Clase 10

jueves, 05 de mayo de 2016 19:43

$f : U \to \mathbb{C}$, $U \subset \mathbb{R}^2$, $\mathbb{C} = \mathbb{R}^2$, $\mathbb{R}^2 = \mathbb{C}$

$f : U \to \mathbb{R}^2$, $U \subset \mathbb{R}^2$

$f(z) = f(x+iy) = u(x+iy) + i\,v(x+iy)$

$f(x,y) = (u(x,y), v(x,y))$

f es analitica ⇔ $\forall z_0 \in U$

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \ \exists$$

$x = x_0 + tp$

$y = y_0 + tq$

$f : \mathbb{R}^2 \to \mathbb{R}^2$, (x_0, y_0)

~~$\forall (x,y) \in \mathbb{R}$~~ $= (p,q)$

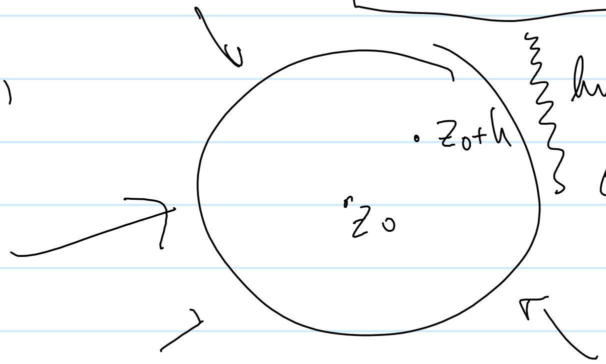
$\forall u \in \mathbb{R}^2$

$$\lim_{t \to 0} \frac{f(x_0 + tp, y_0 + tq) - \dots}{t} \ \exists = (df)_{(x_0,y_0)}(p,q)$$

$$\exists \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$\boxed{\exists \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}}$$

$h = (p,q)$



$$\lim_{(p,q) \to (0,0)} \frac{f(x_0 + p, y_0 + q) - f(x_0, y_0)}{p + iq}$$

$f : U \to \mathbb{C}$ tiene derivada ⇒ f tiene deriv real (es diferenciable) y tiene derivada compleja $\Longleftarrow$ $\Longrightarrow$ (es diferenciable) y además verifica la ecuación de Cauchy Riemann

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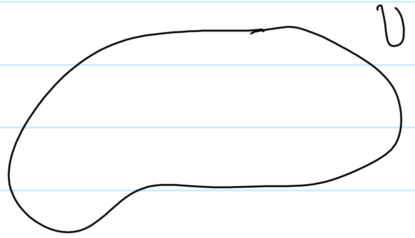
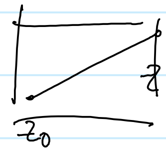


$U \xrightarrow{f} \mathbb{C}$

U (*)

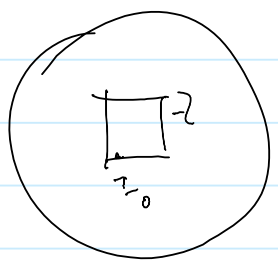
$\Downarrow$

$\exists z_0 \in U : \forall z \in U$



el rectángulo que tiene al segmento como diagonal

$\Longrightarrow$ el rectángulo está en U



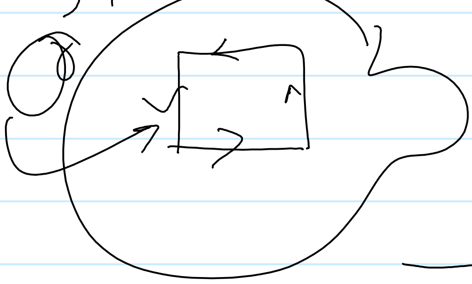
U verifica (*) $\Longrightarrow$ $\forall \gamma : [\alpha, \beta] \to U$ $\cancel{\forall}$

$\gamma(\alpha) = \gamma(\beta)$ $\Longrightarrow$ $\int_\gamma f(z)\,dz = 0$

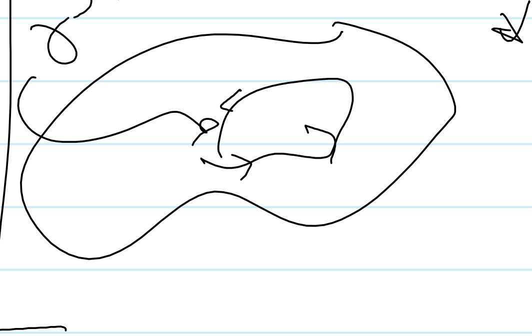
diferenciable

$$\int_\gamma f(z)\,dz = \int_\alpha^\beta f(\gamma(t))\,\gamma'(t)\,dt$$

$\int_\gamma f(z)\,dz = 0$



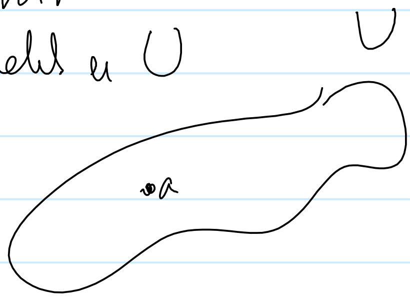
$\int_\gamma f(z)\,dz$



$$f: U \to \mathbb{C}, \quad U \subset \mathbb{C}$$

U abierto

diferenciable en U



$$f: U - \{a\} \to \mathbb{C}$$

f tiene una singularidad

Defn La singularidad en a es evitable si

$$\exists \lim_{z \to a} (z-a)f(z) = 0$$

Ejemplo $f(z) = \dfrac{e^z - 1}{z}$ en 0 tiene una singularidad evitable

$$\lim_{z \to 0} z f(z) = \lim_{z \to 0} e^z - 1 = 0$$

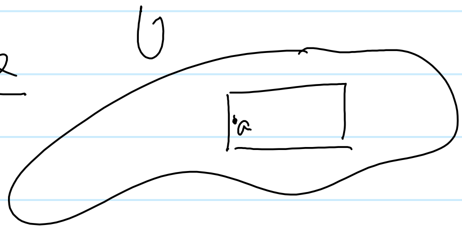
$\cancel{f(z) = \frac{z f(z)}{z}}$

$$f(z) = \frac{(z-a)f(z)}{(z-a)} \quad \forall z \neq a$$

$\to 0$

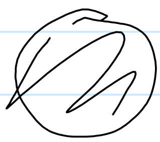
$\to 0$

Teorema

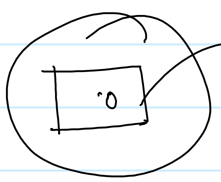


$\cancel{\int_{\partial R} f(z)\,dz = 0}$

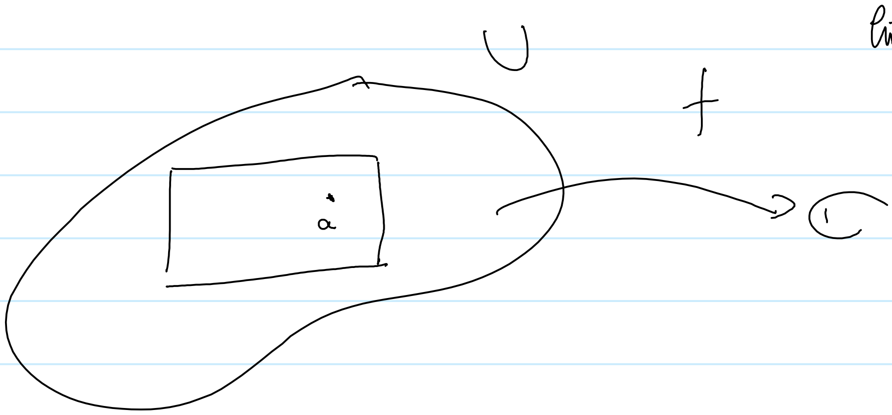
f analítica en $U - a$ donde $a \in \operatorname{int} R$ y la singularidad es evitable $\Rightarrow \int_{\partial R} f(z)\,dz = 0$



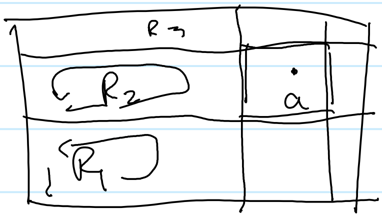
$$\int_\gamma \frac{\sin z}{z}\,dz = 0$$



$\lim_{z\to a}(z-a)f(z)=0$



$\int_\gamma f(z)dz=0$



$(z-a)f(z) \to 0$ cuando $z \to a$

$\int_{\partial R} f(z)dz$ Paso 1 cuadradito alrededor de a $R_7 \ni a$

Paso 2 completamos el rectángulo

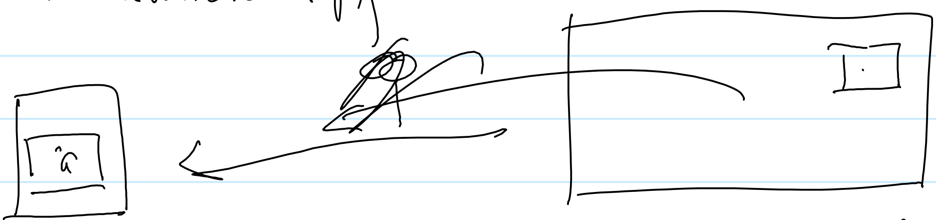
$R_1 \dots R_9$ R

$$\int_{\partial R} f(z)dz = \int_{\partial R_1} f(z)dz + \int_{\partial R_2} f(z)dz + \dots + \int_{\partial R_9} f(z)dz$$

$$\int_{\partial R} f(z)dz = \int_{\partial R_7} f(z)dz$$

3) Para un cuadrado R alrededor de a $\int_R f(z)dz=0$

4) → arbitrariamente pequeño



(*) $\forall$ R cuadrado alrededor de a $\int f(z)dz = \int$

5) $\forall$ R cuadrado alrededor de a $\int_{\partial R} f(z)\,dz = \int_{\text{el original}}$

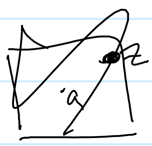
$\rightarrow \int_{\partial R} f(z) = 0$

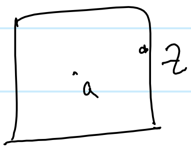
6) Lo probamos sobre un cuadrado arbitrariamente pequeño R_ε

$\forall \varepsilon\ \exists$ cuadrado alrededor de a : $\forall z \in$ cuadrado

$$|(z-a)f(z)| < \varepsilon$$

$$\left|\int_{\partial R_\varepsilon} f(z)\,dz\right| \le \int_{\partial R_\varepsilon} |f(z)|\,|dz| \le \varepsilon \int_{\partial R_\varepsilon} \frac{|dz|}{|z-a|}$$





$$|z-a| \ge l(R_\varepsilon)/2$$

$$\frac{1}{|z-a|} \le \frac{2}{l(R_\varepsilon)}$$

$$\le \frac{2}{l(R_\varepsilon)}\,\varepsilon \int_{\partial R_\varepsilon} |dz|$$

$\cancel{\int |f(z)|\,|dz|}$

$$\int_\gamma |dz| = \text{long}(\gamma)$$

$$= \frac{2}{l(R_\varepsilon)}\,\varepsilon\, \text{long}(R_\varepsilon) = \frac{2\varepsilon\, 4\, l(R_\varepsilon)}{l(R_\varepsilon)}$$

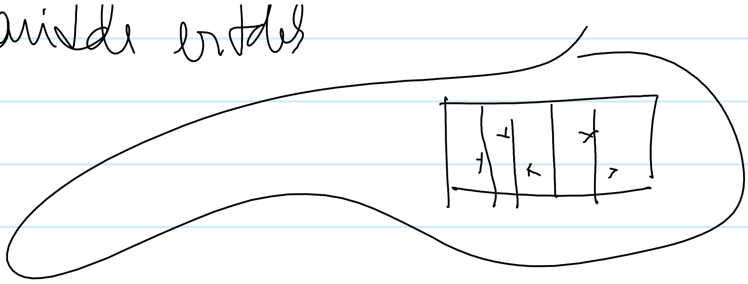
$$\left|\int_{R_\varepsilon} f(z)\,dz\right| \le 8\varepsilon \frac{\cancel{l(R_\varepsilon)}}{\cancel{l(R_\varepsilon)}} = 8\varepsilon$$

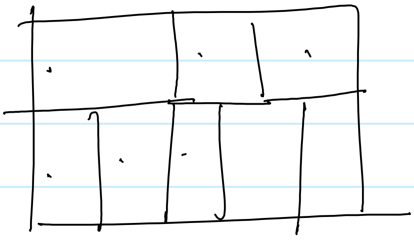
$$\left|\int_{\partial R} f(z)\,dz\right| \le 8\varepsilon \Longrightarrow \int_{\partial R} f(z)\,dz = 0$$

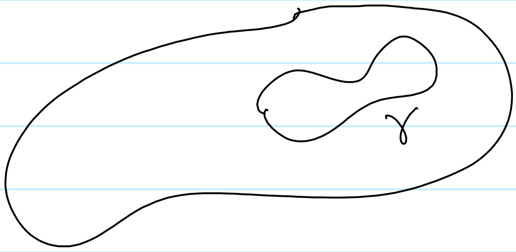
QED

El mismo teorema con un nº finito de singularidades aisladas

Singularidades evitables





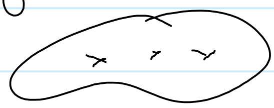


U

U verifica ✩

$$\int_\gamma f(z)\,dz = 0$$

Si en el interior de la curva hay una cantidad finita de singularidades evitables $\int_\gamma f(z)\,dz = 0$



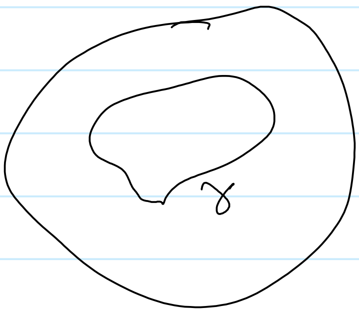
Teorema integral de Cauchy

Si $f : U \to \mathbb{C}$ es una función analítica con un nº finito de singularidades evitables en U.

Si U verifica además la condición ✩

$\Longrightarrow$ $\forall\, \gamma$ cerrado en U que no pasa por ninguna singularidad

$\Longrightarrow$ $\int_\gamma f(z)\,dz = 0$ | Un círculo, un simple, un rectángulo etc



$\xrightarrow{f} \mathbb{C}$ analítica

en D disco

Fijamos $a \in D$ y consideramos la función analítica

$$F(z) = \frac{f(z) - f(a)}{z - a}$$

a es una singularidad evitable de F

$$(z-a)F(z) = f(z) - f(a) \to 0 \quad z \to a$$

porque f es analítica continua

$$\int_\gamma F(z)\,dz = 0$$

si γ no pasa por a

$$\int_\gamma \frac{f(z) - f(a)}{z - a}\,dz = 0 ;\quad \int_\gamma \frac{f(a)}{z-a}\,dz = \int_\gamma \frac{f(z)}{z-a}\,dz$$

$$\underbrace{f(a) \int_\gamma \frac{dz}{z-a}}_{N(\gamma, a)} = \int_\gamma \frac{f(z)}{z-a}$$

$$f(a) = \frac{1}{N(\gamma, a)} \int_\gamma \frac{f(z)}{z-a}$$

si $N(\gamma, a) \neq 0$

Fórmula integral de Cauchy

$$f|_\gamma = g|_\gamma$$

$$f(a) = g(a) \quad \forall a \notin \gamma$$