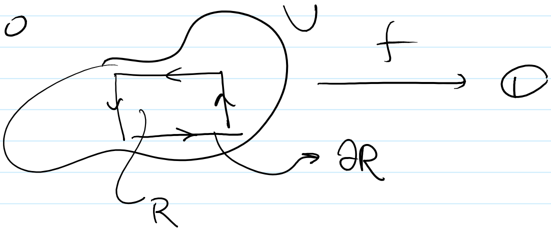
Clase 7

jueves, 28 de abril de 2016 20:18

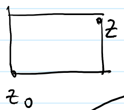
Formula Integral de Cauchy

Demostraremos que si $f: U \to \mathbb{C}$ es analítica y R es un rectángulo totalmente contenido en U entonces

$$\int_{\partial R} f(z)\,dz = 0$$

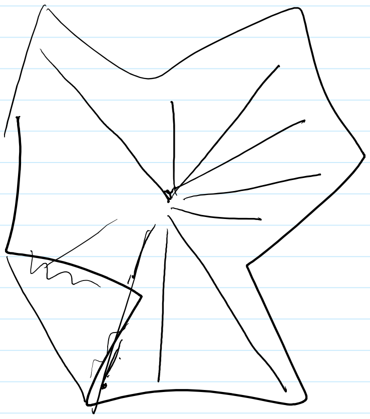


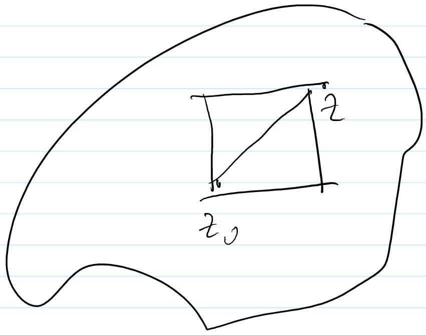
Corolario (Sí) $f: U \to \mathbb{C}$ es analítica – holomorfa en una región U con la propiedad que existe un punto $z_0 = (x_0, y_0)$: $\forall z \in U$ el rectángulo de diagonal



$z_0 z$ está en U $\Longrightarrow$ $\forall$ curva γ definida en U cerrada se tiene que $\int_\gamma f(z)\,dz = 0$ (*)

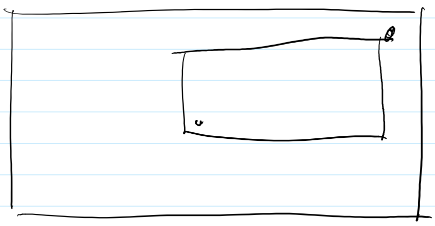
Obs Este teorema es conceptualmente mucho más fuerte que el que primero formulamos pues habla de curvas cualesquiera.

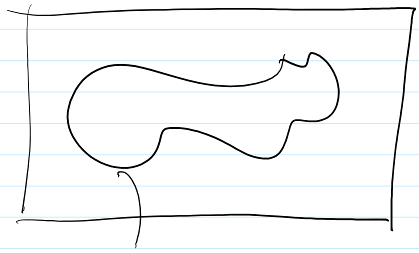




Un rectángulo cumple la condición

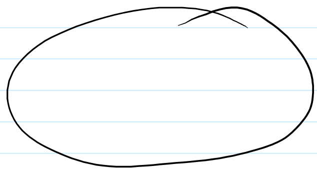
(*)



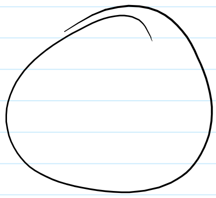


$\mathbb{C}$ completo (*) $\qquad 0 = \int_\gamma f(z)\,dz$

Un cuadrante, un semiplano



el interior de una elipse



el interior de un círculo

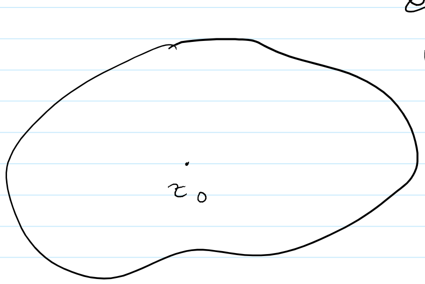
etc

D es un disco abierto del plano complejo y $f : D \to \mathbb{C}$ es analítica $\Rightarrow$

$\forall\, \gamma : [a,b] \to D$ cerrada

se tiene que $\int_\gamma f(z)\,dz = 0$

U



$f : U \to \mathbb{C}$

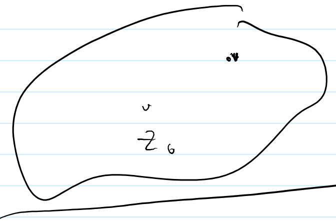
$\int_\gamma f(z)\,dz = 0$

Si F es holomorfa en una región U y

γ es una curva cerrada en U $\Rightarrow$ $\int_\gamma F'(z)\,dz = 0$

En un dominio que cumple la condición ($*$) γ
si $f : U \to \mathbb{C}$ es holomorfa (analítica) $\exists$
$F : U \to \mathbb{C}$ también holomorfa : $F' = f$

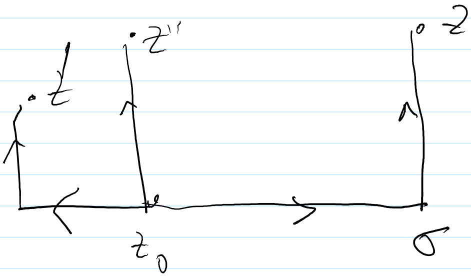
Toda función analítica en una región con la propiedad $*$ es la derivada de una función analítica (tiene una primitiva analítica)



$\xrightarrow{f} \mathbb{C}$

queremos que de la primitiva

$\boxed{\text{Construir la primitiva}}$ definimos F en $z \in U$

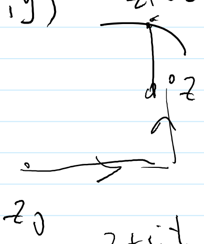


$$F(z) = \int_\sigma f(\zeta)\, d\zeta$$

Queremos probar que F' existe y es f

$$\frac{\partial F}{\partial y} \qquad \frac{\partial F}{\partial x}$$

(x,y)

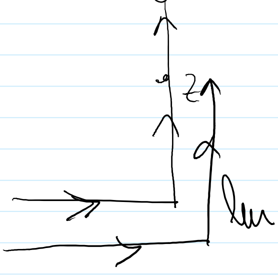


$$\frac{\partial F}{\partial y}(z) = \lim_{t \to 0} \frac{-F(x,y) + F(x,y+t)}{t}$$

z

$F(z+it)$

$F(z)$



$$\lim \frac{F(x,y+t) - F(x)}{t} = \frac{1}{t} \int_{z,z+it} f(\zeta)\, d\zeta$$

t t

σ_t from z to $z+it$

$$\lim_{t\to 0} \frac{F(x, y+t) - F(x)}{t} = \lim_{t\to 0} \frac{\int_{\sigma_t} f(\zeta)\,d\zeta}{t}$$

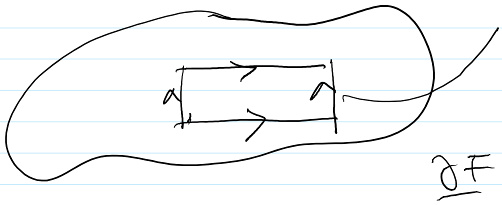
$\gamma(s) = z+is$ $\quad [0,t]$ $\quad \gamma'(s) = i$ $\qquad \int_{\sigma_t} f(\zeta)\,d\zeta = \int_0^t f(z+is)\,i\,ds$

$$\frac{\partial F}{\partial y}(z) = i \lim_{t\to 0} \frac{\int_0 f(z+is)\,ds}{t} \qquad 0 \le \theta \le 1$$

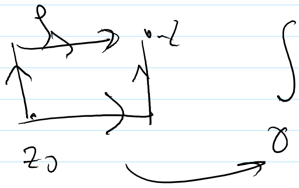
$$\lim_{t\to 0} \frac{\int_0^t f(x, y+s)\,ds}{t} \qquad \int_0^t f(x, y+s)\,ds = f(x, y+\theta t)\,t$$

$$\lim_{t\to 0} \frac{f(x, y+\theta t)\,\cancel{t}}{\cancel{t}} = f(x,y)$$

$$\boxed{\frac{\partial F}{\partial y} = if}$$



$\frac{\partial F}{\partial y}$ $\quad \frac{\partial F}{\partial x}$



$$\int_\gamma f(\zeta)\,d\zeta = F(z) = \int_\delta f(\zeta)\,d\zeta$$

Teorema integral de Cauchy en su versión rectángulo.

$$\int_\gamma f(\zeta)\,d\zeta = \int_\delta f(\zeta)\,d\zeta$$

pues $\boxed{\int_{\gamma-\delta} f(\zeta)\,d\zeta = 0}$ pues $\gamma - \delta = \partial R$ y sabemos el primer teorema integral

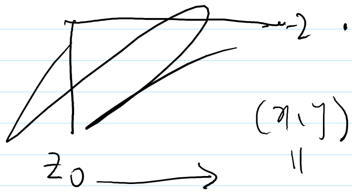
$$\Downarrow$$

$$\int f(\zeta)\,d\zeta + \int -f(\zeta)\,d\zeta = 0 \Rightarrow \int f(\zeta)\,d\zeta - \int f(\zeta)\,d\zeta = 0$$

$$\int_\gamma f(\zeta)\,d\zeta + \int_{-\delta} f(\zeta)\,d\zeta = 0 \Rightarrow \int_\gamma f(\zeta)\,d\zeta - \int_\delta f(\zeta)\,d\zeta = 0$$

$$\Longrightarrow \int_\gamma f(\zeta)\,d\zeta = \int_\delta f(\zeta)\,d\zeta$$

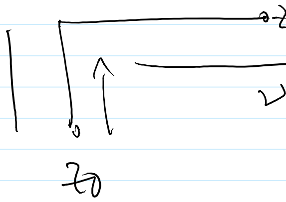
$$\frac{\partial F}{\partial y} = if \qquad \frac{\partial F}{\partial x} = f \quad \leftarrow 1^{\text{o}} \text{ basic}$$



$\cancel{\lim_{t\to 0} F(z_0}$

$(x,y) \qquad (x+t,y)$

$\parallel$



$z \quad z+t \qquad t \in \mathbb{R}$

$F(z+t)$

$$\lim_{t\to 0} \frac{\int_\gamma f(\zeta)\,d\zeta - \int_{\gamma'} f(\zeta)\,d\zeta}{t} \longleftrightarrow F(z)$$

$$\boxed{\lim_{t\to 0} \int_{[z,z+t]} f(\zeta)\,d\zeta \,\cancel{/t}}$$

$[z, z+t]$

$\gamma(s) = z+s$

$0 \le s \le t$

$\gamma'(s) = 1$

$$\lim_{t\to 0} \frac{\int_0^t f(z+s)\gamma'(s)\,ds}{t} = \lim_{t\to 0} \frac{\int_0^t f(z+s)\,ds}{t}$$

$$\int_0^t f(z+s)\,ds = \int_0^t f(x+s,y)\,ds = f(x+\theta t, y)\,t \qquad 0 \le \theta \le 1$$

$$\lim \frac{\int_0^t f(z+s)\,ds}{t} = \lim f(x+\theta t, y) = f(x,y)$$

$$\lim_{t\to 0} \frac{\int_0^t f(z+s)\,ds}{t} = \lim_{t\to 0} f(x+\theta t, y) = f(x,y)$$

$$\boxed{\frac{\partial F}{\partial x} = f \qquad \frac{\partial F}{\partial y} = if} \longleftrightarrow i\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y}$$

$F = U + iV \quad i(U_x + iV_x) = U_y + iV_y$

$\Longrightarrow U_x = V_y \quad U_y = -V_x$ Cauchy Riemann

F es analítica

$$\frac{dF}{dz} = \frac{\partial F}{\partial x} = f$$

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f f es analítica si $\exists\ \frac{\partial u}{\partial x}\ \frac{\partial u}{\partial y}\ \frac{\partial v}{\partial x}\ \frac{\partial v}{\partial y}$

y $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = f'(z)$$

$z, h \in \mathbb{C}$

$$\boxed{\frac{\partial f}{\partial x} = f'(z) \;\Big|\; \frac{\partial f}{\partial y} = if'(z)}$$

$f'(z) = \frac{\partial f}{\partial x}(z)$

$= u_x + iv_x$

$= v_y - iu_y$

$= -i(u_y + iv_y) = -i\frac{\partial f}{\partial y}$

En una región que verifica $(*)$ se tiene que $\forall \gamma : [a,b] \to \mathbb{C}$ y $\forall f : U \to \mathbb{C}$ analítica

$$\int_\gamma f(z)\,dz = 0$$