Clases 8 y 9

viernes, 29 de abril de 2016 19:15

Práctico 1

Problema 1

$z = x + iy \qquad \mathbb{C} \simeq \mathbb{R}^2$

1) $(1+2i)^3 = 1^3 + 3\,1^2.(2i) + 3\,1\,(2i)^2 + (2i)^3$

$= 1 + 6i + 3\cancel{4}.i^2 + 8i^3 \qquad i^2 = -1$

$= 1 + 6i - 12 - 8i = -11 - 2i \qquad i^3 = -i$

2) $\dfrac{5}{-3+4i} \qquad \dfrac{1}{z} = \dfrac{\bar{z}}{|z|^2}$

$= 5\left(\dfrac{-3-4i}{25}\right) = \dfrac{1}{5}(-3-4i) = \dfrac{-3}{5} - \dfrac{4}{5}i$

4) $(1+i)^n - (1-i)^n \qquad z = \rho e^{i\alpha}$

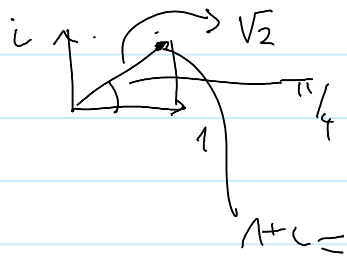
$zz' = \rho\rho' e^{i(\alpha+\alpha')} \qquad z' = \rho' e^{i\alpha'}$

$z = x + iy \quad z' = x' + iy'$

$zz' = (xx' - yy') + i(xy' + yx'); \; z + z' = (x + x') + i(y + y')$

$z = \rho e^{i\alpha} \quad z' = \rho e^{i\alpha'} \quad \| \; z + z' = \rho e^{i\alpha} + \rho e^{i\alpha'} \; \| \; \rho = \rho'$

$z + z' = \rho\left(e^{i\alpha} + e^{i\alpha'}\right)$



$\sqrt{2}\, e^{i\pi/4}$

$\|$

$1 + i = \sqrt{2}\; e^{i\pi/4}$

$1 - i = \sqrt{2}\; e^{-i\pi/4}$

$(1+i)^n = (\sqrt{2})^n e^{in\pi/4}$

$(1-i)^n = (\sqrt{2})^n e^{-in\pi/4}$

$(1+i)^n - (1-i)^n = (\sqrt{2})^n \left(e^{in\pi/4} - e^{-in\pi/4} \right)$

$(1-i)^n = (\sqrt{2})^n e^{-in\pi/4}$

$(1+i)^n - (1-i)^n = (\sqrt{2})^n \left(e^{in\pi/4} - e^{-in\pi/4} \right)$

$= (\sqrt{2})^n i\, 2\, \mathrm{Im}\left(e^{in\pi/4}\right) = (\sqrt{2})^n 2 i \,\mathrm{sen}\left(n\pi/4\right)$

$0, 1, 2, 3\ 4\ \underbrace{5\ 6\ 7}$

2)

$$\boxed{\begin{array}{cccccccc} 0 & \pi/4 & \pi/2 & 3\pi/4 & \pi & 5\pi/4 & 3\pi/2 & 7\pi/4 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ 0 & 2i & 4 & 4i & 0 & -8i & -16i & 32i \end{array}}$$

$\cancel{2\left(\binom{n}{1} - \binom{n}{3} + \binom{n}{5}\right)}$

$2\left[\binom{n}{1} - \binom{n}{3} + \binom{n}{5} - \binom{n}{7} \qquad \right] i$

2) $\dfrac{1}{z^2} = \dfrac{\bar{z}^2}{|z|^4} = \dfrac{(x-iy)^2}{(x^2+y^2)^2} = \dfrac{x^2-y^2-2ixy}{(x^2+y^2)^2}$

$= \dfrac{x^2-y^2}{(x^2+y^2)^2} + i\left(\dfrac{-2xy}{(x^2+y^2)^2}\right)$

$\begin{cases} z = x+iy = \rho e^{i\alpha} \\ z^2 = \rho^2 e^{2i\alpha} = (x^2+y^2)(\cos 2\alpha + i \,\mathrm{sen}\, 2\alpha) \end{cases}$ $\quad \cos\alpha = \dfrac{x}{\sqrt{x^2+y^2}} \quad \mathrm{sen}\,\alpha = y/\sqrt{x^2+y^2}$

$\mathrm{sen}(2\alpha) = 2\,\mathrm{sen}\,\alpha\cos\alpha$

$\cos 2\alpha = \cos^2\alpha - \mathrm{sen}^2\alpha$

$z^2 = (x^2+y^2)\left(\dfrac{x^2-y^2}{} + 2xyi\right)$ $\quad z = x+iy$

$z^2 = (x^4-y^4) + 2xy\,(x^2+y^2)\,i$ $\quad z^2 = x^2-y^2+2ixy$

$z = x+iy = \rho e^{i\alpha} = \rho\cos\alpha + i\rho\,\mathrm{sen}\,\alpha$

$$z = x+iy = \rho e^{i\alpha} = \rho\cos\alpha + i\rho\sin\alpha$$

$$\begin{cases} x = \rho\cos\alpha \\ y = \rho\sin\alpha \end{cases} \qquad \cos\alpha = x/\rho \qquad \cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

$$\sin\alpha = y/\rho \qquad \sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$z^2 = \rho^2 e^{2i\alpha} = \rho^2(\cos 2\alpha + i\sin 2\alpha)$$

$$\cos 2\alpha = x^2/\rho^2 - y^2/\rho^2 \qquad \sin 2\alpha = \frac{2xy}{\rho\rho} = \frac{2xy}{\rho^2}$$

$$z^2 = \rho^2\left(\frac{x^2-y^2}{\rho^2} + 2xyi/\rho^2\right) = x^2-y^2+2ixy$$

$$z^2 = (x+iy)^2 = x^2-y^2+2ixy$$

3)

$$|z-\lambda w|^2 = (z-\lambda w)(\bar{z}-\bar{\lambda}\bar{w})$$

$$= z\bar{z} + \lambda w\bar{\lambda}\bar{w} - z\bar{\lambda}\bar{w} - \lambda w\bar{z}$$

$$= |z|^2 + |\lambda|^2|w|^2 - (\bar{\lambda}z\bar{w} + \overline{\bar{\lambda}z\bar{w}}) = |z|^2 + |\lambda|^2|w|^2 - 2\mathrm{Re}(\bar{\lambda}z\bar{w})$$

$$|a+b|^2 = (a+b)(\bar{a}+\bar{b}) = a\bar{a} + b\bar{b} + \cancel{}\, a\bar{b} + b\bar{a}$$

$$= |a|^2 + |b|^2 + 2\mathrm{R}(a\bar{b})$$

$\bar{a}$ $\qquad \overline{(a\bar{b})}$

$$|z_i - \lambda\bar{w}_i|^2 = |z_i|^2 + |\lambda|^2|w_i|^2 - 2\mathrm{Re}(\bar{\lambda}z_iw_i) \qquad i=1\ldots n$$

$$\sum_i |z_i - \lambda\bar{w}_i|^2 = \sum |z_i|^2 + |\lambda|^2\sum|w_i|^2 - 2\mathrm{Re}\left(\bar{\lambda}\sum z_iw_i\right)$$

$$\lambda = \frac{\sum z_i w_i}{\sum |w_i|^2}$$

$$\sum_i |z_i - \lambda \bar{w}_i|^2 = \sum |z_i|^2 + \frac{|\sum z_i w_i|^2}{(\sum |w_i|^2)^2} \sum |w_i|^2$$

$$-2\,\mathrm{Re}\left(\frac{\overline{\sum z_i w_i} \sum z_i w_i}{\sum |w_i|^2} \right)$$

$$\sum_i |z_i - \lambda \bar{w}_i| = \sum |z_i|^2 + \frac{|\sum z_i w_i|^2}{\sum |w_i|^2} - 2\frac{|\sum z_i w_i|^2}{\sum |w_i|^2}$$

$$\boxed{\sum_i |z_i - \lambda \bar{w}_i|^2 = \sum |z_i|^2 - \frac{|\sum z_i w_i|^2}{\sum |w_i|^2} \geq 0}$$

substituindo o valor

$$|\sum z_i w_i|^2 \leq \left(\sum |z_i|^2\right)\left(\sum |w_i|^2\right)$$

iii) Si $|\sum z_i w_i|^2 = \sum |z_i|^2 \sum |w_i|^2$

$$\Downarrow$$

$$\sum_{i=1}^{n} |z_i - \lambda \bar{w}_i|^2 = 0$$ suma de reales positivos ~~al cuadrado~~

$\Longrightarrow |z_i - \lambda \bar{w}_i| = 0 \Longleftrightarrow z_i = \lambda \bar{w}_i$ OK

$$\lambda = \overline{\sum z_i w_i} \Big/ \sum |w_i|^2$$

iv) $$|\sum z_i w_i|^2 = \left(\sum |z_i|^2\right)\left(\sum |w_i|^2\right) - \sum_{1 \leq i < j \leq n} |z_i \bar{w}_j - z_j \bar{w}_i|^2$$

$1 \le i < j \le n$

$$\boxed{|a+b|^2 = |a|^2 + |b|^2 + 2\operatorname{Re}(a\bar{b})}$$

$$|ab - cd|^2 = |a|^2|b|^2 + |c|^2|d|^2 - 2\operatorname{Re}(ab\bar{c}\bar{d})$$

$z_1 \dots z_n \quad w_1 \dots w_n \qquad j, k \in \{1, \dots, n\}$

$$\operatorname{Re}(z_j w_j \bar{z}_k \bar{w}_k) = \frac{1}{2}\left(|z_j|^2|w_k|^2 + |z_k|^2|w_j|^2 - |z_j\bar{w}_k - z_k\bar{w}_j|^2\right)$$

$a = z_j \quad b = \bar{w}_k \quad c = z_k \quad d = \bar{w}_j$

$\forall j, k$

$$\left(\sum |z_j|^2\right)\left(\sum |w_k|^2\right) \qquad \left(\sum |z_k|^2\right)\left(\sum |w_j|^2\right)$$

$$\sum_{j,k=1..n} \operatorname{Re}(z_j w_j \bar{z}_k \bar{w}_k) = \frac{1}{2}\left(\sum_{jk} |z_j|^2|w_k|^2 + \sum_{jk} |z_k|^2|w_j|^2 - \sum_{j,k} |z_j\bar{w}_k - z_k\bar{w}_j|^2\right)$$

$$= \left(\sum_j |z_j|^2\right)\left(\sum_k |w_k|^2\right) - \sum_{1 \le j < k \le n} |z_j\bar{w}_k - z_k\bar{w}_j|^2$$

$$\sum_{jk} \operatorname{Re}(z_j w_j \bar{z}_k \bar{w}_k) = \left|\sum z_i w_i\right|^2$$

$$\left(\sum_j z_j w_j\right)\overline{\left(\sum_k z_k w_k\right)}$$

$$f : \mathcal{C} \to \mathbb{C} \quad / \quad f\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = a + bi$$

$$f\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0 + 0i$$

$$f\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 + 0i = 1$$

Además f es biyectiva y preserva las operaciones $f(M+N) = f(M) + f(N)$

y $f(MN) = f(M)\,f(N)$

p.e. $M = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ $N = \begin{pmatrix} a' & b' \\ -b' & a' \end{pmatrix}$

$$\Rightarrow MN = \begin{pmatrix} aa' - bb' & ab' + ba' \\ -ba' - ab' & aa' - bb' \end{pmatrix}$$

$$f(MN) = (aa' - bb') + i\,(ab' + ba') = f(M) \cdot f(N)$$

$f(M) = a + ib$

$f(N) = a' + ib'$ luego