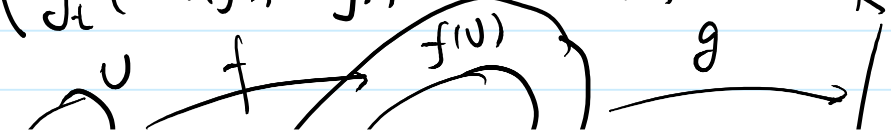
$\cancel{|f'|^2 \neq 0}$

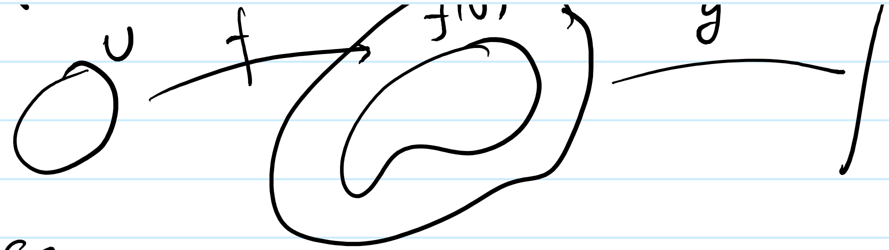
$|f'|^2 = 0$

$|f'| = 0$

$g_s = 0 \quad g_t = 0$

Rup. $\left\{\begin{array}{l} g_s(u(x,y), v(x,y)) = 0 \\ g_t(u(x,y), v(x,y)) = 0 \end{array}\right.$ D $\Rightarrow f = cte$





~~$g(u(x,y),v(x,y))$~~

$g(u(x,y), v(x,y)) = 0$ $\quad g \circ f = 0$

$\boxed{g \text{ en } f(U) \text{ tiene derivada cero}}$ $\Big) \; g = \text{cte}$

¿Si $f: U \to \mathbb{C}$ es analítica cuánta
Inf sobre cuál es la estructura
de $f(U)$ es?

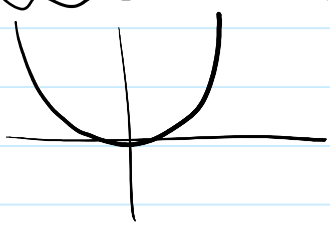
$\boxed{f(U) \text{ es abierto}}$ $\quad$ $\boxed{\text{Teorema de la función abierta}}$

U abierto conexo

$\Longrightarrow f(U)$ abierto conexo

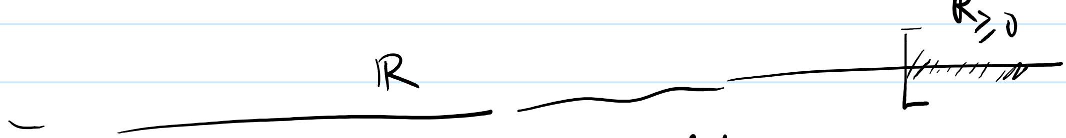
$\Longrightarrow g = \text{cte}$ $\quad g \neq 0$

<u>Teorema</u> Si $g(u(x,y), v(x,y)) = 0$ $\quad f = u + iv$
analítica $\Longrightarrow$ ~~g es no constante~~ en $f(U)$
f es constante



$y = x^2 \qquad f(x) = x^2$

$f(\mathbb{R}) = \mathbb{R}_{\geq 0}$



Dos teoremas sobre funciones analíticas

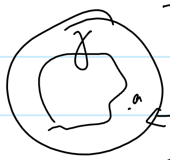
Teorema de Morera $\qquad$ (Liouville)

Teoremas sobre funciones analíticas acotadas

Fórmula integral de Cauchy

$f: D \longrightarrow \mathbb{C}$ D abierto disco

$\gamma: [\alpha,\beta] \longrightarrow D$ curva cerrada $\gamma(a)=\gamma(b)=0$



$D \xrightarrow{f} \mathbb{C}$

$a \in D$

$a \notin$ traza γ

$$\boxed{n(\gamma,a) f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz}$$

$$n(\gamma,a) f(a) = \frac{1}{2\pi i}\int_\alpha^\beta \frac{f(\gamma(t))}{(\gamma(t)-a)}\,\gamma'(t)\,dt$$

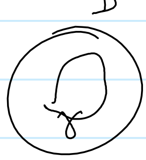
$$n(\gamma,z) f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta-z}\,d\zeta$$

F continua definida sobre γ

$$\boxed{\int_\gamma \frac{F(\zeta)}{\zeta-z}\,d\zeta = G(z)}$$

y definimos una nueva función G por la fórmula

Situación:



$D \xrightarrow{F} \mathbb{C}$ F sea continua

γ curva en D $\forall z$ fuera del trazo de γ definimos

$$G(z) = \int_\gamma \frac{F(\zeta)}{\zeta-z}\,d\zeta$$

$F \longrightarrow G = \mathcal{J}(F)$

Si F es analítica en el disco $\boxed{G(z) = 2\pi i\, n(\gamma,z) F(z)}$

$$n(\gamma,z) f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta-z}\,d\zeta$$

$F \longrightarrow G = \mathcal{J}(F)$

$n(\gamma, z) f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta$

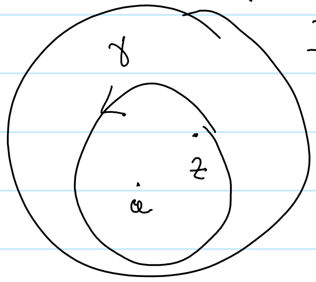
$F \longrightarrow G = \mathcal{F}(F)$

$$\underbrace{2\pi i\, n(\gamma, z) f(z)}_{F} = \underbrace{\int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta}_{G}$$

$\mathcal{F}(f) = 2\pi i\, n(\gamma, z) f$

f analítica

Formula integral de Cauchy para γ particular



D (una circunferencia dentro de D)

$n(\gamma, z) = 1$

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta$$

$$G(z) = \int_\gamma \frac{F(\zeta)}{\zeta - z} d\zeta$$

En la misma situación de la curva γ los puntos etc tomemos

Calcular $G'(z)$

G es analítica con respecto a z

$$\varphi(t) = \int_a^b f(x, t)\, dx$$

f es derivable con respecto a t

$\Rightarrow$ la φ también.

G es derivable

$$\frac{G(z+h) - G(z)}{h} = \frac{1}{h} \int_\gamma F(\zeta) \left(\frac{1}{\zeta - z - h} - \frac{1}{\zeta - z} \right) d\zeta$$

$$G(z) = \int_\gamma \frac{F(\zeta)}{\zeta - z} d\zeta$$

$$\frac{G(z+h) - G(z)}{h} = \frac{1}{h} \int_\gamma F(\zeta) \frac{h}{(\zeta - z)(\zeta - z - h)} d\zeta$$

$$\varphi(t) = \int_a^b f(x, t)\, dx$$

$$\frac{G(z+h) - G(z)}{h} = \int_\gamma \frac{F(\zeta)}{(\zeta - z)(\zeta - z - h)} d\zeta$$

$\lim_{t \to t_0} \varphi(t) = \varphi(t_0)$

$$\lim_{t\to t_0}\int_a^b f(x,t)\,dt = \int_a^b f(x,t_0)\,dt$$

$$\cancel{\lim_{t\to t_0}\int_a^b f(x,t)\,dt \to \int_a^b f(x,t_0)\,dt}$$

$$\lim_{h\to 0}\frac{G(z+h)-G(z)}{h} = \int_\gamma \frac{F(\zeta)\,d\zeta}{(\zeta-z)^2}$$

Conclusión: Si $G(z)=\int_\gamma \frac{F(\zeta)}{\zeta-z}\,d\zeta$

$$\Rightarrow \boxed{G'(z)=\int_\gamma \frac{F(\zeta)}{(\zeta-z)^2}\,d\zeta}$$

$$\int_\gamma \frac{F(\zeta)}{(\zeta-z)^2}\,d\zeta \to \int_\gamma \frac{F(\zeta)}{(\zeta-z)^3}\,d\zeta$$

$$\frac{d}{dz}\left(\frac{F(\zeta)}{\zeta-z}\right) = F(\zeta)\frac{d}{dz}\left(\frac{1}{\zeta-z}\right) = F(\zeta)\frac{1}{(\zeta-z)^2}$$

F

$$f(z)=\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{\zeta-z}\,d\zeta$$

G

$$G'(z)=f'(z)=\frac{1}{2\pi i}\int \frac{f(z)}{(\zeta-z)^2}\,d\zeta$$

$$f(z)=\frac{1}{2\pi i}\boxed{\int_\gamma \frac{f(\zeta)}{\zeta-z}\,d\zeta} \Rightarrow f'(z)=\frac{1}{2\pi i}\boxed{\int_\gamma \frac{f(\zeta)}{(\zeta-z)^2}\,d\zeta}$$

f con derivada

f' tiene derivada

Conclusión en la situación de ambos teoremas

cerrada

$f(z) = \frac{1}{2\pi i}\int \frac{f(\zeta)}{...}\,d\zeta$ ... existen todas las derivadas n-ésimas

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta$$

y existen todas las derivadas n-ésimas $\forall n$ y valen: