Clase 11

viernes, 06 de mayo de 2016 19:27

$|v|^2 = v\cdot\bar{v}$

Ej 5 $|z|, |w| < 1 \Rightarrow \left|\dfrac{z-w}{1-z\bar{w}}\right|^2 \overset{??}{\leq} 1^2$

$\left(\dfrac{z-w}{1-z\bar{w}}\right)\left(\dfrac{\bar{z}-\bar{w}}{1-\bar{z}w}\right) \leq 1 \iff \underbrace{(z-w)(\bar{z}-\bar{w})}_{\mathbb{R}^+} \overset{?}{\leq} \underbrace{(1-z\bar{w})(1-\bar{z}w)}_{\mathbb{R}^+}$

$z\bar{z} + w\bar{w} - \cancel{w\bar{z}} - \cancel{\bar{w}z} \overset{?}{\leq} 1 - z\bar{z}w\bar{w} - \cancel{z\bar{w}} - \cancel{\bar{z}w}$

$\alpha = z\bar{z} = |z|^2 < 1$ $\quad \boxed{\beta = w\bar{w} = |w|^2 < 1}$

$\boxed{\alpha + \beta \leq 1 - \alpha\beta}\ ? \iff \alpha - \alpha\beta \overset{?}{\leq} 1 - \beta$

$\alpha\cancel{(1-\beta)} \overset{?}{\leq} \cancel{1-\beta}$

Si $\beta < 1 \Rightarrow \alpha \leq 1$ ✓

1) Si $\beta < 1$ y $\alpha < 1 \Rightarrow \left|\dfrac{z-w}{1-z\bar{w}}\right| < 1$

2) Si $\beta = 1$ (o sea: $|w| = 1$) habíamos llegado a la desig.:

$\alpha + 1 \leq 1 - \alpha$

Pero aquí tenemos igualdad $\Rightarrow$ antes también

O sea que $\beta = 1 \Rightarrow \left|\dfrac{z-w}{1-z\bar{w}}\right| = 1$

3) Cuando $|z|$ y $|w|$ tienen módulo 1 hay que verificar que $\boxed{1 - z\bar{w} \neq 0.}$

Recordemos la representación polar de z y w:

$\begin{cases} z = e^{i\theta} \\ w = e^{i\theta'} \Rightarrow \bar{w} = e^{i(-\theta')} \Rightarrow z\bar{w} = e^{i(\theta-\theta')} \end{cases}$

Vemos que $z\bar{w} = 1 \iff e^{i(\theta-\theta')} = 1 \iff e^{i\theta} = e^{i\theta'} \iff z = w$

En este caso el cociente es $\frac{0}{0} \Rightarrow$ no está definido.

Supongamos entonces que $|z|, |w| = 1$ pero $z \neq w$

$\dfrac{z-w}{1-z\bar{w}} = \dfrac{e^{i\theta} - e^{i\theta'}}{1 - e^{i(\theta-\theta')}} = \cancel{\dfrac{e^{i\theta}(1 - e^{i(\theta'-\theta)})}{1 - e^{i(\theta-\theta')}}} = \dfrac{e^{i\theta'}(e^{i(\theta-\theta')} - 1)}{1 - e^{i(\theta-\theta')}}$

$e^{i\theta'}$

$$= - e^{i\theta'} = -w$$

Ej. 6 $\quad e^{ix} = \cos x + i \operatorname{sen} x \Rightarrow e^{inx} = \underbrace{\cos nx + i \operatorname{sen} nx = (\cos x + i \operatorname{sen} x)^n}$

$$n=2 \Rightarrow \cos 2x + i \operatorname{sen} 2x = (\cos x + i \operatorname{sen} x)^2 = \cos^2 x - \operatorname{sen}^2 x + i\,2\cos x \operatorname{sen} x$$

$$\Rightarrow \begin{cases} \boxed{\cos 2x} = \cos^2 x - \operatorname{sen}^2 x = \boxed{1 - 2\operatorname{sen}^2 x} \\ \operatorname{sen} 2x = 2\cos x \operatorname{sen} x \end{cases}$$

$$\cos^2 x + \operatorname{sen}^2 x = 1 \Rightarrow \cos^2 x = 1 - \operatorname{sen}^2 x$$

$\cos 3x$ se puede presentar como polinomio de grado 3 en los valores $\cos x$ y $\operatorname{sen} x$

Ej. 9 Recuerden que: $\cos 2x = 1 - 2\operatorname{sen}^2 x$

$$2\operatorname{sen}^2 x = 1 - \cos 2x$$

$$\Rightarrow \boxed{\operatorname{sen}^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x}$$

$$\int \operatorname{sen}^2 x \, dx = \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx = \frac{1}{2}\int 1\,dx - \frac{1}{2}\int \cos 2x \, dx$$

$$= \frac{x}{2} - \frac{1}{2}\cdot\frac{\operatorname{sen} 2x}{2}$$

$$\begin{cases} \operatorname{sen} z = \dfrac{e^{iz} - e^{-iz}}{2i} \\ \cos z = \dfrac{e^{iz} + e^{-iz}}{2} \end{cases}$$

$$e^{ix} = \cos x + i \operatorname{sen} x$$

$$e^{-ix} = \cos x - i \operatorname{sen} x$$

$$\frac{e^{ix} + e^{-ix}}{2} = \cos x \qquad \frac{e^{ix} - e^{-ix}}{2i} = \operatorname{sen} x$$

$$\operatorname{sen}^3 x = \frac{(e^{ix} - e^{-ix})^3}{(2i)^3} = \frac{e^{i3x} - 3e^{ix} + 3e^{-ix} - e^{-i3x}}{-8i}$$

Recuerden que: $\dfrac{e^{i3x} - e^{-i3x}}{2i} = \operatorname{sen} 3x$

$$\Rightarrow \operatorname{sen}^3 x = \frac{2i \operatorname{sen} 3x - 3\overbrace{(e^{ix} - e^{-ix})}^{2i \operatorname{sen} x}}{\ldots} \;\; \ldots 2i \ldots \; (+6\cancel{i}\operatorname{sen} x)$$

$$\sin^3 x = \frac{2i\sin 3x - 3(e^{ix}-e^{-ix})}{-8i} = \frac{2i}{-8i}\cdot\sin 3x + \frac{(+6i\sin x)}{+8i}$$

$$\boxed{\sin^3 x = -\frac{\sin 3x}{4} + \frac{3}{4}\sin x}$$

Formula integral de Cauchy

γ

$$\xrightarrow{f} \mathbb{C} \quad \left(\int_\gamma \frac{dz}{z-a}\right) f(a) = \int_\gamma \frac{f(z)}{z-a}\,dz$$

$$\int_\gamma \frac{dz}{z-a} = 2\pi i\, n(\gamma,a) = \cancel{N(\gamma,a)}$$

$$\boxed{n(\gamma,a) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}}$$

índice de γ con respecto a a

$a \neq z$

$a \notin \gamma$

$$\boxed{\int_\gamma f(z)\,dz = \int_\alpha^\beta f(\gamma(t))\gamma'(t)\,dt}$$

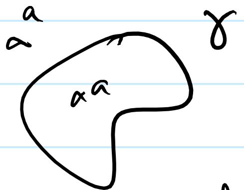
$\gamma : [\alpha,\beta] \to \mathbb{C}$

$z = \gamma(t)$

$dz = \gamma'(t)\,dt$

$$\int_\gamma f(z)\,dz = \int_\alpha^\beta f(\gamma(t))\gamma'(t)\,dt$$

$\alpha \le t \le \beta$



$$n(\gamma,a) = \frac{1}{2\pi}\int_\gamma \frac{dz}{z-a}$$

$$= \frac{1}{2\pi}\int_\alpha^\beta \frac{\gamma'(t)\,dt}{\gamma(t)-a}$$

$f(z) = \frac{1}{z-a}$

$$\frac{\;}{2\pi} \int_{\gamma} \gamma(t) - a$$

Probaremos que $n(\gamma, a) \in \mathbb{Z}$

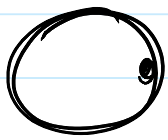
$$n(S^1, 0) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z}$$

$$\gamma(t) = e^{it} \qquad 0 \leq t \leq 2\pi \qquad \gamma'(t) = ie^{it}$$

$$n(S^1, 0) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\gamma'(t)\,dt}{\gamma(t)} = \frac{1}{2\pi i} \int_0^{2\pi} \frac{ie^{it}}{e^{it}}\,dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} dt = \frac{2\pi}{2\pi} = 1$$

$$n(S^1, 0) = \frac{1}{2\pi} \int_0^{4\pi} dt = \frac{4\pi}{2\pi} = 2$$



$t \in [0, 4\pi]$

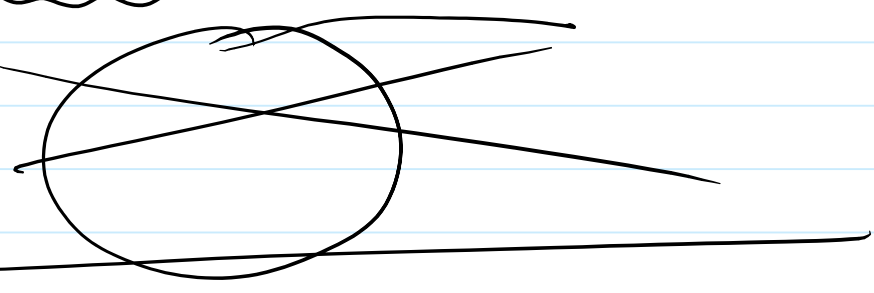
~~$[0, 4\pi]$~~

$$\frac{1}{2\pi i} \int_{-\gamma} \frac{dz}{z-a} = n(-\gamma, a)$$

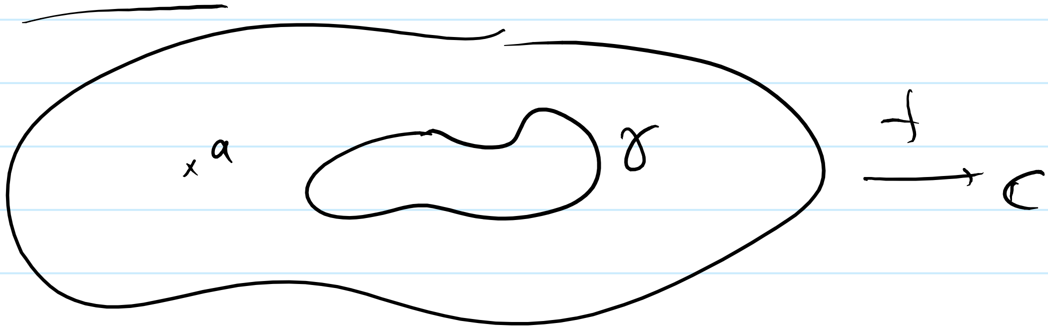
$$\| $$

$$-\frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z-a} = -n(\gamma, a)$$

~~Teorema~~



Teorema



$\gamma : [\alpha, \beta] \to \mathbb{C}$

$\frac{1}{2\pi i} \int_\alpha^\beta \frac{\gamma'(t)\,dt}{\gamma(t)-a}$ es entero

$$\boxed{H(s) = \int_\alpha^s \frac{\gamma'(t)}{\gamma(t)-a}\,dt} \qquad \alpha \le s \le \beta$$

¿$H(\beta)$?

$H(\alpha) = 0$

$$\boxed{H'(s) = \frac{\gamma'(s)}{\gamma(s)-a}}$$

$$\left(e^{-H(s)}(\gamma(s)-a)\right)' = -H'(s)e^{-H(s)}(\gamma(s)-a) + e^{-H(s)}(\gamma'(s))$$

$$= -\frac{\gamma'(s)}{\cancel{\gamma(s)-a}} e^{-H(s)}(\cancel{\gamma(s)-a}) + e^{-H(s)}\gamma'(s) = 0$$

$$e^{-H(s)}(\gamma(s)-a) = e^{-H(\alpha)}(\gamma(\alpha)-a)$$

$$e^{-H(s)} = \frac{\gamma(\alpha)-a}{\gamma(s)-a} \qquad s = \beta$$

$$e^{-H(\beta)} = \frac{\gamma(\alpha)-a}{\gamma(\beta)-a} \qquad \gamma(\alpha) = \gamma(\beta)$$

$$e^{-H(\beta)} = 1 \longleftrightarrow e^{H(\beta)} = 1$$

$$\Downarrow$$

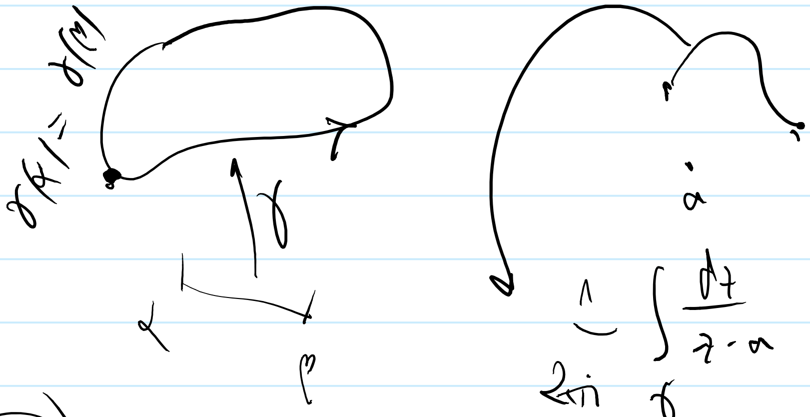
$\cancel{H(\beta)} =$

$$\cancel{\oint \frac{dz}{z-a}}\ \int \frac{dz}{z-a} = H(\beta) = 2\pi i\, n \longrightarrow \text{entero}$$

QED

$$\forall \gamma \quad \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a} = n(\gamma, a) \in \mathbb{Z}$$

$\forall \gamma \quad \frac{1}{2\pi i}\int_\gamma \frac{\quad}{z-a} = n(\gamma, a) \in \mathbb{Z}$

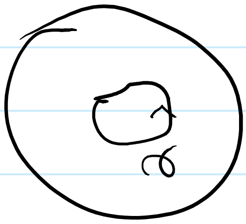


$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}$$

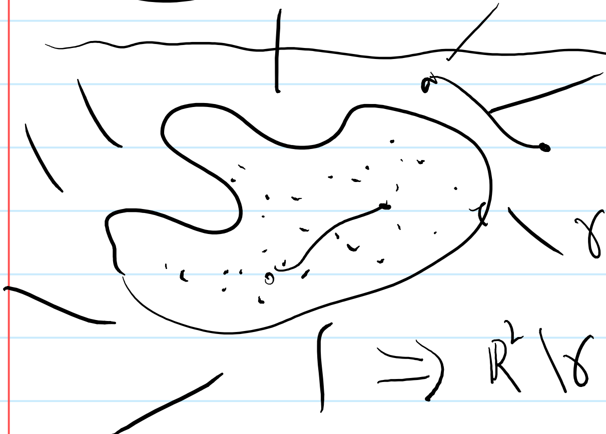
$$n(\gamma,a)\, f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\, dz$$

$\in \mathbb{Z}$

$a \qquad U = D \qquad \gamma \qquad a \notin D$



$n(\gamma, a) = 0 \qquad n(\gamma, a) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}$



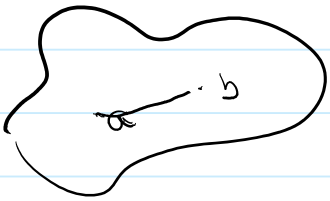
Teorema de Jordan

γ cerrada en el plano

$\Longrightarrow \mathbb{R}^2 \setminus \gamma$ tiene dos componentes

conexas una acotada y otra no acotada

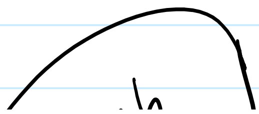
Lema



- el segmento que une
- a con b no corta a la

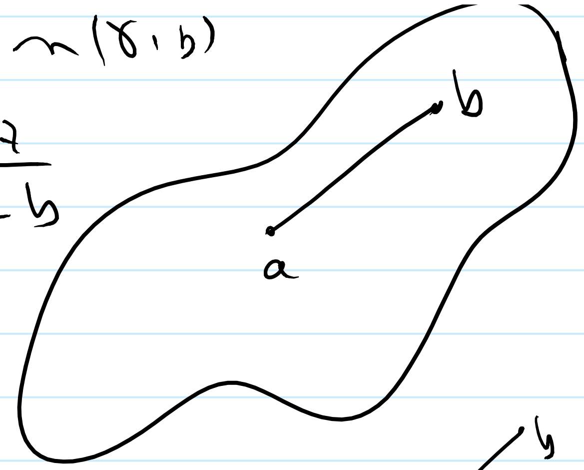
a y b en la misma componente curva γ

$n(\gamma, a)$ y $n(\gamma, b)$



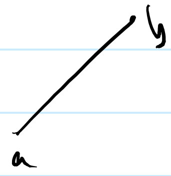
$n(\gamma, a)$ y $n(\gamma, b)$

$$\int_\gamma \frac{dz}{z-a} \qquad \int_\gamma \frac{dz}{z-b}$$



$\frac{z-a}{z-b}$ valores

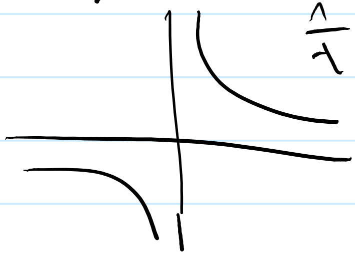
Qué tomo sobre el segmento



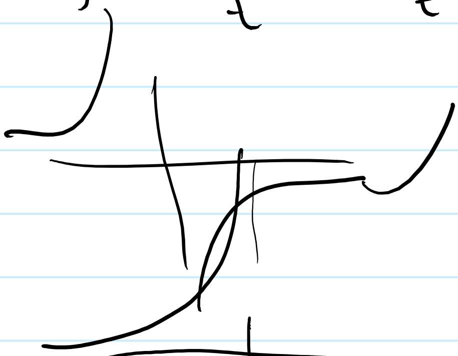
$ta+(1-t)b \qquad 0 \leq t \leq 1$

$$\frac{ta+(1-t)b-a}{ta+(1-t)b-b} = \frac{(1-t)(b-a)}{t\ \ (a-b)} = \frac{t-1}{t}$$

$\frac{1}{t}$

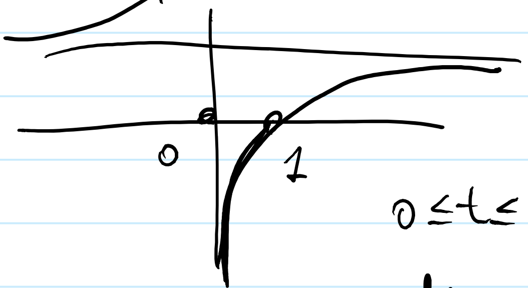


$f(t) = \frac{t-1}{t} = 1 - \frac{1}{t}$



$-\frac{1}{t}$

$t \in \mathbb{R}$

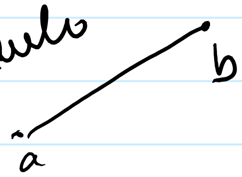


$1 = 1$

$t=1 \quad f(1)=0$

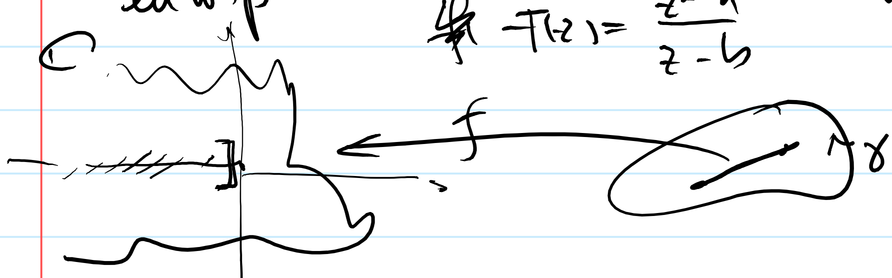
$0 \leq t \leq 1 \quad -\infty < f(t) \leq 0$

$\frac{z-a}{z-b}$ en los puntos del intervalo



toma valores ~~reales~~ reales negativos

$T(z) = \frac{z-a}{z-b}$

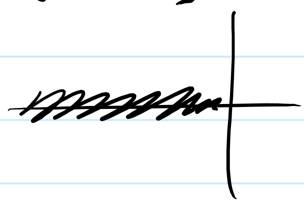


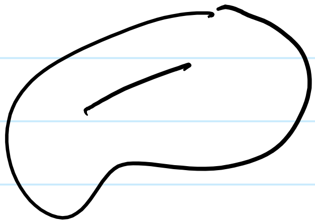
Saltito en la rama $\mathbb{C}-(-\infty, 0]$

Saltito | en la región $\mathbb{C} - (-\infty, 0]$

existe una función $\lg z$

es la inversa de la exponencial



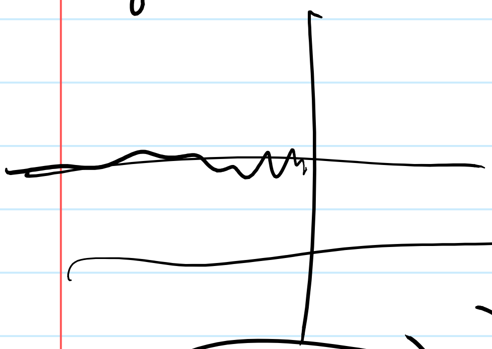


$$\lg\left(\frac{z-a}{z-b}\right) = \lg(z-a) - \lg(z-b)$$

$$(\quad)' = \frac{1}{z-a} - \frac{1}{z-b}$$

$\frac{1}{z-a} - \frac{1}{z-b}$ tiene una primitiva

$$\int_\gamma \frac{dz}{z-a} - \frac{dz}{z-b} = 0 \Longrightarrow \boxed{\int_\gamma \frac{dz}{z-a} = \int_\gamma \frac{dz}{z-b}}$$

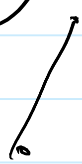


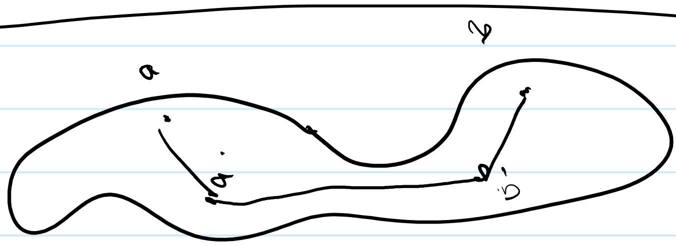
$\lg w = z$

$e^z = w$

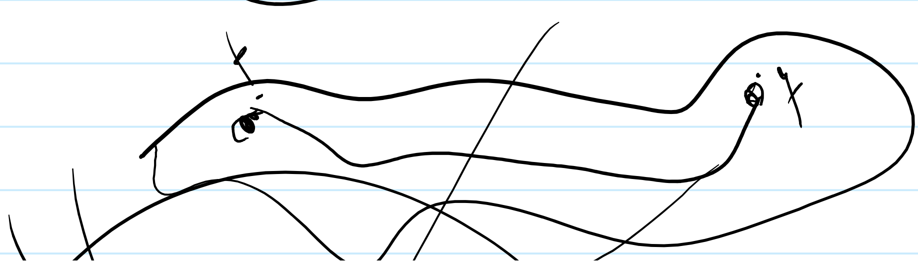
γ

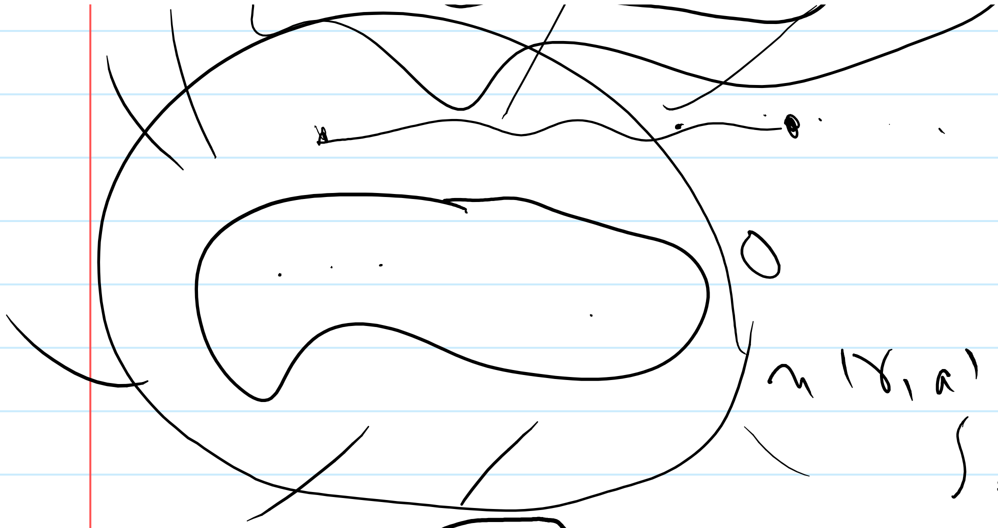
$$n(\gamma, a) = n(\gamma, b)$$





$$\int \frac{dz}{z-a}$$





$n(\gamma, a)$

$n(\gamma, a)$

$\int_{\gamma \cup \delta} \frac{dz}{z-a} =$