Clase 5

viernes, 08 de abril de 2016 19:03

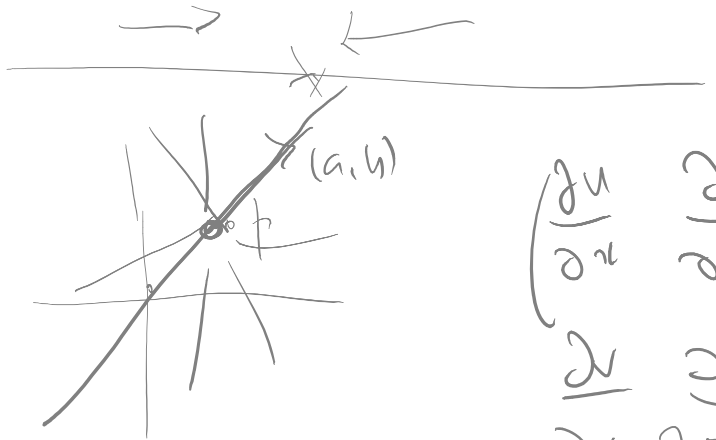
$f: U \to \mathbb{C}$ derivada

$\mathbb{C} \supset U \xrightarrow{f} \mathbb{R}^2$, $U \subset \mathbb{R}^2$

$f(x,y) = (u(x,y), v(x,y))$

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

$f: \mathbb{R} \to \mathbb{R}$



$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

Cauchy Riemann

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Decimos que $f: U \to \mathbb{C}$ es holomorfa

si es diferenciable y además

en U y es diferenciable y ...

verifica las condiciones de CR

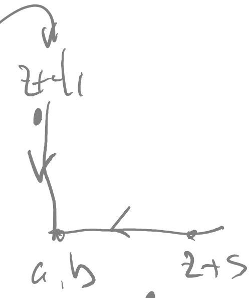
Definimos $f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \quad \forall z \in U$

f tiene derivada

$z = (a, b) = a + bi$

$z + ti = a + (b+t)i \qquad t \in \mathbb{R}$

$z + s = a + s + bi \qquad s \in \mathbb{R}$



$$\lim_{t \to 0} \frac{f(a + (b+t)i) - f(a+bi)}{ti} =$$

$$\lim_{s \to 0} \frac{f(a+s+bi) - f(a+bi)}{s} = f'(z) \qquad z = (a,b) = a+bi$$

U

$f(z) = u(x,y) + iv(x,y)$

$$\lim \frac{u(a, b+t) - u(a,b) + i(v(a,b+t) - v(a,b))}{ti} =$$

$$\lim \frac{u(a+s,b) - u(a,b) + i(v(a+s,b) - v(a,b))}{s}$$

$\frac{u_y(a,b)}{} + v_y(a,b) = u_x(a,b) + iv_x(a,b)$

$v_y(a,b) - i u_y(a,b) = u_x(a,b) + i v_x(a,b)$

~~$f'(z) = v_y(a,b) - i u_y$~~

$$\boxed{f' = v_y - i u_y = u_x + i v_x}$$

teorema

Si f admite derivada compleja en U entonces

1) $v_y = u_x$ & $-u_y = v_x$

2) $f' = v_y - i u_y = u_x + i v_x$

Corolario

Si f admite derivada . . . .

$\Longrightarrow f$ es holomorfa

$|f'|^2 = v_y^2 + u_y^2 = u_x^2 + v_x^2$

$d_x f = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} u_x & u_y \\ -u_y & u_x \end{pmatrix}$ | $\det(d_x f) = |f'(z)|^2$

$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$ $J_x(f) = \det$

$|f'(z)|^2 = \det(d_x f) = J_x(f)$

def

$f: \hat{U} \to \mathbb{C} \mid f = u + iv$ u, v son derivables

$f : \underset{\mathbb{C}}{U} \to \mathbb{C}$ | $f = u + iv$ u, v son derivables

mas u_y y sus der continuas

f es holomorfa $\longleftrightarrow$ y satisfacen CR

$\Rightarrow$ existe $f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$

$f = u + iv$ $\quad z \quad w = h + ik$

$\frac{f(z+h+ik) - f(z)}{h+ik}$ $\quad w \to 0 \quad h, k \to 0$

$u(x+h, y+k) - u(x,y) = \frac{\partial u}{\partial x} h + \frac{\partial u}{\partial y} k + \varepsilon \qquad \frac{\varepsilon}{\sqrt{h^2+k^2}} \to 0$

$v(x+h, y+k) - v(x,y) = \frac{\partial v}{\partial x} h + \frac{\partial v}{\partial y} k + \varepsilon' \qquad \frac{\varepsilon'}{\sqrt{h^2+k^2}} \to 0$

$f(z + h + ik) - f(z) = (u_x + iv_x) h + (u_y + iv_y) k + \varepsilon + i\varepsilon'$

$= (u_x + iv_x) h + (v_y - iu_y)\, ik + \varepsilon + i\varepsilon'$

$= (u_x + iv_x)(h + ik) + \varepsilon + i\varepsilon'$

$\frac{f(z+h+ik) - f(z)}{h+ik} = u_x + iv_x + \frac{\varepsilon + i\varepsilon'}{h+ik} \to 0$

$w = h + ik$

$\lim_{w \to 0} \frac{f(z+w) - f(z)}{w} = \underbrace{u_x + iv_x}_{f'(z)}$

$\forall w$

$\frac{\varepsilon}{\quad} \to 0 \qquad \frac{\varepsilon'}{\quad} \to 0$ | $\alpha_n \to \alpha$

$\alpha f(\alpha_n) \to \alpha$.

$$\frac{\varepsilon}{\sqrt{h^2+k^2}} \to 0 \qquad \frac{\varepsilon'}{\sqrt{h^2+k^2}} \to 0$$

$$\frac{\varepsilon+i\varepsilon'}{h+ik} \to 0$$

$\sqrt{a_n^2+b_n^2}$

$\alpha_n \to \alpha$

$\alpha_n - \alpha \to 0$

$\alpha_n \to 0 \overset{\text{def}}{\Leftrightarrow} |\alpha_n| \to 0$

$\alpha_n = a_n + i b_n$

$a_n \to 0 \quad b_n \to 0$

$\sqrt{a_n^2+b_n^2} \to 0$

$$|a_n| \leq \sqrt{a_n^2+b_n^2}$$

$\downarrow \qquad \downarrow$

$0 \qquad 0$

Hip $\left(\frac{\varepsilon}{\sqrt{h^2+k^2}} \to 0 \quad \frac{\varepsilon'}{\sqrt{h^2+k^2}} \to 0 \right) \Rightarrow \left(\frac{\varepsilon+i\varepsilon'}{h+ik} \to 0 \right)$ T

$$\frac{\sqrt{\varepsilon^2+\varepsilon'^2}}{\sqrt{h^2+k^2}} = \sqrt{\underbrace{\frac{\varepsilon^2}{h^2+k^2}}_{\to 0} + \underbrace{\frac{\varepsilon'^2}{h^2+k^2}}_{\to 0}}$$

Integración compleja

f admite derivada $\Rightarrow$ la derivada es continua

$\Downarrow$ primera $\nearrow$

admite todas las derivadas

$f : [a,b] \to \mathbb{C} \qquad \int_a^b f(t)\,dt$

$f = u + iv$

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

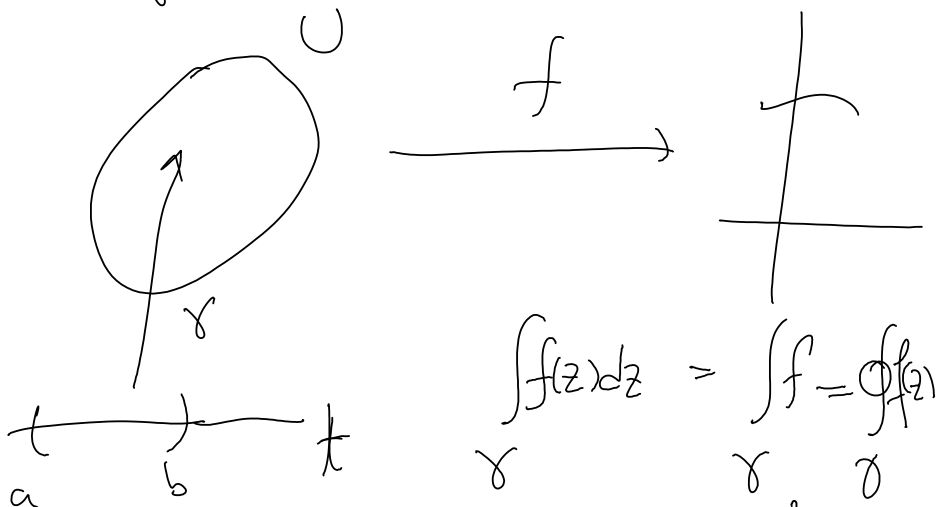
$f:\mathbb{R}^n \to \mathbb{C}$

$a \le b \quad \left| \int_a^b f(t)\,dt \right| \le \int_a^b |f(t)|\,dt$

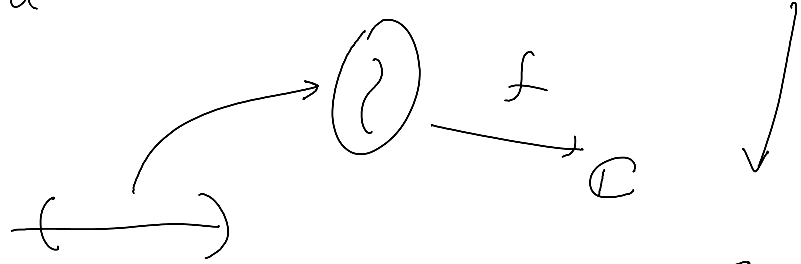
$\int_a^b (f+g) = \int_a^b f + \int_a^b g \quad \Big| \quad \int_a^b \lambda f = \lambda \int_a^b f$

$\lambda \in \mathbb{C}$

Integrales curvilíneas



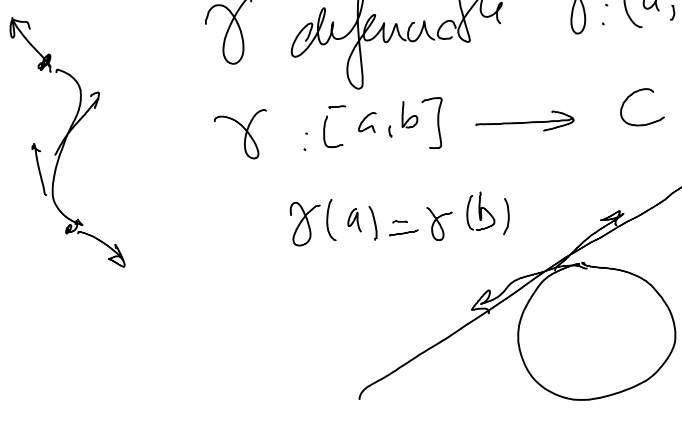
$\int_\gamma f(z)\,dz = \int_\gamma f = \oint_\gamma f(z)\,dz$



γ diferenciable $\gamma:(a,b) \to \mathbb{C}$

$\gamma:[a,b] \to \mathbb{C} \quad \gamma|_{(a,b)}$ diferenciable

$\gamma(a)=\gamma(b)$



$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

$$\gamma(t) = p(t) + i\,q(t) \qquad \gamma'(t) = p'(t) + i\,q'(t)$$

$$f(z) = u(x,y) + i\,v(x,y)$$

$$\int_a^b f(\gamma(t))\gamma'(t)\,dt = \int_a^b (u+iv)(p'+iq')\,dt$$

$$= \int_a^b \left(u(\gamma(t))\,p'(t) - v(\gamma(t))\,q'(t)\right) dt + i\int_a^b u(\gamma(t))\,q'(t) + v(\gamma(t))\,p'(t)\,dt$$

$dz = dx + i\,dy$

$x = p(t)$

$y = q(t)$

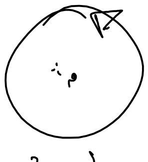
$$\int_\gamma f(z)\,dz = \int_\gamma (u+iv)(dx+i\,dy)$$

$$= \int_\gamma (u\,dx - v\,dy) + i\int_\gamma v\,dx + u\,dy$$

$$\gamma(t) = \cos t + i \sin t = e^{it}$$

$\cos t = p$, $\sin t = q$, $t \in [0, 2\pi]$

$$\int_{S^1} \frac{dz}{z} = \int_0^{2\pi} \frac{\gamma'(t)}{\gamma(t)}\,dt$$



$$\int_{S^1} \frac{dz}{z} = \int_0^{2\pi}$$

$x^2 + y^2 = 1 \qquad S^1$

$$\gamma(t) = (\cos t, \sin t) \qquad \gamma'(t) = (-\sin t, \cos t)$$

$$\frac{\gamma'(t)}{\gamma(t)} = \cancel{\frac{(\cos t, \sin t)}{(-\sin t, \cos t)}}$$

$$i e^{it} \qquad i e^{it} = (-\sin t, \cos t)$$

$\gamma'(t)$

$$= \frac{ie^{it}}{e^{it}} \qquad ie^{it} = (-\sin t, \cos t)$$

$$\gamma(t) = \cos t + i \sin t$$

$$\frac{\gamma'(t)}{\gamma(t)} = i \qquad \int_0^{2\pi} d \qquad i\gamma(t) = -\sin t + i\cos t$$

$$\int_{S^1} \frac{dz}{z} = \int_0^{2\pi} i\,dt = 2\pi i$$

$$\int_\gamma f(z)\,dz = \int_a^b \big(u(\gamma(t))\varphi'(t) - v(\gamma(t))q'(t)\big)dt$$

$$+ i\int_a^b \big(v(\gamma(t))\varphi'(t) + u(\gamma(t))\,q'(t)\big)dt$$

$$\int_\gamma f(z)\,dz = \int_\gamma z\,dz \qquad \gamma = S^1$$

$$= \int_0^{2\pi} (\cos t + i\sin t)(-\sin t + i\cos t)\,dt$$

$$= \int_0^{2\pi} \big(\cos t(-\sin t) - \sin t\cos t\big) + i\,(\cos^2 t - \sin^2 t)\,dt$$

$$= \int_0^{2\pi} (-\sin 2t + i\cos 2t)\,dt \qquad ie^{2it}$$

$$= 0$$

$$\gamma(t) = e^{it} \qquad \gamma'(t) = ie^{it}$$

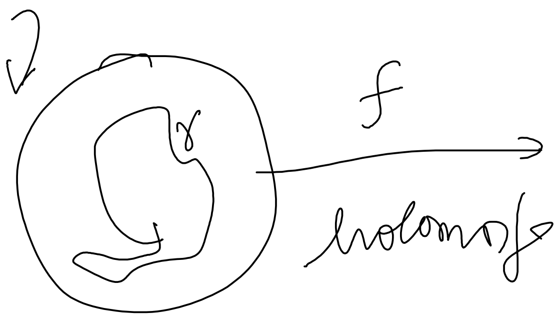
$\gamma(t) = e^{it} \quad \gamma'(t) = ie^{it}$

$$i\int_0^{2\pi} e^{it} e^{it}\, dt = \int_\gamma z\, dz$$

$$\underbrace{i\int_0^{2\pi} e^{2it}\, dt}_{0 \quad 1} = 0$$

Formula integral de Cauchy

Teorema integral de Cauchy



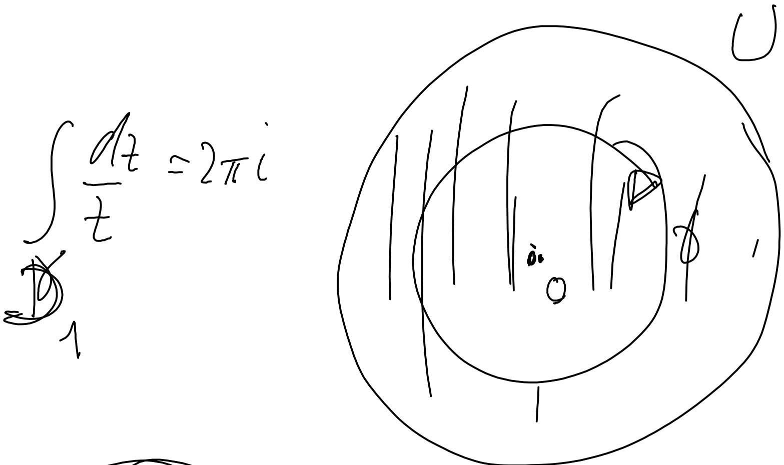
$\mathbb{C}$

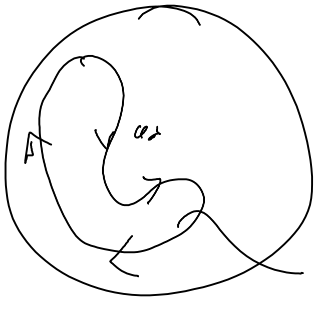
holomorfa

Disco

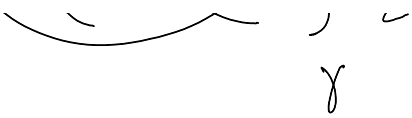
$$\int_\gamma f(z)\, dz = 0$$

$$\int_\gamma \frac{dz}{z} = 2\pi i$$



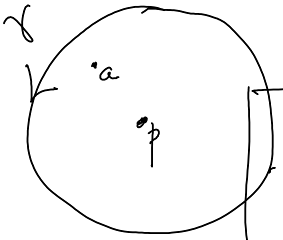


$$\int_\gamma \frac{dz}{z} = 0$$



Fórmula integral de Cauchy

γ disco de centro p y radio r



$$\boxed{f(a)=\frac{1}{2\pi i}\int_{\gamma}\frac{f(z)}{z-a}\,dz}$$

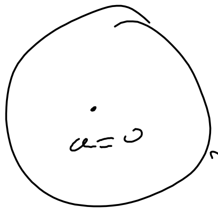
$p=0$

$$=0 \qquad a\notin D$$

$$D(0,1)=\{e^{it} : t\in[0,2\pi]\}$$

$a=0$

$$f(0)=\frac{1}{2\pi i}\int_{D}\frac{f(z)}{z}\,dz$$



$\gamma(t)=e^{it}$

$$f(0)=\frac{1}{2\pi i}\int_{0}^{2\pi}\frac{f(e^{it})\,ie^{it}}{e^{it}}\,dt$$

$$f(0)=\frac{1}{2\pi}\int_{0}^{2\pi}f(e^{it})\,dt,$$

$$\boxed{f(0)=\frac{1}{2\pi}\int_{0}^{2\pi}f(e^{it})\,dt}$$

$f=u+iv$

$$u(0,0)=\frac{1}{2\pi}\int_{0}^{2\pi}u(\cos t,\operatorname{sen} t)\,dt$$

$$v(0,0)=\frac{1}{2\pi}\int_{0}^{2\pi}v(\cos t,\operatorname{sen} t)\,dt$$