Practico III. Si la curva γ no es cerrada hay que utilizar la definición

O sea $\int_\gamma f(z)dz = \int_a^b \underbrace{f(\gamma(t))\cdot\gamma'(t)}dt$

$\gamma:[a,b]\longrightarrow \mathbb{C}$ $\quad t\longmapsto \gamma(t)$

$\gamma(t)=x(t)+iy(t)$ $\quad \gamma'(t)=x'(t)+iy'(t)$

Se supone que las dos integrales en cuestión

$\int_a^b Re\left(f(\gamma(t))\cdot\gamma'(t)\right)dt$ y $\int_a^b Im(\quad)dt$

son calculables

Si en cambio el camino γ es cerrado entonces es posible que sea absolutamente necesario utilizar la fórmula de Cauchy o la de Green

$f(z)=f(x+iy)=u(x,y)+iv(x,y)$ y γ es el borde de un dominio K (parametrizados en sentido positivo) entonces

$$\int_{\gamma=\partial K} f(z)dz = \iint_K (v_x-u_y)\,dxdy$$

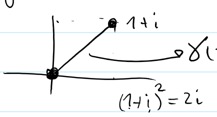
Si f es derivable en un entorno de K

$v_x-u_y=0$

Practico III Ej. 1

$\underbrace{f(z)=z(z-1)}$, $Re\,z=x$

γ es el segmento $[0,1+i]$



$\gamma'(t)=1+i$

$\int_0^1 t(1+i)\left(t(1+i)-1\right)(1+i)\,dt=(1+i)^2\int_0^1\left((1+i)t^2-t\right)dt$ $\quad \gamma(t)=(1+i)t=t+it$

$(1+i)^2\left[(1+i)\frac{t^3}{3}-\frac{t^2}{2}\right]_0^1=2i\cdot\left(\frac{1+i}{3}-\frac{1}{2}\right)$

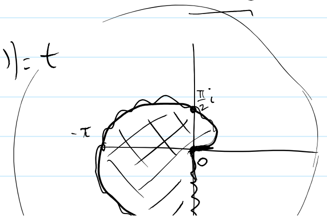
Si en cambio $f(z)=Re(z)=x \Rightarrow f(\gamma(t))=Re(\gamma(t))=t$

$\int_0^1 t\,(1+i)\,dt=(1+i)\frac{t^2}{2}\Big]_0^1=\frac{1+i}{2}$

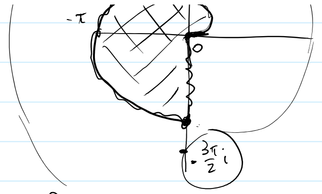
~~Ej. 3~~ $\cancel{f(z)=\frac{e^z-e^{-z}}{z^n}}$

Ej. 2 $f(z)=\frac{1}{z+2i}$ $\quad \gamma(t)=e^{it}\cdot t$ donde $t\in\left[0,\frac{3\pi}{2}\right]$

$|\gamma(t)|=t$ $\quad arg(\gamma(t))=t$



$\gamma(t)=te^{it}$



$\gamma(t)=te^{it}$

$\gamma'(t)=e^{it}(1+it)$

$f(\gamma(t))=\dfrac{1}{te^{it}-2i}$

$$\int_0^{\frac{3\pi}{2}}\frac{e^{it}(1+it)}{te^{it}-2i}\,dt$$

Pequeño cambio en la letra:

$$\boxed{f(z)=\frac{1}{z-2i}}$$

Coment. de Román: f es derivable en $\mathbb{C}\setminus\{2i\}$ y se puede invocar la fórmula de Cauchy

$$\Longrightarrow \int \frac{1}{z-2i}\,dz=0$$

$\Longrightarrow$ nos reducimos a calcular $\int \frac{1}{z-2i}\,dz$ a lo largo del segmento $[0,-\frac{3\pi}{2}i]$

$\tau(t)=t\left(-\frac{3\pi i}{2}\right)\quad t\in[0,1]\qquad \tau'(t)=-\frac{3\pi i}{2}$

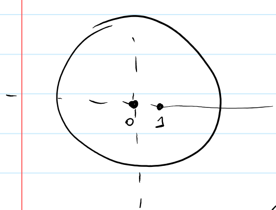
$$\int_\tau f(z)\,dz=\int_0^1\frac{1}{\left(-\frac{3\pi i}{2}\right)t-2i}\left(-\frac{3\pi i}{2}\right)dt=\frac{-\frac{3\pi i}{2}}{-i}\int_0^1\frac{dt}{\frac{3\pi}{2}t+2}=\frac{3\pi}{2}\int_0^1\frac{dt}{\frac{3\pi}{2}\left(t+\frac{4}{3\pi}\right)}=\int_0^1\frac{dt}{t+\frac{4}{3\pi}}=$$

$$=\log\left(t+\frac{4}{3\pi}\right)\Big|_0^1=\log\left(1+\frac{4}{3\pi}\right)-\log\left(\frac{4}{3\pi}\right)$$

$$=\log\left(\frac{3\pi+4}{4}\right)$$

Ej. 5 $\gamma(t)=3e^{it}\quad t\in[0,2\pi]$

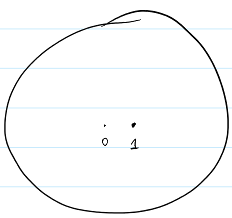
$$f(z)=\frac{5z-2}{z(z-1)}=\frac{a}{z}+\frac{b}{z-1}=\frac{a(z-1)+bz}{z(z-1)}=\frac{(a+b)z-a}{z(z-1)}$$



$$\begin{cases}a+b=5 \Rightarrow b=3\\ a=2\end{cases}$$

$$f(z)=\frac{2}{z}+\frac{3}{z-1}\Longrightarrow\int_\gamma f\,dz=2\int_\gamma\frac{1}{z}\,dz+3\int_\gamma\frac{1}{z-1}\,dz$$

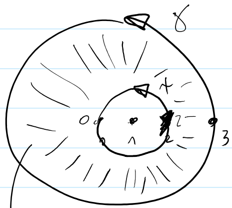
$$2\int_\gamma\frac{1}{z}\,dz\ \Big|\ \gamma(t)=3e^{it},\ \gamma'(t)=3ie^{it}\qquad 2\int_0^{2\pi}\frac{1}{3e^{it}}3ie^{it}\,dt=i2\int_0^{2\pi}dt=2it\Big|_0^{2\pi}=4\pi i$$



$$\int_\gamma\frac{1}{z-1}\,dz=\int_0^{2\pi}\frac{1}{3e^{it}-1}3ie^{it}\,dt=3i\int_0^{2\pi}\frac{\cos t+i\,\mathrm{sen}\,t}{3\cos t-1+i3\,\mathrm{sen}\,t}\,dt=3i\int_0^{2\pi}\frac{(\cos t+i\,\mathrm{sen}\,t)(3\cos t-1-i3\,\mathrm{sen}\,t)}{(3\cos t-1)^2+9\,\mathrm{sen}^2t}\,dt$$

$$=3i\underbrace{\int_0^{2\pi}\frac{[\cos t(3\cos t-1)+3\,\mathrm{sen}^2t]}{(3\cos t-1)^2+9\,\mathrm{sen}^2t}\,dt}+3i\cdot i\int_0^{2\pi}\dots$$

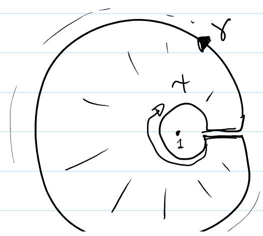
muy complicado



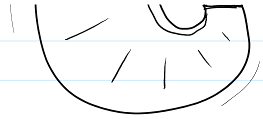
$\gamma(t)=3e^{it},\ t\in[0,2\pi]$

$\tau(s)=1+e^{is},\ s\in[0,2\pi]$

$K\Rightarrow\partial K=\gamma-\tau$



Como $1\notin K$ la fórmula de Cauchy nos dice que



Como $1 \notin K$ la fórmula de Cauchy nos dice que

$$\int_{\partial K} f(z) = \int_{\partial K} \frac{1}{z-1}\,dz = 0 = \int_\gamma \frac{1}{z-1}\,dz - \int_\tau \frac{1}{z-1}\,dz$$

En resumen: $$\int_\gamma \frac{dz}{z-1} = \int_\tau \frac{dz}{z-1} = \left|\begin{array}{l} \tau(s) = 1+e^{is} \\ \tau'(s) = i e^{is} \\ s\in[0,2\pi] \end{array}\right. \int_0^{2\pi} \frac{1}{e^{is}}\, i e^{is}\,ds = i\int_0^{2\pi} ds = 2\pi i$$

En total obtenemos:

$$\int_\gamma \frac{5z-2}{z(z-1)}\,dz = 4\pi i + 6\pi i = 10\pi i$$

Ej. 3 sugerencias:

$$f(z) = \frac{e^z - e^{-z}}{z^n} \qquad \gamma(t) = e^{it} \;\Big|\; \gamma'(t) = i e^{it}$$

$$\int_\gamma f(z)\,dz = i\int_0^{2\pi} \left(\frac{e^{e^{it}} - e^{-e^{it}}}{e^{nit}}\right) e^{it}\,dt$$

$$\cancel{e^{e^{it}} = e^{\cos t + i \operatorname{sen} t} = e^{\cos t}(\cos t + i \operatorname{sen} t) \text{ etc}}$$

$$e^z = 1 + z + \tfrac{1}{2}z^2 + \tfrac{1}{3!}z^3 + \dots$$

$$\int \frac{e^z}{z^2} = \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2} + \frac{z}{3!} + \frac{z^2}{4!} + \dots$$

$$e^z - 1 - z \qquad \Big\| \qquad \frac{\operatorname{sen} x}{x}$$

$$e^z - 1 - z = \tfrac{1}{2}z^2 + \dots$$

$\dfrac{e^z - 1 - z}{z^2} = \dfrac{1}{2} + O(z)$ es analítica en todo $\mathbb{C}$

$\longrightarrow$ sing. evitable en $0 \in \mathbb{C}$

$$\frac{e^z}{z^2} = \frac{1+z}{z^2} + \underbrace{\text{analít.}}$$

$$\int_\gamma \frac{e^z}{z^2}\,dz = \int_\gamma \frac{1+z}{z^2}\,dz + \overbrace{\int_\gamma \text{analit.}}^{=0}$$

$$= \int_\gamma \frac{1}{z^2}\,dz + \int_\gamma \frac{1}{z}\,dz$$